

An infinitesimal criterion for topological triviality of families of sections of analytic varieties

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Abstract.

We present sufficient conditions for the topological triviality of families of germs of functions defined on an analytic variety V . The main result is an infinitesimal criterion using the integral closure of a convenient ideal as the tangent space to a subset of the set of topologically trivial deformations of a given germ. Applications to the problem of equisingularity of families of sections of V are also discussed.

§1. Introduction

Let $V, 0$ be the germ of an analytic subvariety of k^n ($k = \mathbb{R}$ or \mathbb{C}) and let \mathcal{R}_V (respectively $C^0\text{-}\mathcal{R}_V$) be the group of germs of diffeomorphisms (respectively homeomorphisms) preserving $V, 0$. In this paper we introduce a sufficient condition for the $C^0\text{-}\mathcal{R}_V$ -triviality of families of map germs $h : k^n \times k, 0 \rightarrow k^p, 0$, based on the integral closure of $T\mathcal{R}_V(h)$, the tangent space to the orbit of h under the action of the group \mathcal{R}_V . Our main result establishes that if $\frac{\partial h}{\partial t} \in \overline{T\mathcal{R}_V(h)}$, then h is topologically \mathcal{R}_V -trivial.

We are specially concerned with the case $p = 1$, that is, with families $h : k^n \times k, 0 \rightarrow k, 0$. In this case $h^{-1}(0)$ defines a family of sections of the analytic variety $V, 0$.

As a corollary of the method, we obtain sharp results when the analytic variety is weighted homogeneous and the family of sections is a deformation of a weighted homogeneous map germ h_0 (consistent with V) by terms of filtration higher than or equal to the filtration of h_0 .

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This result was previously proved by Damon in [8]. In the final section, we introduce a notion of V -equisingularity of the family of sections and we show that the hypothesis of the main theorem implies this geometric condition. A weighted approach for the topological triviality of families of sections of analytic varieties was presented in [16]. For other results related to the subject discussed in this paper, see for instance [2], [8], [19].

§2. Basic results

Let \mathcal{O}_n be the ring of germs of analytic functions $h : k^n, 0 \rightarrow k$, $k = \mathbb{R}$ or \mathbb{C} .

A germ of a subset $V, 0 \subset k^n, 0$ is the germ of an analytic variety if there exist germs of analytic functions f_1, \dots, f_r such that $V = \{x : f_1(x) = \dots = f_r(x) = 0\}$.

Our aim is to study map germs $h : k^n, 0 \rightarrow k^p, 0$ under the equivalence relation that preserves the analytic variety $V, 0$. We say that two germs h_1 and $h_2 : k^n, 0 \rightarrow k^p, 0$ are \mathcal{R}_V -equivalent (respectively C^0 - \mathcal{R}_V -equivalent) if there exists a germ of a diffeomorphism (respectively homeomorphism) $\phi : k^n, 0 \rightarrow k^n, 0$ with $\phi(V) = V$ and $h_1 \circ \phi = h_2$. That is,

$$\mathcal{R}_V = \{\phi \in \mathcal{R} : \phi(V) = V\},$$

where \mathcal{R} is the group of germs of diffeomorphisms of $k^n, 0$.

A one parameter deformation $h : k^n \times k, 0 \rightarrow k^p, 0$ of $h_0 : k^n, 0 \rightarrow k^p, 0$ is topologically \mathcal{R}_V -trivial (or C^0 - \mathcal{R}_V -trivial) if there exists homeomorphism $H : k^n \times k, 0 \rightarrow k^n \times k, 0$, $H(x, t) = (\bar{h}(x, t), t)$, such that $h \circ H(x, t) = h_0(x)$ and $H(V \times k) = V \times k$.

We denote by θ_n the set of germs of tangent vector fields in $k^n, 0$; θ_n is a free \mathcal{O}_n module of rank n . Let $I(V)$ be the ideal in \mathcal{O}_n consisting of germs of analytic functions vanishing on V . We denote by $\Theta_V = \{\eta \in \theta_n : \eta(I(V)) \subseteq I(V)\}$, the submodule of germs of vector fields tangent to V (see [2] for more details).

