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Sobolev type spaces on metric measure spaces

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Dedicated to Professor C. David Minda on his 61st birthday

Abstract.

The aim of this note is to summarise some of the approaches to extending Sobolev space theory to metric measure spaces. In particular, we will give a brief survey of Hajłasz-Sobolev, Newton-Sobolev, and the Korevaar-Schoen type Sobolev spaces on metric measure spaces.

§1. Introduction

Many developments in the study of quasiconformal mappings and quasiregular mappings between domains in manifolds were aided by Sobolev space theory. The groundbreaking paper [19] by Heinonen and Koskela already had indications that an analog of Sobolev space theory for metric measure spaces is desirable in the study of quasiconformal mappings. Meanwhile, certain degenerate elliptic partial differential equations were reformulated in terms of elliptic partial differential equations on Carnot groups such as the Heisenberg groups and were then studied using modifications of standard techniques of elliptic PDE theory; see for example [16], [15], [11], [7], and the references therein. It was therefore clear that a viable Sobolev space theory on metric measure spaces would aid in further development of the study of quasiconformal mappings between metric measure spaces and of the study of a wide class of partial differential equations.

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Using a characterization of Sobolev functions on Euclidean spaces, Hajłasz formulated a theory of Sobolev type function spaces on metric measure spaces in [12]; this theory was developed further in [13] and [9]. Following the definition of upper gradients given by Heinonen and Koskela in [19], the author and Cheeger independently proposed a theory of Sobolev type spaces on metric measure spaces; see [37] and [8]. Concurrently, using the theory of strongly local Dirichlet forms as a model, Korevaar and Schoen developed a theory of Sobolev mappings from Riemannian domains into metric space targets in [29], and they used this theory to study harmonic mappings in [30]. Their approach was modified by Koskela and MacManus in [33] to obtain another version of Sobolev type space of functions on metric measure spaces; see also [34] for a discussion connecting this Korevaar-Schoen type Sobolev space theory with the theory of Dirichlet forms on metric measure spaces.

In this note we will describe the above-mentioned function spaces and the connections between them without proofs. This note is arranged as follows. The next section will summarise the notations used throughout this paper. The third and fourth sections will discuss the Hajłasz and Newtonian approaches to defining Sobolev type spaces on metric measure spaces. The final section will describe a Korevaar-Schoen approach to constructing Sobolev type spaces on metric measure spaces and discuss the relationships between the three Sobolev type spaces under certain conditions. While no proofs are provided in this note, references to articles where the proofs can be found are given. However, the references given are not exhaustive, and many good references are left out for brevity of exposition.

§2. Notations

In this note $X = (X, d, \mu)$ denotes a metric measure space with metric d and measure μ. Given $r > 0$ and $x \in X$, the (open) metric ball centered at x with radius r is denoted $B(x, r)$. We will assume throughout that μ is a Borel regular measure such that bounded sets have finite measure and non-empty open sets have positive measure. The Lebesgue measure of sets $A \subset \mathbb{R}^n$ is denoted $|A|$.

We fix an index $1 \leq p < \infty$. Measurable functions $f : X \to \mathbb{R}$ are said to be in the class $L^p(X)$ if the integral $\|f\|^p_{L^p(X)} := \int_X |f|^p \, d\mu$ is finite. We say that $f \in L^p_{loc}(X)$ if $f \in L^p(Y)$ for every bounded subset $Y \subset X$. The integral average of a function $f \in L^1_{loc}(X)$ on a measurable set $A \subset X$ with $\mu(A) > 0$ is denoted f_A:

$$f_A := \frac{1}{\mu(A)} \int_A f(y)\, d\mu(y).$$

Given functions $f \in L^1_{loc}(X)$, we define the Hardy-Littlewood maximal function Mf on X by

$$Mf(x) = \sup_{r>0} |f|_{B(x,r)}.$$

The measure μ is said to be *doubling* if there is a constant $C \geq 1$ such that for every $x \in X$ and $r > 0$ we have $\mu(B(x, 2r)) \leq C\, \mu(B(x,r))$. The standard Lebesgue measure on $\mathbb{R}^n$ for example is a doubling measure. It is known that if μ is a doubling measure, then the Hardy-Littlewood maximal function operator $M : L^p(X) \to L^p(X)$ is a bounded sublinear operator for all $p > 1$; see for example [17]. Furthermore, $M : L^1(X) \to$ wk$-L^1(X)$ boundedly. Here wk$-L^1(X)$ is the collection of all functions f on X for which there is a constant $C_f > 0$ such that for all $t > 0$,

$$\mu(\{x \in X \,:\, |f(x)| \geq t\}) \leq \frac{C_f}{t}.$$

The norm on wk $- L^1(X)$ is obtained by associating to each function f in this class the infimum/minimum of all such numbers C_f.

