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Continuity of weakly monotone Sobolev functions of variable exponent

Toshihide Futamura and Yoshihiro Mizuta

Abstract.

Our aim in this paper is to deal with continuity properties for weakly monotone Sobolev functions of variable exponent.

§1. Introduction

This paper deals with continuity properties of weakly monotone Sobolev functions. We begin with the definition of weakly monotone functions. Let D be an open set in the n-dimensional Euclidean space $\mathbf{R}^n$ $(n \geq 2)$. A function u in the Sobolev space $W^{1,q}_{loc}(D)$ is said to be weakly monotone in D (in the sense of Manfredi [12]), if for every relatively compact subdomain G of D and for every pair of constants $k \leq K$ such that

$$(k-u)^+ \qquad \text{and} \qquad (u-K)^+ \in W^{1,q}_0(G),$$

we have

$$k \leq u(x) \leq K \qquad \text{for a.e. } x \in G,$$

where $v^+(x) = \max\{v(x), 0\}$. If a weakly monotone Sobolev function is continuous, then it is monotone in the sense of Lebesgue [11]. For monotone functions, see Koskela-Manfredi-Villamor [9], Manfredi-Villamor [13, 14], the second author [17], Villamor-Li [20] and Vuorinen [21, 22].

Following Kováčik and Rákosník [10], we consider a positive continuous function $p(\cdot) : D \to (1, \infty)$ and the Sobolev space $W^{1,p(\cdot)}(D)$ of

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all functions u whose first (weak) derivatives belong to $L^{p(\cdot)}(D)$. In this paper we consider the function $p(\cdot)$ satisfying

$$|p(x) - p(y)| \le \frac{a \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \frac{b}{\log(1/|x-y|)}$$

whenever $|x-y| < 1/2$, for $a \ge 0$ and $b \ge 0$.

Our first aim is to discuss the continuity for weakly monotone functions u in the Sobolev space $W^{1,p(\cdot)}(D)$. For the properties of Sobolev spaces of variable exponent, we refer the reader to the papers by Diening [2], Edmunds-Rákosník [3], Kováčik-Rákosník [10] and Růžička [19].

We know that if $p(x) \ge n$ for all $x \in D$, then all weakly monotone functions in $W^{1,p(\cdot)}(D)$ are continuous in D (see Manfredi [12] and Manfredi-Villamor [13]). We show that u is continuous at $x_0 \in D$ when $p(\cdot)$ is of the form

$$p(x) = n - \frac{a \log(\log(1/|x-x_0|))}{\log(1/|x-x_0|)} \qquad (p(x_0) = n)$$

for $x \in B(x_0, r_0)$, where $0 < r_0 < 1/2$ and $a \le 1$.

Our second aim is to prove the existence of boundary limits of weakly monotone Sobolev functions on the unit ball B, when $p(\cdot)$ satisfies the inequality

$$\left| p(x) - \left\{ n + \frac{a \log(e + \log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))}$$

for $a \ge 0$ and $b \ge 0$, where $\rho(x) = 1 - |x|$ denotes the distance of x from the boundary ∂B. Continuity of Sobolev functions has been obtained by Harjulehto-Hästö [7] and the authors [4]. Of course, our results extend the non-variable case studied in [17].

§2. Weakly monotone Sobolev functions

Throughout this paper, let C denote various constants independent of the variables in question.

We use the notation $B(x,r)$ to denote the open ball centered at x of radius r. If u is a weakly monotone Sobolev function on D and $q > n-1$, then

$$(1) \qquad |u(x) - u(x')|^q \le C r^{q-n} \int_{A(y,2r)} |\nabla u(z)|^q dz$$

for almost every $x, x' \in B(y,r)$, whenever $B(y,2r) \subset D$ (see [12, Theorem 1]) and $A(y,2r) = B(y,2r) \setminus B(y,r)$. If we define $u^*(x)$ by

$$u^*(x) = \limsup_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)dy,$$

then we see that u^* satisfies (1) for all $x, x' \in B(y,r)$. Note here that u^* is a quasicontinuous representative of u and it is locally bounded on D. Hereafter, we identify u with u^*.