The tangent space to the action of the group \mathcal{R}_V is $T\mathcal{R}_V(h) = dh(\Theta_V^0)$, where Θ_V^0 is the submodule of Θ_V given by the vector fields that are zero at zero.

The group \mathcal{R}_V is a geometric subgroup of the contact group, as defined by J.Damon [5], [6], hence the infinitesimal criterion for \mathcal{R}_V -determinacy holds (see [2] for a proof).

Theorem 2.1. *The germ h is \mathcal{R}_V -finitely determined if and only if there exists a positive integer k such that $T\mathcal{R}_V(h) \supset \mathcal{M}_n^k$.*

The following theorem is the geometric criterion for the \mathcal{R}_V -finite determinacy.

Theorem 2.2. ([2]) *Let $V, 0 \subseteq \mathbb{C}^n, 0$ be the germ of an analytic variety and let $h : \mathbb{C}^n, 0 \rightarrow \mathbb{C}, 0$ be the germ of an analytic function. Let*

$$V(h) = \{x \in \mathbb{C}^n : \xi h(x) = 0 \text{ for all } \xi \in \Theta_V\}.$$

Then h is \mathcal{R}_V -finitely determined if and only if $V(h) = \{0\}$ or \emptyset .

As a consequence of this result, it follows that if h is \mathcal{R}_V -finitely determined, then $h^{-1}(c)$ is transverse to V away from 0, for sufficiently small values of c .

In the real case, the necessary condition remains true, that is, if h is \mathcal{R}_V -finitely determined then the set $\{x \in \mathbb{R}^n : \xi h(x) = 0 \text{ for all } \xi \in \Theta_V\}$ is $\{0\}$ or \emptyset .

§3. Basic facts on integral closure of ideals

Let I be an ideal in a ring A . An element $h \in A$ is said to be integral over I if it satisfies an integral dependence relation $h^n + a_1 h^{n-1} + \dots + a_n = 0$ with $a_i \in I^i$. The set of such elements form an ideal in A , called the integral closure of I .

When $A = \mathcal{O}_{X, x_0}$, the local ring of a complex analytic set, Teissier gives in [18] various notions equivalent to the above concept.

Theorem 3.1. ([11], Proposition 1.2) *Let I be an ideal in \mathcal{O}_{X, x_0} and \bar{I} its integral closure, where X is a complex analytic space. The following statements are equivalent:*

- (a) $h \in \bar{I}$.
- (b) *For each choice of generators $\{g_i\}$ of I there exist a neighbourhood U of x_0 and a constant $C > 0$ such that for all $x \in U$:*

$$|h(x)| \leq C \sup_i |g_i(x)|.$$

- (c) *For each analytic curve $\varphi : \mathbb{C}, 0 \rightarrow X, x_0$, $h \circ \varphi$ lies in $(\varphi^*(I))\mathcal{O}_1$.*
- (d) *There exists a faithful \mathcal{O}_{X, x_0} module L of finite type such that $h.L \subset I.L$.*

In the real case, the above algebraic definition of integral closure is not appropriate. But, one can use condition (c) above as a definition. More precisely,

Definition 3.2. *Let I be an ideal of the ring \mathcal{O}_{X, x_0} , where X is a real analytic set. The real integral closure \bar{I} of I is the set of h such that for all analytic $\varphi : \mathbb{R}, 0 \rightarrow X, x_0$, we have $h \circ \varphi \in (\varphi^*(I))\mathcal{O}_1$.*