A metric space is said to be proper if closed and bounded subsets of the space are compact in the metric topology. An easy topological argument shows that if X is a complete metric space and μ is a doubling measure on X then X is proper.

§3. The Hajłasz-Sobolev spaces

The following theorem was proven by Hajłasz in [12]. Recall that a domain Ω is a p-extension domain if and only if there is a bounded linear extension operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ with $Ef = f$ on Ω.

Theorem 3.1 (Hajłasz). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain or $\Omega = \mathbb{R}^n$, and $1 < p < \infty$. Suppose in addition that Ω is a p-extension domain. Then a function $f : \Omega \to \mathbb{R}$ is in the class $W^{1,p}(\Omega)$ if and only if there is a non-negative function $g \in L^p(\mathbb{R}^n)$ and a set $Z \subset \Omega$ with $|Z| = 0$ such that whenever $x, y \in \Omega \setminus Z$,*

$$|f(x) - f(y)| \leq |x-y|\, (g(x) + g(y))\,. \tag{3.1}$$

The function g is called a *Hajłasz gradient* of f. For functions $f \in W^{1,p}(\mathbb{R}^n)$ it can be shown via a "telescoping sequence of balls" argument that $M|\nabla f|$ is a Hajłasz gradient of f.

It is clear that equation (3.1) can be used to extend the notion of Sobolev spaces to metric measure spaces. This is done in [12] as follows.

Definition 3.1. Given a function $f : X \to \mathbb{R}$, we say that a non-negative function $g : X \to \mathbb{R}$ is a *Hajłasz gradient* for f if there exists a set $Z \subset X$ with $\mu(Z) = 0$ so that whenever $x, y \in X \setminus Z$,

$$|f(x) - f(y)| \le d(x,y)\ (g(x) + g(y))\,. \tag{3.2}$$

For functions $f \in L^p(X)$ we define the *Hajłasz-Sobolev norm* of f by

$$\|f\|_{M^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all Hajłasz gradients g of f. We denote by $M^{1,p}(X)$ the collection of all (equivalence classes of) functions $f \in L^p(X)$ for which the norm $\|f\|_{M^{1,p}(X)}$ is finite.

Lipschitz functions in $M^{1,p}(X)$ form a dense subclass of $M^{1,p}(X)$. If the measure μ is doubling, then functions in the class $M^{1,p}(X)$ always satisfy a weak $(1,p)$-Poincaré inequality: there are constants $C > 0$ and $\lambda \ge 1$ such that for all $f \in M^{1,p}(X)$ and all Hajłasz gradients $g \in L^p(X)$ of f,

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \le C\, r \left(\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,\lambda r)} g^p\, d\mu\right)^{1/p}$$

whenever $B(x,r) \subset X$ is a ball in X with radius r. Such Poincaré inequalities play a crucial role in potential theory. For example, using such inequalities it can be shown that solutions to the Dirichlet problem with boundary data from $M^{1,p}(X)$ always exist. Poincaré inequalities are also useful in the study of p-extension domains; see [14] for an elegant discussion of extension and trace theorems.

It should be noted however that by definition, if $f \in M^{1,p}(X)$ and $F \in L^p(X)$ such that $F = f$ μ-a.e. in X, then $F \in M^{1,p}(X)$. This is one of the differences between the Hajłasz-Sobolev space and the Newton-Sobolev space discussed in the next section. Another crucial difference is as follows. If $U \subset X$ is a non-empty open set and $f \in M^{1,p}(X)$ is constant on U, it is not clear that we can choose a Hjłasz gradient g of f in $L^p(X)$ so that $g = 0$ μ-a.e. in U. Such a truncation property for gradients is crucial in the current techniques used in the study of PDEs; for example, the truncation property is essential in the Nash-Moser proof and in the DeGiorgi proof of Harnack inequalities for energy minimizers and harmonic functions, and the lack of this truncation property in the Hajłasz-Sobolev space makes the related potential theory difficult.