EXAMPLE 2.1. Let $1 < q < \infty$ and $\mathcal{A} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be a mapping satisfying the following assumptions for some measurable function α and constant β such that $0 < \alpha(x) \leq \beta < \infty$ for a.e. $x \in \mathbf{R}^n$:

(i) the mapping $x \mapsto \mathcal{A}(x,\xi)$ is measurable for all $\xi \in \mathbf{R}^n$,
(ii) the mapping $\xi \mapsto \mathcal{A}(x,\xi)$ is continuous for a.e. $x \in \mathbf{R}^n$,
(iii) $\mathcal{A}(x,\xi) \cdot \xi \geq \alpha(x)|\xi|^q$ for all $\xi \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$,
(iv) $|\mathcal{A}(x,\xi)| \leq \beta|\xi|^{q-1}$ for all $\xi \in \mathbf{R}^n$ and a.e. $x \in \mathbf{R}^n$.

Then a weak solution of the equation

$$-\operatorname{div}\mathcal{A}(x, \nabla u(x)) = 0 \tag{2}$$

in an open set D is weakly monotone (see [9, Lemma 2.7]). In the special case $\alpha(x) \geq \alpha > 0$, according to the well-known book by Heinonen-Kilpeläinen-Martio [8], a weak solution of (2) is monotone in the sense of Lebesgue.

§3. Continuity of weakly monotone functions

For an open set G in $\mathbf{R}^n$, define the $L^{p(\cdot)}(G)$ norm by

$$\|f\|_{p(\cdot)} = \|f\|_{p(\cdot),G} = \inf\left\{\lambda > 0 : \int_G \left|\frac{f(y)}{\lambda}\right|^{p(y)} dy \leq 1\right\}$$

and denote by $L^{p(\cdot)}(G)$ the space of all measurable functions f on G with $\|f\|_{p(\cdot)} < \infty$. We denote by $W^{1,p(\cdot)}(G)$ the space of all functions $u \in L^{p(\cdot)}(G)$ whose first (weak) derivatives belong to $L^{p(\cdot)}(G)$. We define the conjugate exponent function $p'(\cdot)$ to satisfy $1/p(x) + 1/p'(x) = 1$.

Let $B(x,r)$ be the open ball centered at x and radius $r > 0$, and let $B = B(0,1)$. Consider a positive continuous function $p(\cdot)$ on $[0,1]$ such that $\inf_{r \in [0,1]} p(r) > 1$ and

$$\left|p(r) - \left\{n - \frac{a\log(e + \log(1/r))}{\log(e/r)}\right\}\right| \leq \frac{b}{\log(e/r)} \qquad (p(0) = n)$$

for $a \geq 0$ and $b \geq 0$.

Our aim in this section is to prove that if $a \leq 1$, then functions in $W^{1,p(\cdot)}(B)$ are continuous at the origin, in spite of the fact that $p_-(B) = \inf_{x \in B} p(x) < n$. For this purpose, we prepare the following result.

LEMMA 3.1. *Let $p(x) = p(|x|)$ for $x \in B$. Let u be a weakly monotone Sobolev function in $W^{1,p(\cdot)}(B)$. If $a < 1$, then*

$$|u(x) - u(0)|^n \leq C(\log(1/r))^{a-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy,$$

and if $a = 1$, then

$$|u(x) - u(0)|^n \leq C(\log(\log(1/r)))^{-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy$$

whenever $|x| < r < 1/4$, where $R = \sqrt{r}$ when $a < 1$ and $R = e^{-\sqrt{\log(1/r)}}$ when $a = 1$.

PROOF. Let u be a weakly monotone Sobolev function in $W^{1,p(\cdot)}(B)$. Set $p_1(r) = p(r)/q$, where $n - 1 < q < n$. Then, as in (1), we apply Sobolev's theorem on the sphere $S(0,r)$ to establish

$$|u(x) - u(0)|^q \leq C r^{q-(n-1)} \int_{S(0,r)} |\nabla u(y)|^q dS(y)$$

for $|x| < r$. By Hölder's inequality we have

$$\begin{aligned} |u(x) - u(0)|^q &\leq C r^{q-(n-1)} \left(\int_{S(0,r)} dS(y) \right)^{1/p_1'(r)} \\ &\quad \times \left(\int_{S(0,r)} |\nabla u(y)|^{qp_1(r)} dS(y) \right)^{1/p_1(r)} \\ &\leq C r^{q-(n-1)/p_1(r)} \left(\int_{S(0,r)} |\nabla u(y)|^{p(r)} dS(y) \right)^{1/p_1(r)}, \end{aligned}$$

which yields

$$|u(x) - u(0)|^{p(r)} \leq C r (\log(1/r))^a \int_{S(0,r)} |\nabla u(y)|^{p(y)} dS(y)$$

for $|x| < r$. Since u is bounded on $B(0,1/2)$, we see that

$$|u(x) - u(0)|^n \leq C r (\log(1/r))^a \int_{S(0,r)} |\nabla u(y)|^{p(y)} dS(y).$$