Gaffney ([11], p. 30) shows that $h \in \bar{I}$ if and only if for each choice of generators $\{g_i\}$ of I there exists a neighbourhood U of x_0 and a constant $C > 0$ such that for all $x \in U$:

$$|h(x)| \leq C \sup_i |g_i(x)|.$$

§4. The main result

Let $h_0 : k^n, 0 \rightarrow k, 0$ be a \mathcal{R}_V -finitely determined germ of an analytic function and let $h : k^n \times k, 0 \rightarrow k, 0$ be an analytic deformation of h_0 . In the sequel, we shall assume $h(0, t) = 0$. The property of being \mathcal{R}_V -finitely determined is open in the sense that the germ $\{x \in k^n : dh_t \xi(x) = 0, \forall \xi \in \Theta_V\}$ at 0 is $\{0\}$ or empty for sufficiently small values of the parameters ([2]). However, this does not guarantee the existence of a neighbourhood U of 0 in $k^n, 0$ and an open ε -ball, B_ε , centered at the origin in k such that the above condition holds $\forall x \in U$ and $\forall t \in B_\varepsilon$. We then need the following definition:

Definition 4.1. *Let $h_0 : k^n, 0 \rightarrow k, 0$ be a \mathcal{R}_V -finitely determined germ. We say that a deformation $h : k^n \times k, 0 \rightarrow k, 0$ of h_0 is a good deformation if $V(h) \subseteq \{0\} \times k, 0$, where $V(h) = \{(x, t) \in k^n \times k, 0; dh_t(x)\xi(x) = 0 \forall \xi \in \Theta_V\}$.*

Example 4.2. *Let V be the x -axis in k^2 ; Θ_V is generated by $(1, 0)$ and $(0, y)$. The germ $h_0(x, y) = x^2 + y^3$ is \mathcal{R}_V -finitely determined. The deformation $h_t(x, y) = x^2 + y^3 + ty^2$ of h_0 has the property that h_t is \mathcal{R}_V -finitely determined for each fixed t , but we cannot find $\varepsilon > 0$ such that the above condition holds for all $t \in B_\varepsilon$.*

Our main result is the following theorem:

Theorem 4.3. *Let $h_0 : k^n, 0 \rightarrow k, 0$ be a \mathcal{R}_V -finitely determined germ and let $h : k^n \times k, 0 \rightarrow k, 0$ be a good deformation of h_0 . If $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$ for all $t \in k$ sufficiently near 0, then h is C^0 - \mathcal{R}_V -trivial.*

The proof of the theorem is a consequence of the following results.

In what follows we can assume that $dh_t \xi(0) = 0, \forall \xi \in \Theta_V$. In fact, if $\xi \in \Theta_V$, then $dh_t \xi \cdot \frac{\partial h}{\partial t} = dh_t(\frac{\partial h}{\partial t} \cdot \xi)$. If $dh_t \xi_0(0) \neq 0$ for some ξ_0 , then

$$\frac{\partial h}{\partial t} = dh_t\left(\frac{\frac{\partial h}{\partial t} \cdot \xi_0}{dh_t \xi_0}\right)$$

and hence the deformation is C^ω - \mathcal{R}_V -trivial (i.e. analytically trivial).

Observe that $\frac{\frac{\partial h}{\partial t} \cdot \xi_0}{dh_t \xi_0} \in \Theta_V^0$.

Lemma 4.4. *Let I and J be ideals in \mathcal{O}_n with $\mathcal{M}_n I \subseteq J \subseteq I$ and $V(I) = \{0\}$, where $V(I)$ is the variety of the ideal I . Then $V(J) = \{0\}$.*

Proof. From the hypothesis, $V(\mathcal{M}_n I) \supseteq V(J) \supseteq V(I)$. Since $V(\mathcal{M}_n I) = V(\mathcal{M}_n) \cup V(I) = \{0\} \cup \{0\}$, we get $V(J) = \{0\}$. Q.E.D.

Let $h_0 : k^n, 0 \rightarrow k, 0$ be a \mathcal{R}_V -finitely determined germ and let $h : k^n \times k, 0 \rightarrow k, 0$ be a good deformation of h_0 . Let $\{\xi_1, \dots, \xi_r\}$ be generators of Θ_V and $I = \langle dh_t \xi_1, \dots, dh_t \xi_r \rangle$ the ideal in \mathcal{O}_{n+1} then $V(I) \subseteq \{0\} \times k$, since h is a good deformation of h_0 . Let $\{\alpha_1, \dots, \alpha_m\}$ be the generators of Θ_V^0 , $dh_t \alpha_i = \rho_i$ and $J = \langle \rho_1, \dots, \rho_m \rangle$. Since the α_i and hence the ρ_i vanish on $\{0\} \times k$, it follows that $V(J) \supseteq \{0\} \times k$. On the other hand, $\mathcal{M}_n I \subseteq J \subseteq I$, and it follows from Lemma 4.4, that $V(J) \subseteq \{0\} \times k$.