§4. Newtonian spaces

In [19], Heinonen and Koskela propose an alternative to distributional derivatives in the setting of metric measure spaces. Recall that if $f : \mathbb{R}^n \to \mathbb{R}$ is a C^1-function, then by the fundamental theorem of calculus, for every pair of points $x, y \in \mathbb{R}^n$ and every rectifiable curve γ joining x and y in $\mathbb{R}^n$,

$$|f(x) - f(y)| \le \int_\gamma |\nabla f|\, ds. \tag{4.1}$$

However, if $\Omega \subset \mathbb{R}^n$ is a domain and $f \in W^{1,p}(\Omega)$, then the collection of non-constant compact rectifiable curves γ in Ω for which (4.1) fails is a *zero p-modulus collection of curves*; that is, there is a non-negative Borel measurable function $\rho_0 \in L^p(\Omega)$ such that for every such curve γ we have $\int_\gamma \rho_0\, ds = \infty$. Using this fact as a motivation, Heinonen and Koskela proposed the following alternative to distributional derivatives for functions on metric measure spaces.

Definition 4.1. A family Γ of non-constant compact rectifiable curves in a metric measure space X is said to be a zero p-modulus family if there exists a non-negative Borel measurable function $\rho_0 \in L^p(X)$ such that for all curves $\gamma \in \Gamma$ the path integral $\int_\gamma \rho_0\, ds = \infty$. Given a function $f : X \to \mathbb{R}$, we say that a non-negative Borel measurable function ρ on X is an *upper gradient* of f if for all non-constant compact rectifiable curves γ in X,

$$|f(x) - f(y)| \le \int_\gamma |\nabla f|\, ds. \tag{4.2}$$

Here x and y denote the endpoints of γ. If (4.2) fails only for a zero p-modulus family of curves, then we say ρ is a *p-weak upper gradient* of f.

It can be shown that if f is a Lipschitz function on X, then the local Lipschitz constant function ρ given by

$$\rho(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)}$$

is an upper gradient of f; see [17]. On the other hand, if $(\rho_n)_n$ is a sequence of upper gradients of a function f on X such that each $\rho_n \in L^p(X)$ and $\rho_n \to \rho$ in $L^p(X)$, then the Borel function ρ, though may not be an upper gradient of f, is necessarily a p-weak upper gradient of f; see for example the discussion in [23].

Using the above definition of p-weak upper gradients given in [19], the author proposed in [37] the following version of Sobolev spaces, called Newtonian spaces.

Definition 4.2. Given a function $f : X \to \mathbb{R}$ such that f belongs to an equivalence class in $L^p(X)$, the *Newtonian norm* of f is given by

$$\|f\|_{N^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_\rho \|\rho\|_{L^p(X)},$$

where the infimum is taken over all upper gradients (or equivalently, all p-weak upper gradients) of f. We say that two functions f_1, f_2 on X are equivalent, denoted $f_1 \sim f_2$, if $\|f_1 - f_2\|_{N^{1,p}(X)} = 0$. It is easy to see that $\sim$ defines an equivalence class on the collection of all functions f on X for which $\|f\|_{N^{1,p}(X)}$ is finite. The *Newton-Sobolev space* $N^{1,p}(X)$ is the collection of all such equivalence classes of functions.

It can be shown that $N^{1,p}(X)$, equipped with the above norm, is indeed a lattice and a normed vector space that is also a Banach space; see [37]. It should be noted that perturbations of functions in the Hajłasz space $M^{1,p}(X)$ on sets of μ-measure zero are again in the Hajłasz space, and hence it is easy to see that $M^{1,p}(X)$ is a Banach space. However, perturbations of functions from $N^{1,p}(X)$ on sets of μ-measure zero usually does not yield a function in $N^{1,p}(X)$; therefore the proof that $N^{1,p}(X)$ is a Banach space is more involved. On the other hand, if two functions f_1 and f_2 are in $N^{1,p}(X)$ and $f_1 = f_2$ μ-a.e. we can see that $f_1 \sim f_2$ and hence they belong to the same equivalence class in $N^{1,p}(X)$.

Using the techniques found in the book [36] by Ohtsuka, it can be shown that whenever X is a domain in $\mathbb{R}^n$, equipped with the Euclidean metric and the standard Lebesgue measure, $N^{1,p}(X) = W^{1,p}(X)$ both isometrically and isomorphically; see [37].

Given $f \in N^{1,p}(X)$, there are infinitely many p-weak upper gradients for f in $L^p(X)$. Indeed, if ρ is a p-weak upper gradient of f and $g \in L^p(X)$ is a non-negative Borel measurable function, then $\rho + g$ is also a p-weak upper gradient of f. The following lemma is very useful in associating to each $f \in N^{1,p}(X)$ a unique p-weak upper gradient.