Hence, by dividing both sides by $r(\log(1/r))^a$ and integrating them on the interval (r, R), we obtain

$$|u(x) - u(0)|^n \le C(\log(1/r))^{a-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy \quad \text{when } a < 1$$

and

$$|u(x) - u(0)|^n \le C(\log(\log(1/r)))^{-1} \int_{B(0,R)} |\nabla u(y)|^{p(y)} dy \quad \text{when } a = 1$$

whenever $|x| < r < 1/4$. □

Lemma 3.1 yields the following result.

THEOREM 3.2. *Let u be a weakly monotone Sobolev function in $W^{1,p(\cdot)}(B)$. If $a < 1$, then u is continuous at the origin and it satisfies*

$$\lim_{x \to 0} (\log(1/|x|))^{(1-a)/n} |u(x) - u(0)| = 0;$$

if $a = 1$, then

$$\lim_{x \to 0} (\log(\log(1/|x|)))^{1/n} |u(x) - u(0)| = 0.$$

REMARK 3.3. Consider the function

$$u(x) = \frac{x_n}{|x|}$$

for $x = (x_1, ..., x_n)$. If we define $u(0) = 0$, then u is a weakly monotone quasicontinuous representative in $\mathbf{R}^n$. Note that u is not continuous at 0 and if $a > 1$, then

$$\int_B |\nabla u(x)|^{p(x)} dx < \infty;$$

if $a \le 1$, then

$$\int_B |\nabla u(x)|^{p(x)} dx = \infty.$$

This shows that continuity result in Theorem 3.2 is good as to the size of a.

REMARK 3.4. Let φ be a nonnegative continuous function on the interval $[0, r_0]$ such that

(i) $\varphi(0) = 0$;
(ii) $\varphi'(t) \ge 0$ for $0 < t < r_0$;

(iii) $\varphi''(t) \le 0$ for $0 < t < r_0$.
Then note that

$$\varphi(s+t) \le \varphi(s) + \varphi(t) \tag{3}$$

for $s, t \ge 0$ and $s + t \le r_0$. Consider

$$\varphi(r) = \frac{\log(\log(1/r))}{\log(1/r)}, \quad \frac{1}{\log(1/r)}$$

for $0 < r \le r_0$; set $\varphi(r) = \varphi(r_0)$ for $r > r_0$. Then we can find $r_0 > 0$ such that φ satisfies (i) - (iii) on $[0, r_0]$, and hence (3) holds for all $s \ge 0$ and $t \ge 0$. Hence if we set

$$p(r) = n + \frac{a \log(e + \log(1/r))}{\log(e/r)} + \frac{b}{\log(e/r)},$$

then we can find $c > 0$ and $r_0 > 0$ such that

$$|p(s) - p(t)| \le \frac{|a| \log(\log(1/|s-t|))}{\log(1/|s-t|)} + \frac{c}{\log(1/|s-t|)}$$

whenever $|s - t| < r_0$.

§4. 0-Hölder continuity of continuous Sobolev functions

Consider a positive continuous function $p(\cdot)$ on the unit ball B such that $p_-(B) = \inf_{x \in B} p(x) > 1$ and

$$\left| p(x) - \left\{ p_0 + \frac{a \log(e + \log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))}$$

for all $x \in B$, where $1 < p_0 < \infty$ and $\rho(x) = 1 - |x|$ denotes the distance of x from the boundary ∂B. Then note that

$$\begin{aligned} p'(x) - p'_0 &= -\frac{p(x) - p_0}{(p(x)-1)(p_0-1)} \\ &= -\frac{p(x) - p_0}{(p_0-1)^2} + \frac{(p(x)-p_0)^2}{(p(x)-1)(p_0-1)^2}, \end{aligned}$$

where $p'_0 = p_0/(p_0 - 1)$. Hence we have the following result.