Let $\rho(x, t) = \sum_{i=1}^m |\rho_i|^2$. The condition $V(J) = \{0\} \times k$ implies that $\rho \geq 0$, and $\rho_t(x) = 0$ is equivalent to $x = 0$. Then, the following result holds.

Lemma 4.5. *Let $h_0 : k^n, 0 \rightarrow k, 0$ be a \mathcal{R}_V -finitely determined germ and let $h : k^n \times k, 0 \rightarrow k, 0$ be a good deformation of h_0 . If $\rho(x, t) = \sum_{i=1}^m |dh_t \alpha_i|^2$, then $V(\rho(x, t)) = \{0\} \times k$.*

Lemma 4.6. *Let $h : k^n \times k, 0 \rightarrow k, 0$ be a deformation of h_0 . Suppose there is a continuous vector field $(W, 1) \in \Theta_{V \times k}$ such that:*

(i) $\rho \frac{\partial h}{\partial t} = dh_t(W)$, where ρ is a control function, that is, $\rho : k^n \times k, 0 \rightarrow \mathbb{R}$ with $\rho(x, t) \geq 0$ and $\rho(x, t) = 0$ if and only if $x = 0$.

(ii) $(-\frac{W}{\rho}, 1)$ is locally integrable.

Then h is topologically \mathcal{R}_V -trivial.

Proof. Let $\phi(x, t, \tau)$ be the flow of the on $k^n \times k, 0$ defined by $(-\frac{W}{\rho}, 1)$, so $\frac{\partial \phi}{\partial \tau} = (-\frac{W}{\rho}, 1) \circ \phi$, $\phi(x, t, 0) = (x, t)$. When $k = \mathbb{R}$, we define

$$\varphi(x, t) = \phi(x, 0, t) = (\bar{\varphi}(x, t), t).$$

Taking the derivative of $h(\varphi(x, t)) = h(\bar{\varphi}(x, t), t)$ with respect to t , we get

$$\begin{aligned} \frac{\partial}{\partial t}(h(\varphi(x, t))) &= \sum_{i=1}^n \frac{\partial h}{\partial x_i}(\bar{\varphi}(x, t), t) \frac{\partial \bar{\varphi}_i}{\partial t}(x, t) + \frac{\partial h}{\partial t}(\bar{\varphi}(x, t), t) \\ &= -\sum_{i=1}^n \frac{\partial h}{\partial x_i}(\bar{\varphi}(x, t), t) \frac{W_i}{\rho}(\bar{\varphi}(x, t), t) + \frac{\partial h}{\partial t}(\bar{\varphi}(x, t), t) \\ &= \left(\frac{\partial h}{\partial t} - \sum_{i=1}^n \frac{W_i}{\rho} \frac{\partial h}{\partial x_i}\right)(\bar{\varphi}(x, t), t) = 0 \end{aligned}$$

where W_i are the components of W . Hence, fixing x , it follows that $h(\varphi(x, t))$ is constant, that is, $h(\varphi(x, t)) = h(\varphi(x, 0)) = h(x, 0) = h_0(x)$

for all t and x . Therefore h is topologically \mathcal{R}_V -trivial. When $k = \mathbb{C}$, we consider the restriction

$$h^1 = h|_{\mathbb{C}^n \times \mathbb{R} \times \{0\}} \rightarrow \mathbb{C}.$$

It is sufficient to show that h is a \mathcal{R}_V -topologically trivial deformation of h^1 , which in turn is a \mathcal{R}_V -topologically trivial deformation of h_0 .