Lemma 4.1. *Let $f \in N^{1,p}(X)$. Then the collection of all p-weak upper gradients of f in $L^p(X)$ forms a convex subset of $L^p(X)$. If $1 < p < \infty$, then there is a unique p-weak upper gradient $\rho_f \in L^p(X)$ of f*

such that whenever $\rho \in L^p(X)$ is another p-weak upper gradient of f, we have $\rho_f \le \rho$ μ-a.e.

Such a p-weak upper gradient is called the *minimal* p-weak upper gradient of f.

The following lemma shows that the truncation property holds true for the class $N^{1,p}(X)$.

Lemma 4.2. *If $U \subset X$ is a closed or an open set and $f \in N^{1,p}(X)$ such that f is constant μ-a.e. in U, then $\rho_f = 0$ μ-a.e. in U.*

On the other hand, unlike the Hajłasz-Sobolev class, functions in $N^{1,p}(X)$ need not satisfy a Poincaré inequality. We say that $N^{1,p}(X)$ satisfies a weak $(1,p)$-Poincaré inequality on X if there exist constants $C > 0$ and $\lambda \ge 1$ such that whenever $f \in N^{1,p}(X)$ and $B(x,r)$ is a ball in X,

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \le C\, r \left(\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,\lambda r)} \rho_f^p\, d\mu\right)^{1/p}.$$

Clearly, if X has no non-constant compact rectifiable curve, then 0 would be an upper gradient for every function on X; in this case, $N^{1,p}(X) = L^p(X)$, and for every $f \in N^{1,p}(X)$ we have $\rho_f = 0$. In this event the above inequality can not be satisfied. Examples of such metric spaces include the so-called snow-flaked Euclidean space $X = \mathbb{R}^n$ with metric $d(x,y) = |x-y|^\epsilon$ for some fixed $0 < \epsilon < 1$. Other examples of metric spaces where Poincaré inequalities do not hold for $N^{1,p}(X)$ include certain fractal sets such as the Sierpinski gasket. However, there are many examples of non-Euclidean metric measure spaces supporting a Poincaré inequality; see [35], [6], [32], [31], and the references therein. Given that functions from $N^{1,p}(X)$ and their upper gradients satisfy the truncation property of Lemma 4.2, whenever the measure on X is doubling and X supports a weak $(1,p)$-Poincaré inequality, many of the classical methods of analysing harmonic functions in Euclidean domains can be modified to study energy minimizers and p-harmonic functions on domains in X; see for example [27], [26], [25], [28], [5], [4], [38], [3], [21], and [22].

While in general the Banach space $N^{1,p}(X)$ may not be reflexive, the following weak closure result from [23] demonstrates that one can almost apply Mazur's lemma to bounded sequences in $N^{1,p}(X)$.

Lemma 4.3. *Let $1 < p < \infty$. If X is complete and $(f_j)_j$ is a sequence of functions in $L^p(X)$ with upper gradients $(g_j)_j$ in $L^p(X)$, such that f_j weakly converges to f and g_j weakly converges to g in $L^p(X)$,* *then g is a weak upper gradient of f after modifying f on a set of measure zero, and there is a convex combination sequence $\tilde{f}_j = \sum_{k=j}^{N(j)} \lambda_{k,j} f_k$ and $\tilde{g}_j = \sum_{k=j}^{N(j)} \lambda_{k,j} g_k$ with $\sum_{k=j}^{N(j)} \lambda_{k,j} = 1$, $\lambda_{k,j} \geq 0$, so that $\tilde{f}_j$ converges to f and $\tilde{g}_j$ converges to g in $L^p(X)$.*

In [8], Cheeger independently developed a theory of Sobolev type spaces on metric spaces using the notion of upper gradients, and he used this theory to prove a Rademacher-type differentiability theorem for Lipschitz functions on metric measure spaces whose measure is doubling and supports a weak $(1,p)$-Poincaré inequality. The approach in [8] is as follows.

Definition 4.3. Given $f \in L^p(X)$, we say that $f \in H^{1,p}(X)$ if the following norm is finite:

$$\|f\|_{H^{1,p}(X)} := \|f\|_{L^p(X)} + \inf_{(f_j,\rho_j)_j} \liminf_{j\to\infty} \|\rho_j\|_{L^p(X)},$$

where the infimum is taken over all sequences of function-upper gradient pairs $(f_j, \rho_j)_j$ with $f_j \to f$ in $L^p(X)$.

