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Radial limits of harmonic functions

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Abstract.

A classical result of Alice Roth characterizes those functions on the unit circle that can arise from taking radial limits of entire functions. This paper describes recent progress on the characterization of radial limit functions of harmonic functions defined either in the unit ball or the whole of space. Some related open problems are posed.

§1. Introduction

Let $\mathbb{T}$ denote the unit circle. The starting point for this article is the following question: which functions $f : \mathbb{T} \to \mathbb{C}$ can be expressed as

$$f(e^{i\theta}) = \lim_{r\to\infty} g(re^{i\theta}) \qquad (0 \le \theta < 2\pi) \tag{1.1}$$

for some entire function g? Such a function f must, of course, be a Baire-one function, that is, the pointwise limit of a sequence from $C(\mathbb{T})$. One would expect, however, that only a restricted class of Baire-one functions on $\mathbb{T}$ can arise in this manner. The answer to the above question is found in the following classical result of Alice Roth [15], [16] (or see Chapter IV, §5 of the book by Gaier [9]).

Theorem A. *Let $f : \mathbb{T} \to \mathbb{C}$. The following statements are equivalent:*

(a) there is an entire function g such that (1.1) holds;
(b) f is Baire-one, and is constant on each component of some relatively open dense subset J of $\mathbb{T}$.

Further, if (b) holds, then (a) holds with the additional property that the convergence in (1.1) is locally uniform on J.

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To see that (a) implies (b) in this result, suppose that (a) holds, let

$$K_k = \{e^{i\theta} : |g(re^{i\theta})| \leq k \text{ for all } r > 0\} \qquad (k \in \mathbb{N}),$$

and let J_k denote the interior of K_k relative to $\mathbb{T}$. Then $\cup_k K_k = \mathbb{T}$, so the set $J = \cup_k J_k$, which is relatively open in $\mathbb{T}$, must also be dense in $\mathbb{T}$ by a Baire category argument. Since g is bounded on the set $\{re^{i\theta} : r > 0, e^{i\theta} \in J_k\}$, the radial limit function f must, by Montel's theorem, be constant on each component arc of J_k. Thus (b) follows (and, indeed, the convergence in (1.1) is locally uniform in J). The more difficult, and hence more interesting, part of the result is the converse. The proof of this involved adapting ideas from Runge's theorem on rational approximation to deal with approximation on non-compact sets, and foreshadowed much later celebrated work of Arakeljan [1], [2].

More recently, Boivin and Paramonov [6] obtained an analogue of Roth's result for radial limits of solutions of homogeneous elliptic partial differential equations of order two with constant complex coefficients in $\mathbb{R}^2$. In the particular case of harmonic functions the radial limit functions are characterized as those Baire-one functions on $\mathbb{T}$ that are first-degree polynomials of θ on each component arc of some relatively open dense subset of $\mathbb{T}$. The arguments used do not apply in higher dimensions.

In Section 2 below we will remain in the context of the plane and consider the nature of radial limit functions of harmonic functions that are defined in the unit disc. The corresponding problem in higher dimensions is still open. Then, in Section 3, we will move to higher dimensions and see a characterization of radial limit functions of entire harmonic functions. New features, and deeper arguments, apply in this setting.

§2. Radial limits of harmonic functions in the disc

We now consider the question: which functions $f : \mathbb{T} \to \mathbb{R}$ can be expressed as

$$f(e^{i\theta}) = \lim_{r \to 1-} h(re^{i\theta}) \qquad (0 \leq \theta < 2\pi) \tag{2.1}$$

for some harmonic function h on the unit disc $\mathbb{D}$? To see what form the result should take we can follow the pattern of the argument outlined in Section 1. Let

$$K_k = \{e^{i\theta} : |h(re^{i\theta})| \leq k \text{ whenever } 0 \leq r < 1\} \qquad (k \in \mathbb{N}),$$

and $J = \cup_k J_k$, where J_k denotes the interior of K_k relative to $\mathbb{T}$. It is not difficult to deduce from (2.1) that

$$H^{\mathbb{D}}_{f\chi_{J_k}}(re^{i\theta}) \to f(e^{i\theta}) \text{ as } r \to 1- \qquad (e^{i\theta} \in J_k),$$

where H^U_g denotes the (generalised) solution to the Dirichlet problem on an open set U with boundary data g (the solution is given by a Poisson integral in the present context) and χ_A denotes the characteristic function of a set A. It follows from a converse of Fatou's theorem, due to Loomis [13] and valid for bounded boundary functions such as $f\chi_{J_k}$, that

$$\text{(2.2)} \qquad \frac{1}{2t}\int_{(-t,t)} f(e^{i(\theta+\phi)})d\phi \to f(e^{i\theta}) \qquad \text{as } t \to 0+$$

when $e^{i\theta} \in J_k$. We will say that f is *asymptotically mean-valued* at $e^{i\theta}$ if (2.2) holds. One implication of the following result, taken from [11], has now been established.