Let $\phi(x, t, \tau)$ be such that $\frac{\partial \phi}{\partial \tau} = (-\frac{W}{\rho}, 1) \circ \phi$ and $\phi(x, t, 0) = (x, t)$. We consider $\phi_1(x, u + iv) = \phi(x, u, v)$ and $\phi_2(x, u) = \phi(x, 0, u)$. It follows that $h \circ \phi_1$ is constant with respect to v and hence $h(\phi_1(x, u + iv)) = h(\phi_1(x, u)) = h(\phi(x, u, 0)) = h(x, u) = h^1(x, u)$. One can also show that $h^1 \circ \phi_2$ is constant with respect to u , therefore $h^1(\phi_2(x, u)) = h^1(\phi_2(x, 0)) = h^1(x, 0) = h_0$ and the result follows. Q.E.D.

Proof of the Theorem 4.3. With the above notations, it follows that

$$|\rho_i|^2 \frac{\partial h}{\partial t} = dh_t(\bar{\rho}_i \frac{\partial h}{\partial t} \alpha_i).$$

Since $\rho = \sum_{i=1}^m |\rho_i|^2$, it follows that

$$\rho \frac{\partial h}{\partial t} = dh_t \left(\frac{\partial h}{\partial t} (\bar{\rho}_1 \alpha_1 + \dots + \bar{\rho}_m \alpha_m) \right)$$

hence

$$\frac{\partial h}{\partial t} = dh_t \left(\frac{\partial h}{\partial t} \frac{1}{\rho} (\bar{\rho}_1 \alpha_1 + \dots + \bar{\rho}_m \alpha_m) \right).$$

From Lemma 4.5, $V(\rho(x, t)) = \{0\} \times k$. We define the vector field X in $k^n \times k, 0$,

$$X(x, t) = \begin{cases} \left(-\frac{\partial h}{\partial t} \frac{1}{\rho} (\bar{\rho}_1 \alpha_1 + \dots + \bar{\rho}_m \alpha_m), 1 \right) & \text{if } x \neq 0 \\ (0, 1) & \text{if } x = 0 \end{cases}$$

The vector field $X(x, t)$ is real analytic away from $\{0\} \times k$.

From the hypothesis, $\frac{\partial h}{\partial t} \in dh_t(\Theta_V^0)$ and hence by item (b) of Theorem 3.1

$$\left| \frac{\partial h}{\partial t} \right| \leq c \sup\{|\rho_i|\}.$$

Then

$$\begin{aligned}
 |X(x, t) - X(0, t)| &= \left| \frac{\partial h}{\partial t} \frac{1}{\rho} (\overline{\rho_1} \alpha_1 + \dots + \overline{\rho_m} \alpha_m) \right| \\
 &\leq \left| \frac{\partial h}{\partial t} \right| \frac{1}{\rho} (|\overline{\rho_1}| |\alpha_1| + \dots + |\overline{\rho_m}| |\alpha_m|) \\
 &\leq c \sup\{|\rho_i|\} \frac{1}{\rho} (|\rho_1| |\alpha_1| + \dots + |\rho_m| |\alpha_m|) \\
 &\leq c(|\alpha_1| + \dots + |\alpha_m|) \leq C|x|.
 \end{aligned}$$

Thus, X satisfies the Lipschitz condition around the solution $(0, t)$, and it follows from [4] or [13] that $X(x, t)$ is locally integrable in a neighbourhood of $(0, 0) \in k^n \times k$. Then, there exists a family of homeomorphisms $\phi(x, t, \tau)$, $\phi : k^n \times k \times \mathbb{R}, 0 \rightarrow k^n \times k, 0$ such that $\frac{\partial \phi}{\partial \tau} = -X \circ \phi$ and $\phi(x, t, 0) = (x, t)$. The proof follows now from Lemma 4.6 (see Lemma 6.2, in [9]). Q.E.D.