LEMMA 4.1. *There exist positive constants r_0 and C such that*

$$|p'(x) - \{p'_0 - \omega(\rho(x))\}| \le C/\log(1/\rho(x))$$

for $x \in B$, *where* $\omega(t) = (a/(p_0-1)^2)\log(\log(1/t))/\log(1/t)$ *for* $0 < r \le r_0 < 1/e$; *set* $\omega(t) = \omega(r_0)$ *for* $r > r_0$.

We see from Sobolev's theorem that all functions $u \in W^{1,p(\cdot)}(B)$ are continuous in B when $p(x) > n$ in B. In what follows we discuss the 0-Hölder continuity of u. Before doing so, we need the following result.

LEMMA 4.2. *Let* $p_0 = n$ *and let* u *be a continuous Sobolev function in* $W^{1,p(\cdot)}(B)$ *such that* $\||\nabla u|\|_{p(\cdot)} \le 1$. *If* $a > n-1$, *then*

$$\int_{B\cap B(x,r)} |x-y|^{1-n}|\nabla u(y)| \le C(\log(1/r))^{-A},$$

where $A = (a-n+1)/n$.

PROOF. Let $f(y) = |\nabla u(y)|$ for $y \in B$ and $f = 0$ outside B. For $0 < \mu < 1$, we have

$$\begin{aligned}
&\int_{B(x,r)} |x-y|^{1-n} f(y)dy \\
\le\ & \mu\left\{\int_{B(x,r)\cap B} (|x-y|^{1-n}/\mu)^{p'(y)}dy + \int_{B(x,r)} f(y)^{p(y)}dy\right\} \\
\le\ & \mu\left\{\mu^{-n/(n-1)}\int_{B(x,r)\cap B} |x-y|^{(1-n)p'(y)}dy + 1\right\}.
\end{aligned}$$

Applying polar coordinates, we have

$$\begin{aligned}
&\int_{B(x,r)\cap B} |x-y|^{(1-n)p'(y)}dy \\
\le\ & C\int_{\{t:|t-\rho(x)|<r\}} |\rho(x)-t|^{(1-n)(n'-\omega_0(t))+n-1}dt \\
=\ & C\int_{\{t:|t-\rho(x)|<r\}} |\rho(x)-t|^{(n-1)\omega_0(t)-1}dt,
\end{aligned}$$

where $\omega_0(t) = \omega(t) - C/\log(1/t)$. If $r \le \rho(x)/2$ and $|\rho(x)-t| < \rho(x)/2$, then

$$\omega_0(t) \ge \omega(r) - C/\log(1/r),$$

so that

$$\int_{\{t:|t-\rho(x)|<r\}} |\rho(x)-t|^{(n-1)\omega_0(t)-1}dt \le C(\log(1/r))^{1-a/(n-1)}.$$

If $r > \rho(x)/2$, then $|t| < 3|\rho(x) - t|$ when $|\rho(x) - t| \geq \rho(x)/2$. Hence, in this case, we obtain

$$\begin{aligned} & \int_{\{t:|t-\rho(x)|<r\}} |\rho(x) - t|^{(n-1)\omega_0(t)-1} dt \\ \leq & \int_{\{t:|t-\rho(x)|<\rho(x)/2\}} |\rho(x) - t|^{(n-1)\omega_0(t)-1} dt \\ & + C \int_{\{t:|t|<3r\}} |t|^{(n-1)\omega_0(t)-1} dt \\ \leq & \ C(\log(1/r))^{1-a/(n-1)}, \end{aligned}$$

so that

$$\int_{B(x,r)\cap B} |x - y|^{(1-n)p'(y)} dy \leq C(\log(1/r))^{1-a/(n-1)}.$$

Consequently it follows that

$$\int_{B(x,r)} |x - y|^{1-n} f(y) dy \leq \mu \left(C\mu^{-n/(n-1)} (\log(1/r))^{1-a/(n-1)} + 1 \right).$$

Now, letting $\mu^{-n/(n-1)}(\log(1/r))^{1-a/(n-1)} = 1$, we establish

$$\int_{B(x,r)} |x - y|^{1-n} f(y) dy \leq C(\log(1/r))^{(n-1-a)/n},$$

as required. □

Now we are ready to show the 0-Hölder continuity of Sobolev functions in $W^{1,p(\cdot)}(B)$.