Theorem 1. *Let $f : \mathbb{T} \to \mathbb{R}$. The following statements are equivalent:*

(a) there is a harmonic function h on $\mathbb{D}$ such that (2.1) holds for all θ;

(b) f is Baire-one, and there is a relatively open dense subset J of $\mathbb{T}$ on which f is locally bounded and asymptotically mean-valued.

Further, if (b) holds, then (a) holds with the additional property that the mapping $w \longmapsto \sup_{0<r<1} |h(rw)|$ is locally bounded on J.

It remains to see why (b) implies (a). Suppose that condition (b) holds, let $\{J_j\}$ be the component arcs of J and let $\{U_j\}$ be the corresponding sectors of $\mathbb{D}$. We write J_j as $\{e^{i\theta} : |\theta - \theta_j| < a_j\}$. A naïve approach would now be to solve the Dirichlet problem in each sector U_j with boundary data f on J_j and $f(e^{i(\theta_j \pm a_j)})$ on the boundary radii. The asymptotic mean value property of f could then be used in conjunction with Fatou's theorem to deduce that the resulting Dirichlet solution had the desired radial limits at points of $\overline{J}_j$. There would remain, of course, the problem of how to "stitch together" the various Dirichlet solutions from different sectors to obtain a function that is harmonic on all of the disc. However, before we even get that far with this approach, an additional obstacle is that $f\,\big|_{J_j}$ need not be integrable with respect to harmonic measure for the sector U_j.

These difficulties can be overcome by refining our approach. Firstly, we modify the region in which we solve the Dirichlet problem by the removal of radial slits from the sector U_j. More precisely, let

$$S_{j,k}(\rho) = \left\{re^{i\theta} : 0 \leq r \leq \rho, |\theta - \theta_j| = a_j\left(1 - 2^{-k}\right)\right\} \quad (0 \leq \rho < 1; k \geq 1).$$

Then it is possible to choose a sequence $(\rho_{j,k})_{k\geq 1}$ in $(0,1)$, with limit 1, such that the function f (interpreted as 0 off $\mathbb{T}$) is integrable with respect to harmonic measure for the set

$$V_j = U_j \backslash \left(\bigcup_{k\geq 1} S_{j,k}(\rho_{j,k}) \right).$$

If we denote the resultant harmonic function on V_j by h_j, then it can be deduced, as above, from the asymptotic mean value property that

$$h_j(rw) \to f(w) \qquad \text{as } r \to 1- \qquad (w \in J_j).$$

Secondly, we must find some way of constructing a harmonic function on $\mathbb{D}$ that imitates the boundary behaviour of h_j in V_j, for each j, and also has the right behaviour along radii ending in the closed set $\mathbb{T}\backslash J$. To do this, let $\rho_j : J_j \to (1 - j^{-1}, 1)$ be a continuous function such that $\rho_j(e^{i\theta}) \to 1$ as $\theta \to \theta_j \pm a_j$ and such that the set

$$E_j = \{rw : w \in J_j \text{ and } \rho_j(w) \leq r < 1\}$$

does not intersect any of the radial slits $\{S_{j,k}(\rho_{j,k}) : k \geq 1\}$. Then E_j is a relatively closed subset of $\mathbb{D}$ such that $E_j \subset V_j$. The set

$$E_0 = \left\{rw : w \in \mathbb{T}\backslash J \text{ and } \frac{1}{2} \leq r < 1\right\}$$

is also closed relative to $\mathbb{D}$ and is, in addition, nowhere dense. Since f is Baire-one, we can choose a continuous function h_0 on E_0 such that

$$h_0(rw) \to f(w) \qquad \text{as } r \to 1- \qquad (w \in \mathbb{T}\backslash J).$$

The disjoint union $E = \cup_{j\geq 0} E_j$ is a relatively closed subset of $\mathbb{D}$. Further, if we define v on E by setting it equal to h_j on E_j $(j \geq 0)$, then v has radial limit function f. The theorem will therefore be established if we can approximate v on E by a harmonic function on $\mathbb{D}$ in such a way that the error of approximation tends to 0 at $\mathbb{T}$. Since v is continuous on E and harmonic on E°, Corollary 3.21 of [10] (based on work of Armitage and Goldstein [4]) tells us that this can be done provided $\mathbb{D}^*\backslash E$ is connected and locally connected, where $\mathbb{D}^*$ denotes the Alexandroff (or one-point) compactification of $\mathbb{D}$. Our construction of E evidently guarantees these connectivity hypotheses (see §3.2 of [10] for a discussion of local connectedness in this context), so the proof is complete.