§5. Weighted homogeneous germs and varieties

Definition 5.1. (a) Given $(w_1, \dots, w_n : d_1, \dots, d_p)$, $w_i, d_j \in \mathbb{Q}^+$, a map germ $f : k^n, 0 \rightarrow k^p, 0$ is weighted homogeneous of type $(w_1, \dots, w_n : d_1, \dots, d_p)$ if for all $\lambda \in k - \{0\}$:

$$f(\lambda^{w_1} x_1, \lambda^{w_2} x_2, \dots, \lambda^{w_n} x_n) = (\lambda^{d_1} f_1(x), \lambda^{d_2} f_2(x), \dots, \lambda^{d_p} f_p(x)).$$

In this case, the value w_i is called weight of the variable x_i and the value d_i , is the filtration of f_i with respect to the weights (w_1, \dots, w_n) . We write: $\text{weight}(x_i) = w(x_i) = w_i$ and $\text{filtration}(f) = \text{fil}(f) = (d_1, \dots, d_p)$.

(b) Given (w_1, \dots, w_n) , and any monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$, we define $\text{fil}(x^\alpha) = \sum_{i=1}^n \alpha_i w_i$.

(c) We define a filtration in the ring \mathcal{O}_n via the function defined by $\text{fil}(f) = \inf_{|\alpha|} \{ \text{fil}(x^\alpha) : \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(0) \neq 0 \}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Definition 5.2. A germ of an analytic variety $V, 0 \subseteq k^n, 0$ is weighted homogeneous if it is defined by a weighted homogeneous map germ $f : k^n, 0 \rightarrow k^p, 0$.

Definition 5.3. Let $V, 0 \subseteq k^n, 0$ be the germ of a weighted homogeneous analytic variety. We say that a set $\{\alpha_1, \dots, \alpha_r\}$ of generators of Θ_V is weighted homogeneous of type $(w_1, \dots, w_n : d_1, \dots, d_r)$ if α_{ij} are weighted homogeneous polynomials of type $(w_1, \dots, w_n : d_i + w_j)$ whenever $\alpha_{ij} \neq 0$, where $\alpha_i = \sum_{j=1}^n \alpha_{ij} \frac{\partial}{\partial x_j}$, $i = 1 \dots r$.

When V is a weighted homogeneous variety, we always can choose weighted homogeneous generators for Θ_V . A proof can be found in [10].

Following [7], we define:

Definition 5.4. *Let V be defined by weighted homogeneous polynomials. We say that h is weighted homogeneous consistent with V if h is weighted homogeneous with respect to the same set of weights assigned to V .*

Example 5.5. *Let $V = \phi^{-1}(0) \subset k^3$ where $\phi(x, y, z) = z^2 - x^2y$. We have that ϕ is weighted homogeneous of type $(1, 2, 2 : 4)$. Let $h(x, y, z) = x^3 + xy + xz$ and $f(x, y, z) = x^3 + xy + z^2$. Then h is consistent with V , f is weighted homogeneous but not consistent with V .*

The following result does not follow as a corollary of the Theorem 4.3, but the proof is similar. It was previously proved by J. Damon in [8], but we include it here for completeness. In [16], we discuss a weighted approach for the topological triviality of families of sections of analytic varieties, which also gives Theorem 5.6 as a corollary.

Theorem 5.6. *Let V be a weighted homogeneous subvariety of $k^n, 0$ and let $h_0 : k^n, 0 \rightarrow k, 0$ be weighted homogeneous consistent with V and \mathcal{R}_V -finitely determined. Then any deformation h of h_0 by terms of filtration greater than or equal to the filtration of h_0 , is C^0 - \mathcal{R}_V -trivial.*

Proof. Under the above conditions, any such h is a good deformation of h_0 (see [15]).