THEOREM 4.3. *Let $p_0 = n$ and u be a continuous Sobolev function in $W^{1,p(\cdot)}(B)$ such that $\||\nabla u|\|_{p(\cdot)} \leq 1$. If $a > n - 1$, then*

$$|u(x) - u(y)| \leq C(\log(1/|x - y|))^{-A}$$

whenever $x, y \in B$ and $|x - y| < 1/2$.

PROOF. Let $x, y \in B$ and $r = |x - y| \leq \rho(x)$. Then we see from Lemma 4.2 that

$$|u(x) - u(y)| \leq C \int_{B(x,r)} |x - z|^{1-n} |\nabla u(z)| dz \leq C(\log(1/r))^{-A}.$$

If $r = |x-y| < 1/2$, $\rho(x) < r$ and $\rho(y) < r$, then we take $x_r = (1-r)x/|x|$ and $y_r = (1-r)y/|y|$ to establish

$$\begin{aligned} |u(x) - u(y)| &\leq |u(x) - u(x_r)| + |u(x_r) - u(y_r)| + |u(y_r) - u(y)| \\ &\leq C(\log(1/r))^{-A}, \end{aligned}$$

which proves the assertion. □

REMARK 4.4. Let $p(\cdot)$ be as above, and consider the function

$$u(x) = [\log(e + \log(1/|x - \xi|))]^{\delta},$$

where $\xi \in \partial B$ and $0 < \delta < (n-1)/n$. We see readily that $u(\xi) = \infty$ and it is monotone in B. Further, if $a \leq n-1$, then

$$\int_B |\nabla u(x)|^{p(x)} dx < \infty,$$

so that Theorem 4.3 does not hold for $a \leq n-1$.

§5. Tangential boundary limits of weakly monotone Sobolev functions

Let G be a bounded open set in $\mathbf{R}^n$. Consider a positive continuous function $p(\cdot)$ on $\mathbf{R}^n$ satisfying

(p1) $p_-(G) = \inf_G p(x) > 1$ and $p_+(G) = \sup_G p(x) < \infty$;

(p2) $|p(x) - p(y)| \leq \dfrac{a \log(\log(1/|x-y|))}{\log(1/|x-y|)} + \dfrac{b}{\log(1/|x-y|)}$ whenever $|x-y| < 1/e$, where $a \geq 0$ and $b \geq 0$.

For $E \subset G$, we define the relative $p(\cdot)$-capacity by

$$C_{p(\cdot)}(E;G) = \inf \int_G f(y)^{p(y)} dy,$$

where the infimum is taken over all nonnegative functions $f \in L^{p(\cdot)}(G)$ such that

$$\int_G |x-y|^{1-n} f(y) dy \geq 1 \qquad \text{for every } x \in E.$$

From now on we collect fundamental properties for our capacity (see Meyers [15], Adams-Hedberg [1] and the authors [6]).

LEMMA 5.1. *For $E \subset G$, $C_{p(\cdot)}(E;G) = 0$ if and only if there exists a nonnegative function $f \in L^{p(\cdot)}(G)$ such that*

$$\int_G |x-y|^{1-n} f(y)dy = \infty \qquad \text{for every } x \in E.$$

For $0 < r < 1/2$, set

$$h(r;x) = \begin{cases} r^{n-p(x)}(\log(1/r))^a & \text{if } p(x) < n, \\ (\log(1/r))^{a-(n-1)} & \text{if } p(x) = n \text{ and } a < n-1, \\ (\log(\log(1/r)))^{-a} & \text{if } p(x) = n \text{ and } a = n-1, \\ 1 & \text{if } p(x) > n \text{ or } p(x) = n,\ a > n-1 \end{cases}$$

LEMMA 5.2. *Suppose $p(x_0) \le n$ and $a \le n-1$. If $B(x_0, r) \subset G$ and $0 < r < 1/2$, then*

$$C_{p(\cdot)}(B(x_0,r);G) \le Ch(r;x_0).$$

LEMMA 5.3. *If f is a nonnegative measurable function on G with $\|f\|_{p(\cdot)} < \infty$, then*

$$\lim_{r\to 0+} h(r;x)^{-1} \int_{B(x,r)} f(y)^{p(y)} dy = 0$$

holds for all x except in a set $E \subset G$ with $C_{p(\cdot)}(E;G) = 0$.