Using the uniform convexity of $L^p(X)$ and Lemma 4.3, it is clear that whenever $1 < p < \infty$ and X is complete we have $H^{1,p}(X) = N^{1,p}(X)$; however, we can modify functions from $H^{1,p}(X)$ on sets of μ-measure zero, whereas (as mentioned above), we cannot do so to functions from $N^{1,p}(X)$. If X is not complete, then $H^{1,p}(X) = N^{1,p}(\widehat{X})$ where $\widehat{X}$ is the completion of X. In general, $H^{1,1}(X) \neq N^{1,1}(X)$ as $H^{1,1}(\mathbb{R}^n)$ corresponds to the class of functions of bounded variation.

As mentioned above, it is not in general true that $N^{1,p}(X)$ is reflexive. However, in the event that $1 < p < \infty$ and the measure on X is doubling and supports a weak $(1,p)$-Poincaré inequality, one of the results in [8] demonstrates the reflexivity of $H^{1,p}(X)$ and hence of $N^{1,p}(X)$. To prove this, a linear derivation operator on $H^{1,p}(X)$ is constructed in [8] as follows.

Theorem 4.1 (Cheeger). *Let the measure on X be doubling, $1 < p < \infty$, and assume that X admits a $(1,p)$-Poincaré inequality. Then there exists a countable collection (U_α, X^α) of measurable sets U_α and Lipschitz "coordinate" functions $X^\alpha = (X_1^\alpha, \dots, X_{k(\alpha)}^\alpha) : X \to \mathbb{R}^{k(\alpha)}$ such that $\mu(X \setminus \bigcup_\alpha U_\alpha) = 0$, $\mu(U_\alpha) > 0$, and for all α the following hold.*

The functions $X_1^\alpha, \dots, X_{k(\alpha)}^\alpha$ are linearly independent on U_α and $1 \leq k(\alpha) \leq N$, where N is a constant depending only on the doubling constant of μ and the constant from the Poincaré inequality. If $f : X \to$

$\mathbb{R}$ *is Lipschitz, then there exist unique bounded measurable vector-valued functions* $d^\alpha f : U_\alpha \to \mathbb{R}^{k(\alpha)}$ *such that for* μ*-a.e.* $x_0 \in U_\alpha$,

$$\lim_{r\to 0^+} \sup_{x\in B(x_0,r)} \frac{|f(x)-f(x_0)-d^\alpha f(x_0)\cdot(X^\alpha(x)-X^\alpha(x_0))|}{r} = 0.$$

Furthermore, there is a constant $C>0$ *such that for all Lipschitz functions* f *on* X, $\frac{1}{C}\, g_f \le |d^\alpha f| \le C\, g_f$ μ*-a.e. on* U_α *for each* α.

Since a weak $(1,p)$-Poincaré inequality holds on X, Lipschitz functions form a dense subclass of $N^{1,p}(X) = H^{1,p}(X)$; see [8] and [37]. Hence the discussion by Franchi, Hajłasz, and Koskela in [9] demonstrates that the linear derivation operator d^α can be extended to operate also on functions in $N^{1,p}(X)$. Thus we have a natural embedding of $N^{1,p}(X)$ into $L^p(X)\times L^p(X:\mathbb{R}^N)$ (which is a uniformly convex space and hence is reflexive), resulting in $N^{1,p}(X)$ being reflexive itself. A further advantage of having this linear derivation operator for functions in $N^{1,p}(X)$ is that associated to (Cheeger) p-harmonic functions there is an Euler-Lagrange equation. The Euler-Lagrange equations are quite useful in the study of potential theory; see for example the discussions in [22] and [18]. It should be noted here that the map $f\mapsto \rho_f$ is rarely a linear map; also in general $\rho_{f_1-f_2}\neq|\rho_{f_1}-\rho_{f_2}|$.

§5. Sobolev spaces of Korevaar-Schoen, and the connection between the various Sobolev type spaces

Using the notion of energy integral proposed by Korevaar and Schoen in [29], Koskela and MacManus studied the following version of Sobolev spaces on metric measure spaces in [33] (see also [20] for a more general discussion).