We have $dh_0(\alpha_i)$ is weighted homogeneous, where $\{\alpha_1, \dots, \alpha_m\}$ is a set of weighted homogeneous generators of Θ_V . Let r_i be the filtration of $dh_0(\alpha_i)$, $i = 1, \dots, m$ and

$$\omega_0(x) = |dh_0(\alpha_1)(x)|^{2s_1} + \dots + |dh_0(\alpha_m)(x)|^{2s_r}$$

with $s_i = k/r_i$, and $k = \text{l.c.m.}\{r_i\}$. Let $\rho_i = dh_t(\alpha_i)$ and $\omega = \sum_{i=1}^m |\rho_i|^{2s_i}$. Since

$$|\rho_i|^2 \frac{\partial h}{\partial t} = dh_t(\bar{\rho}_i \frac{\partial h}{\partial t} \alpha_i),$$

it follows that

$$\omega \frac{\partial h}{\partial t} = dh_t \left(\frac{\partial h}{\partial t} (\bar{\rho}_1 |\rho_1|^{2s_1-2} \alpha_1 + \dots + \bar{\rho}_m |\rho_m|^{2s_m-2} \alpha_m) \right).$$

Then

$$\frac{\partial h}{\partial t} = dh_t \left(\frac{\partial h}{\partial t} \frac{1}{\omega} (\bar{\rho}_1 |\rho_1|^{2s_1-2} \alpha_1 + \dots + \bar{\rho}_m |\rho_m|^{2s_m-2} \alpha_m) \right).$$

The proof now follows analogously to the proof of Theorem 4.3. Q.E.D.

Example 5.7. Let $V, 0 \subset \mathbb{R}^3, 0$ (or $\mathbb{C}^3, 0$) be defined by $\varphi(x, y, z) = 2x^{k+1}y^2 + y^3 - z^2 + x^{2(k+1)}y = 0$. This is the implicit equation for the S_k -singularities classified by D. Mond [14]. The function-germ φ is weighted homogeneous of weights 2, $2k + 2$ and $3k + 3$ for x, y and z respectively. We have that $h(x, y, z) = y + a_{k+1}x^{k+1}$ is \mathcal{R}_V -finitely determined for $a_{k+1} \neq 0, 1$ and consistent with V . Therefore deformations of h by terms of order higher than or equal to $\text{fil}(h)$ are C^0 - \mathcal{R}_V -trivial. For k odd, $h_1(x, y, z) = z + ax^{3(k+1)/2}$ and $h_2(x, y, z) = z + bx^{(k+1)/2}y$ are consistent with V and \mathcal{R}_V -finite for all $a^2 \neq -4/27$ and $b \neq \pm 2$. Thus deformations of h_1 and h_2 , respectively by terms of order higher than or equal to $\text{fil}(h_1)$ and $\text{fil}(h_2)$ are C^0 - \mathcal{R}_V -trivial.

§6. V-Equisingularity

Bernard Teissier developed in [18] an infinitesimal theory and a theory of geometrical invariants to study the equisingularity of families of complex analytic hypersurfaces X_t^d with isolated singularities. The integral closure of an ideal I is the right object to the infinitesimal part of that theory. T. Gaffney in [11] extended Teissier results, using the integral closure of a convenient module to obtain necessary and sufficient conditions for the equisingularity of families of complete intersections with isolated singularities.

Definition 6.1. Suppose (X, x) is a complex analytic germ, $\mathcal{O}_{X,x}$ its local ring and M a submodule of $\mathcal{O}_{X,x}^p$. Then an element $h \in \mathcal{O}_{X,x}^p$ is in \overline{M} if and only if for all $\phi : \mathbb{C}, 0 \rightarrow X, x, h \circ \phi$ is in $(\phi^*(M))\mathcal{O}_1$.

Theorem 6.2. ([11], Theorem 2.5) Let $F : \mathbb{C}^t \times \mathbb{C}^N \rightarrow \mathbb{C}^p, 0$, defining $X = F^{-1}(0)$ with reduced structure, $Y = \mathbb{C}^t \times 0$ and X_0 the smooth part of X . Then $\frac{\partial F}{\partial s} \in \left\langle z_i \frac{\partial F}{\partial z_j} \right\rangle_{\mathcal{O}_X}$ for all tangent vectors $\frac{\partial}{\partial s}$ to $\mathbb{C}^t \times 0$ iff (X_0, Y) are Whitney regular.