Since our motivation came originally from classical function theory, it is natural to pose the following question.

Problem 1. *Which functions $f : \mathbb{T} \to \mathbb{C}$ can be expressed as*

$$f(e^{i\theta}) = \lim_{r \to 1-} g(re^{i\theta}) \qquad (0 \leq \theta < 2\pi)$$

for some holomorphic function g on $\mathbb{D}$?

Another obvious question concerns higher dimensions. Let $\mathbb{S}$ denote the unit sphere in $\mathbb{R}^n$.

Problem 2. *Which functions $f : \mathbb{S} \to \mathbb{R}$ can be expressed as*

$$f(w) = \lim_{r \to 1-} h(rw) \qquad (w \in \mathbb{S})$$

for some harmonic function h on the open unit ball of $\mathbb{R}^n$?

We will see in the next section that moving from the context of the plane to higher dimensions is not routine.

§3. Radial limits of harmonic functions in space

Now we develop the discussion of Section 1 in another direction by asking the question: which functions $f : \mathbb{S} \to \mathbb{R}$ can be expressed as

$$f(w) = \lim_{r \to \infty} h(rw) \qquad (w \in \mathbb{S}) \tag{3.1}$$

for some harmonic function h on $\mathbb{R}^n$? Let δ denote the Laplace-Beltrami operator on $\mathbb{S}$; thus the Laplacian on $\mathbb{R}^n$ can be expressed in polar co-ordinates as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\delta.$$

If I is a relatively open subset of $\mathbb{S}$ and the entire harmonic function h in (3.1) is bounded on the conical set $\{rw : r > 0, w \in I\}$, then a simple dilation argument shows that $\delta f = 0$ on I. This observation, together with a Baire category argument, shows that a function f of the form (3.1) must, in addition to being Baire-one, satisfy the Laplace-Beltrami equation $\delta f = 0$ on a relatively open dense subset of $\mathbb{S}$. In the case of two dimensions the latter equation reduces to $\partial^2 f/\partial\theta^2 = 0$ and so we arrive at the answer to the above question obtained by Boivin and Paramonov (see Section 1). However, this Baire category argument can be applied also in relation to the δ-fine topology on $\mathbb{S}$, that is, the coarsest topology that makes all supersolutions of the Laplace-Beltrami equation continuous. This allows us to conclude that any function f of the form (3.1) must be a δ-fine solution of the Laplace-Beltrami equation on a δ-finely open δ-finely dense subset of $\mathbb{S}$. (We refer to Fuglede [8] for these notions from fine potential theory.) The important point here is that, when $n \geq 3$, there exist compact subsets of $\mathbb{S}$ that are nowhere dense in $\mathbb{S}$ and yet have non-empty δ-fine interior. This shows that the answer to our question in higher dimensions will be more delicate.

In order to proceed we need some additional notation and terminology. Given a compact subset J of $\mathbb{S}$ we write $u \in \mathcal{L}(J)$ if u is a function on a relatively open subset I of $\mathbb{S}$ such that $J \subset I$ and $\delta u = 0$ on I. Further, given $z \in J$, we denote by $\mathcal{N}_z(J)$ the collection of all $\mathcal{L}(J)$-representing measures for z, that is, probability measures μ on J satisfying

$$u(z) = \int_J u\, d\mu \qquad \text{for every } u \in \mathcal{L}(J).$$

A bounded Borel function f on J will be called *$\mathcal{L}$-affine* on J if

$$f(z) = \int f\, d\mu \qquad \text{whenever } z \in J \text{ and } \mu \in \mathcal{N}_z(J).$$

Clearly the collection of $\mathcal{L}$-affine functions on J contains $\mathcal{L}(J)$.

We are now in a position to formulate the answer to our question in the following result, which is taken from [12].