Let $p(\cdot)$ be as in Section 4; that is, we assume that $p(x) > n$ and

$$\left| p(x) - \left\{ n + \frac{a\log(e+\log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))} \tag{4}$$

for $x \in B$, where $a \ge 0$ and $b > 0$. Then $p_1(x) \le p(x) \le p_2(x)$ for $x \in B$, where

$$\begin{aligned} p_1(x) &= n + \frac{a\log(e+\log(1/\rho(x)))}{\log(e/\rho(x))} - \frac{b}{\log(e/\rho(x))}, \\ p_2(x) &= n + \frac{a\log(e+\log(1/\rho(x)))}{\log(e/\rho(x))} + \frac{b}{\log(e/\rho(x))}. \end{aligned}$$

For simplicity, set

$$p(x) = p_1(x) = p_2(x) = n$$

outside B. Then we can find $b' > b$ such that for $i = 1, 2$

$$\begin{aligned} |p_i(x) - p_i(y)| &\leq \frac{a\log(e + \log(1/|x-y|))}{\log(e/|x-y|)} + \frac{b}{\log(e/|x-y|)} \\ &\leq \frac{a\log(\log(1/|x-y|))}{\log(1/|x-y|)} + \frac{b'}{\log(1/|x-y|)} \end{aligned}$$

whenever $|x - y|$ is small enough, say $|x - y| < r_0 < 1/e$.

Since G has finite measure, we find a constant $K > 0$ such that

$$C_{p(\cdot)}(E; G) \leq K C_{p_2(\cdot)}(E; G) \tag{5}$$

whenever $E \subset G$. Hence, in view of Lemma 5.2, we obtain

$$C_{p(\cdot)}(B(x_0, r); 2B) \leq Ch(r; x_0) \tag{6}$$

for $x_0 \in \partial B$, where $2B = B(0, 2)$.

COROLLARY 5.4. *If f is a nonnegative measurable function on $2B$ with $\|f\|_{p(\cdot)} < \infty$, then*

$$\lim_{r \to 0+} h(r; x)^{-1} \int_{B(x,r)} f(y)^{p(y)} dy = 0$$

holds for all $x \in \partial B$ except in a set $E \subset \partial B$ with $C_{p(\cdot)}(E; 2B) = 0$.

If u is a weakly monotone function in $W^{1,p(\cdot)}(B)$, then, since $p(x) > n$ for $x \in B$ by our assumption, we see that u is continuous in B and hence monotone in B in the sense of Lebesgue. We now show the existence of tangential boundary limits of monotone Sobolev functions u in B when $a \leq n - 1$.

For $\xi \in \partial B$, $\gamma \geq 1$ and $c > 0$, set

$$T_\gamma(\xi, c) = \{x \in B : |x - \xi|^\gamma < c\rho(x)\}.$$

THEOREM 5.5. *Let $p(\cdot)$ be a positive continuous function on $2B$ such that $p(x) \geq n$ for $x \in 2B$ and*

$$\left| p(x) - \left\{ n + \frac{a\log(e + \log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \leq \frac{b}{\log(e/\rho(x))}$$

for $x \in B$, where $a \geq 0$ and $b > 0$. If u is a monotone function in $W^{1,p(\cdot)}(B)$ (in the sense of Lebesgue), then there exists a set $E \subset \partial B$ such that

(i) $C_{p(\cdot)}(E; 2B) = 0$;

(ii) *if* $\xi \in \partial B \setminus E$, *then* $u(x)$ *has a finite limit as* $x \to \xi$ *along the sets* $T_\gamma(\xi, c)$.

If $a > n-1$, then the above function u has a continuous extension on $\overline{B} = B \cup \partial B$ in view of Theorem 4.3, and hence the exceptional set E can be taken as the empty set.

To prove Theorem 5.5, we may assume that

$$p(x) = n + \frac{a \log(e + \log(e/\rho(x)))}{\log(e/\rho(x))} - \frac{b}{\log(e/\rho(x))}$$

for $x \in B$.

We need the following two results. The first one follows from inequality (1) (see e.g. [9] and [5]).