Definition 5.1. Given $f : X\to\mathbb{R}$, we define the Korevaar-Schoen energy of f to be the number $E(f)$, where

$$E(f) := \sup_B \left(\limsup_{\epsilon\to 0} \int_B \int_{B(x,\epsilon)} \frac{|f(x)-f(y)|^p}{\mu(B(x,\epsilon))\,\epsilon^p}\, d\mu(y)\, d\mu(x) \right),$$

the supremum being taken over all balls $B\subset X$. We say that $f\in KS^p(X)$ if the norm $\|f\|_{KS^{1,p}(X)} := \|f\|_{L^p(X)} + E(f)^{1/p}$ is finite.

The motivation behind such an energy construction is the theory of Dirichlet forms. The early work of Beurling and Deny in [2] and [1], applied to strongly local Dirichlet forms, yields a representation of such Dirichlet forms associated with the above energy for $p=2$.

In the study of the relationships between the Hajłasz-Sobolev spaces, the Newtonian spaces, the collection of all pairs of functions satisfying a weak $(1,p)$-Poincaré inequality, and in the case of $p=2$ the theory of Dirichlet forms, the Sobolev spaces $KS^{1,p}(X)$ of Korevaar-Schoen play an important connective role; see [20] and [34]. The paper [34] studies the connection between $N^{1,p}(X)$ and domains of various Dirichlet forms on X using the space $KS^{1,p}(X)$; a discussion of Dirichlet forms is beyond the scope of this note, but an excellent discussion can be found in the book [10] by Fukushima, Ōshima, and Takeda.

In what follows, we say that the metric measure space X supports a weak $(1,p)$-Poincaré inequality if there are constants $C>0$ and $\lambda \geq 1$ such that whenever $f : X \to \mathbb{R}$ is a measurable function with p-weak upper gradient ρ and $B(x,r)$ is a ball in X,

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \leq C\, r\left(\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,\lambda r)} \rho_f^p\, d\mu\right)^{1/p}.$$

In what follows, $P^{1,p}(X)$ consists of all functions $f \in L^p(X)$ for which there exists a non-negative function $g \in L^p(X)$ so that whenever $B(x,r)$ is a ball in X,

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f - f_{B(x,r)}|\, d\mu \leq C\, r\left(\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,\lambda r)} g^p\, d\mu\right)^{1/p}.$$

Theorem 5.1. *Fix $1 < p < \infty$. If X is complete and the measure on X is doubling and supports a $(1,p)$-Poincaré inequality, then the natural mapping between the following spaces are isometric isomorphisms as Banach spaces:*

$$H^{1,p}(X) = N^{1,p}(X) = M^{1,p}(X) = KS^{1,p}(X) = P^{1,p}(X).$$

If $p=1$, then $M^{1,p}(X) \subset N^{1,p}(X) \subset H^{1,p}(X)$.

The fact that $H^{1,p}(X) = N^{1,p}(X)$ holds true even without the assumption of the doubling property of the measure nor the Poincaré inequality; see for example [37]. In [37] it is also proven that even without the assumption of a Poincaré inequality $M^{1,p}(X) \subset N^{1,p}(X)$; however, we do need the measure μ to be doubling here. It is also shown in [37] that if X supports a weak $(1,q)$-Poincaré inequality in addition for some $1 \leq q < p$, then $M^{1,p}(X) = N^{1,p}(X)$. The proof of this fact uses a telescoping sequence of balls concentric with points in the metric space, and when these points are Lebesgue points of the function $f \in N^{1,p}(X)$ the weak $(1,q)$-Poincaré inequality is applied to these balls in order to control the values of f at these points in terms of the Hardy-Littlewood maximal function $M\rho_f^q$ of ρ_f^q. If $q < p$ and $\rho_f \in L^p(X)$, then $(M\rho_f^q)^{1/q} \in L^p(X)$. We need this better Poincaré inequality $q < p$ since it is not in general true that $(M\rho_f^p)^{1/p} \in L^p(X)$. However, it is a deep result of Keith and Zhong [24] that if X is complete as a metric space and the measure on X is doubling and supports a weak $(1,p)$-Poincaré inequality, then there exists $1 \le q < p$ such that X supports a weak $(1,q)$-Poincaré inequality. Hence we have the validity in the above theorem of the statement that $N^{1,p}(X) = M^{1,p}(X)$ under the assumptions that X is proper and supports a weak $(1,p)$-Poincaré inequality. See Theorem 4.5 of [33] for a proof of the equality $KS^{1,p}(X) = M^{1,p}(X) = P^{1,p}(X)$. Again, in [33] Koskela and MacManus require X to support a weak $(1,q)$-Poincaré inequality in addition for some $1 \le q < p$, but because of the results of Keith and Zhong in [24] we have the validity of the above theorem.

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