Theorem 2. *Let $f : \mathbb{S} \to \mathbb{R}$. The following statements are equivalent:*

(a) there is a harmonic function h on $\mathbb{R}^n$ such that (3.1) holds;
(b) there is a sequence of compacts $J_k \uparrow \mathbb{S}$ such that, for each k, the restriction $f\,|_{J_k}$ is bounded, Baire-one and $\mathcal{L}$-affine on J_k.

We will briefly outline below the main ideas of the proof and refer to [12] for full details. Although some things have to be verified, the implication $(a) \Longrightarrow (b)$ is not difficult. As usual, the main interest lies in the proof of the converse. A key ingredient here is the result stated below. It follows from an abstract result of Lukeš *et al.* [14] that deals with approximation of bounded Baire-one functions in the context of simplicial function spaces. It can be applied in the present situation because of work of Bliedtner and Hansen [5] concerning simpliciality in potential theory.

Theorem B. *Let J be a compact subset of $\mathbb{S}$ and let $f : J \to \mathbb{R}$ be a bounded Baire-one function. If f is $\mathcal{L}$-affine on J, then there is a* *bounded sequence* (u_m) *in* $C(J)$ *such that each function* u_m *is* $\mathcal{L}$*-affine on* J *and* $u_m \to f$ *pointwise on* J.

Now let $f : \mathbb{S} \to \mathbb{R}$ and suppose that condition (b) of Theorem 2 holds. We fix k temporarily. By Theorem B there is a sequence $(u_{k,m})_{m\geq 1}$ in $C(J_k)$, and a positive constant c_k, such that

- $u_{k,m}$ is $\mathcal{L}$-affine on J_k for each m,
- $|u_{k,m}| \leq c_k$ on J_k for each m, and
- the sequence $(u_{k,m})_{m\geq 1}$ converges to f pointwise on J_k.

Further, by an approximation result of Debiard and Gaveau [7], we may assume that each function $u_{k,m}$ satisfies $\delta u_{k,m} = 0$ on a neighbourhood of $\overline{I}_{k,m}$, where $I_{k,m}$ is some relatively open neighbourhood of J_k in $\mathbb{S}$. (We may also assume that the sequence $(I_{k,m})_{m\geq 1}$ is decreasing.) Let ω_k denote the open set defined by

$$\omega_k = \bigcup_{m\geq 1} \{rz : z \in I_{k,m+1} \text{ and } ((m-1)!)^4 < r < ((m+1)!)^4\}$$

and let h_k denote the solution to the Dirichlet problem on ω_k with boundary data g_k where, for each $m \geq 1$, the function g_k is defined on the boundary subset

$$\{x \in \partial\omega_k : ((m-1)!)^4 m^2 \leq \|x\| < (m!)^4(m+1)^2\}$$

by

$$g_k(x) = \frac{1}{m}\sum_{l=1}^{m} u_{k,l}\left(\frac{x}{\|x\|}\right).$$

Careful estimation of harmonic measure can be used to show that

$$h_k(rz) \to f(z) \quad \text{as } r \to \infty \qquad (z \in J_k). \tag{3.2}$$

We now consider general values of k and define $E = \cup_k E_k$, where

$$E_k = \begin{cases} \{rz : z \in J_1 \text{ and } r \geq 1\} & (k=1) \\ \{rz : z \in J_k, \operatorname{dist}(z, J_{k-1}) \geq \frac{1}{r} \text{ and } r \geq k\} & (k \geq 2). \end{cases}$$

Clearly the set E is closed. We can obtain a harmonic function v on a neighbourhood of E by defining $v = h_k$ on an appropriate neighbourhood of E_k for each k. Further, it is readily seen that the set $(\mathbb{R}^n \cup \{\infty\}) \setminus E$ is connected and locally connected, where ∞ denotes the point at infinity for $\mathbb{R}^n$. Under these circumstances we can appeal to another result from the theory of harmonic approximation (see [3], or Corollary 5.10 of [10]) to conclude that there is an entire harmonic function h satisfying

$$|v(x) - h(x)| < \frac{1}{\|x\|} \qquad (x \in E).$$

Now let $z \in \mathbb{S}$ and $k_0 = \min\{k : z \in J_k\}$. For all sufficiently large values of r we have $rz \in E_{k_0}$ and so

$$\begin{aligned} |f(z) - h(rz)| &\leq |f(z) - v(rz)| + \frac{1}{r} \\ &= |f(z) - h_{k_0}(rz)| + \frac{1}{r} \\ &\to 0 \quad \text{as } r \to \infty \end{aligned}$$

by (3.2), as required.

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