LEMMA 5.6. *Let* u *be a monotone Sobolev function in* $W^{1,p(\cdot)}(B)$. *If* $\xi \in \partial B$, $x \in B$ *and* $n-1 < q < n$, *then*

$$|u(x) - u(\tilde{x})|^q \leq C(\log(2r/\rho(x)))^{q-1} \int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy,$$

where $\tilde{x} = (1-r)\xi$, $r = |\xi - x|$ *and* $E(x) = \cup_{y \in \overline{x\tilde{x}}} B(y, \rho(y)/2)$ *with* $\overline{x\tilde{x}} = \{tx + (1-t)\tilde{x} : 0 < t < 1\}$.

LEMMA 5.7. *Let* u *be a monotone Sobolev function in* $W^{1,p(\cdot)}(B)$. *Let* $\xi \in \partial B$ *and* $a \geq 0$. *Suppose*

$$(\log(1/r))^{n-1-a} \int_{B \cap B(\xi, 2r)} |\nabla u(y)|^{p(y)} dy \leq 1.$$

If $x \in T_\gamma(\xi, c)$, $\tilde{x} = (1-r)\xi$ *and* $r = |\xi - x|$, *then*

$$|u(x) - u(\tilde{x})|^n \leq C(\log(1/r))^{n-1-a} \int_{B \cap B(\xi, 2r)} |\nabla u(y)|^{p(y)} dy.$$

PROOF. First note that

$$\rho(y) \geq C(\rho(x) + |x-y|) \qquad \text{for } y \in E(x).$$

Take q such that $n-1<q<n$; when $a>0$, assume further that $a>(n-q)/q$. Set $p_1(x)=p(x)/q$. Then we have for $\mu>0$

$$\begin{aligned}
&\int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy \\
\le\ & \mu\left\{\int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy + \int_{E(x)} |\nabla u(y)|^{qp_1(y)} dy\right\} \\
=\ & \mu\left\{\int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy + F\right\},
\end{aligned}$$

where $F=\int_{E(x)} |\nabla u(y)|^{p(y)} dy$. Note from Lemma 4.1 that

$$|p_1'(y) - \{n/(n-q) - \omega(\rho(y))\}| \le C/\log(1/\rho(y))$$

for $y\in E(x)$, where $\omega(t)=(aq^2/(n-q)^2)\log(\log(1/t))/\log(1/t)$. Hence

$$n/(n-q) - \omega_1(\rho(y)) \le p_1'(y) \le n/(n-q) - \omega_2(\rho(y)),$$

where $\omega_1(t)=\omega(t)+C/\log(1/t)$ and $\omega_2(t)=\omega(t)-C/\log(1/t)$. Suppose

$$(\log(1/r))^{-1+aq/(n-q)} F > 1.$$

Since $p_1'(y)\le n/(n-q)$, we have for $0<\mu<1$,

$$\begin{aligned}
&\int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy \\
\le\ & C\mu^{-n/(n-q)} \int_{E(x)} (\rho(x)+|x-y|)^{(q-n)(n/(n-q)-\omega_2(\rho(y)))} dy \\
\le\ & C\mu^{-n/(n-q)} \int_0^{2r} (\rho(x)+t)^{-n} (\log(1/(\rho(x)+t)))^{-aq/(n-q)} t^{n-1} dt \\
\le\ & C\mu^{-n/(n-q)} (\log(1/r))^{1-aq/(n-q)}
\end{aligned}$$

whenever $x\in T_\gamma(\xi,c)$. Considering

$$\mu^{-n/(n-q)}(\log(1/r))^{1-aq/(n-q)} = F,$$

we obtain

$$\begin{aligned}
&\int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy \\
\le\ & C\left\{(\log(1/r))^{-1+aq/(n-q)} F\right\}^{-(n-q)/n} F \\
=\ & C\left\{(\log(1/r))^{(n-q)/q-a} \int_{E(x)} |\nabla u(y)|^{p(y)} dy\right\}^{q/n}.
\end{aligned}$$

Consequently it follows from Lemma 5.6 that

$$|u(x) - u(\tilde{x})|^n \le C(\log(1/r))^{n-1-a} \int_{B\cap B(\xi,2r)} |\nabla u(y)|^{p(y)} dy$$

whenever $x \in T_\gamma(\xi, c)$.

Next consider the case when $(\log(1/r))^{-1+aq/(n-q)} F \le 1$. Set $p^+ = \sup_{B\cap B(\xi,2r)} p(y)$ and and $p_1^+ = \sup_{B\cap B(\xi,2r)} p_1(y) = p^+/q$. For $\mu > 1$, we apply the above considerations to obtain

$$\begin{aligned} & \int_{E(x)} (\rho(y)^{(q-n)}/\mu)^{p_1'(y)} dy \\ \le\ & C\mu^{-(p_1^+)'} \int_{E(x)} (\rho(x) + |x-y|)^{(q-n)(n/(n-q)-\omega_2(\rho(y)))} dy \\ \le\ & C\mu^{-(p_1^+)'} (\log(1/r))^{1-aq/(n-q)}. \end{aligned}$$

If we take μ satisfying $\mu^{-(p_1^+)'}(\log(1/r))^{1-aq/(n-q)} = F$, then we have

$$\begin{aligned} & \int_{E(x)} |\nabla u(y)|^q \rho(y)^{q-n} dy \\ \le\ & C\Big\{(\log(1/r))^{(n-q)/q-a} \int_{E(x)} |\nabla u(y)|^{p(y)} dy\Big\}^{1/p_1^+}. \end{aligned}$$

Since $(\log(1/r))^{\omega(r)}$ is bounded above for small $r > 0$, Lemma 5.6 yields

$$|u(x) - u(\tilde{x})|^{p^+} \le C(\log(1/r))^{n-1-a} \int_{B\cap B(\xi,2r)} |\nabla u(y)|^{p(y)} dy$$

whenever $x \in T_\gamma(\xi, c)$, which proves the required assertion. □

PROOF OF THEOREM 5.5. Consider $E = E_1 \cup E_2$, where

$$E_1 = \{\xi \in \partial B : \int_B |\xi - y|^{1-n} |\nabla u(y)| dy = \infty\}$$

and

$$E_2 = \{\xi \in \partial B : \limsup_{r\to 0+} (\log(1/r))^{n-1-a} \int_{B(\xi,r)} |\nabla u(y)|^{p(y)} dy > 0\}.$$

We see from Lemma 5.1 and Corollary 5.4 that $E = E_1 \cup E_2$ is of $C_{p(\cdot)}$-capacity zero. If $\xi \notin E_1$, then we can find a line L along which u has a finite limit ℓ. In view of inequality (1), we see that u has a radial limit ℓ at ξ, that is, $u(r\xi)$ tends to ℓ as $r \to 1-0$. Now we insist from Lemma

5.7 that if $\xi \in \partial B \setminus E$, then $u(x)$ tends to ℓ as x tends to ξ along the sets $T_\gamma(\xi, c)$. □

REMARK 5.8. If $a > n-1$, then we do not need the monotonicity in Theorem 5.5, because of Theorem 4.3.

Finally we show the nontangential limit result for weakly monotone Sobolev functions. Recall that a quasicontinuous representative is locally bounded.

THEOREM 5.9. *Let $p(\cdot)$ be a positive continuous function on B such that*

$$\left| p(x) - \left\{ p_0 + \frac{a \log(e + \log(1/\rho(x)))}{\log(e/\rho(x))} \right\} \right| \le \frac{b}{\log(e/\rho(x))},$$

where $-\infty < a < \infty$, $b \ge 0$ and $n-1 < p_0 \le n$. If u is a weakly monotone function in $W^{1,p(\cdot)}(B)$ (in the sense of Manfredi), then there exists a set $E \subset \partial B$ such that

(i) $C_{p(\cdot)}(E; 2B) = 0$;
(ii) *if $\xi \in \partial B \setminus E$, then $u(x)$ has a finite limit as $x \to \xi$ along the sets $T_1(\xi, c)$.*

To prove this, we need the following lemma instead of Lemma 5.7, which can be proved by use of (1) with $q = p_- = \inf_{z \in B(x,\rho(x)/2)} p(z)$.

LEMMA 5.10. *Let p and u be as in Theorem 5.9. If $y \in B(x,r)$ with $r = \rho(x)/4$, then*

$$|u(x) - u(y)|^{p_-} \le C r^{p_0 - n} (\log(1/r))^{-a} \left(r^n + \int_{B(x,2r)} |\nabla u(z)|^{p(z)} dz \right).$$

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Toshihide Futamura
Department of Mathematics
Daido Institute of Technology
Nagoya 457-8530
Japan
E-mail address: futamura@daido-it.ac.jp

Yoshihiro Mizuta
The Division of Mathematical and Information Sciences
Faculty of Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima 739-8521
Japan
E-mail address: mizuta@mis.hiroshima-u.ac.jp