Our purpose in this section is to show that the infinitesimal condition in Theorem 4.3 gives a sufficient condition for equisingularity of families of sections of analytic varieties. We also show with an example that it is not a necessary condition.

Let $V \subset \mathbb{C}^n$ be an analytic variety. The family of sections of V is defined by $h(x, t) = 0$, where $h : \mathbb{C}^n \times \mathbb{C}, 0 \rightarrow \mathbb{C}, 0, h(0, t) = 0$, is a good deformation of a \mathcal{R}_V -finitely determined map germ $h_0 : \mathbb{C}^n, 0 \rightarrow \mathbb{C}, 0$.

In order to define the notion of V -equisingularity, we will construct a stratified diagram of mappings which satisfies Thom's second isotopy lemma.

From now on, we assume that V admits a Whitney stratification \mathcal{S}_V in a neighbourhood U of the origin, for which $\{0\}$ is a stratum. We can also extend this stratification to the neighbourhood U of the origin in a natural way, that is, the strata are the strata of \mathcal{S}_V and the complement of V in U . We denote by \tilde{V} the subvariety of $\mathbb{C}^n \times \mathbb{C}, 0$ defined by $\tilde{V} = V \times \mathbb{C}$. The product stratification is clearly Whitney regular. Since the germ $h : \mathbb{C}^n \times \mathbb{C}, 0 \rightarrow \mathbb{C}, 0$ is a good deformation, we can choose a representative, which we also denote by h , given by $h : U \times B_r, 0 \rightarrow \mathbb{C}, 0$, where B_r is an open ball in \mathbb{C} centered at the origin with the property that $h^{-1}(0)$ is transversal to the strata of \tilde{V} away from $0 \times B_r$.

We refine the stratification $\tilde{\mathcal{S}}$ of $U \times B_r$ as follows. Given a stratum S of \mathcal{S} , we define the new strata \tilde{S} of $\tilde{\mathcal{S}}$ as one of the following types: $(S \times B_r) - h^{-1}(0)$ and $(S \times B_r) \cap h^{-1}(0)$. This refinement defines a new stratification $U \times B_r$, since h is transversal to \tilde{V} away from zero. We denote this new stratification by the same notation $\tilde{\mathcal{S}}$.

Definition 6.3. *With the above notation, h is V -equisingular if there exists $\varepsilon > 0$ such that:*

- (1) $(B_\varepsilon \times B_r, \tilde{\mathcal{S}})$ is Whitney regular;
- (2) $B_\varepsilon \times B_r \xrightarrow{F} \mathbb{C} \times B_r \xrightarrow{\pi} B_r$ satisfies the second isotopy lemma, where B_ε is the closed ball in \mathbb{C}^n with radius ε , B_r is the closed ball in \mathbb{C} of radius r , and $F : \mathbb{C}^n \times \mathbb{C}, 0 \rightarrow \mathbb{C} \times \mathbb{C}, 0$ is given by $F(x, t) = (h(x, t), t)$.

In the following theorem we show that $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$ is a sufficient condition for V -equisingularity.

Theorem 6.4. *Let $V = \phi^{-1}(0)$, $\phi : \mathbb{C}^n, 0 \rightarrow \mathbb{C}, 0$, $h_0 : \mathbb{C}^n, 0 \rightarrow \mathbb{C}, 0$ \mathcal{R}_V -finitely determined and $h : \mathbb{C}^n \times \mathbb{C}, 0 \rightarrow \mathbb{C}, 0$ a good deformation of h_0 . Let $h^{-1}(0) \cap \Sigma_\phi = \{0\} \times \mathbb{C}$, where Σ_ϕ is the singular set of ϕ . If $\frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}$, then h is V -equisingular.*

J.W. Bruce in [1] considers an analogous question. He describes the topological type of generic families of sections of a semialgebraic stratification \mathcal{T} of a neighbourhood of the origin in \mathbb{R}^n , with 0 being a stratum. Such families are *generalised transverse* (G.T) with respect to the stratification, that is, for every pair of strata S_1 and S_2 , and a sequence of points $(x_i) \in S_1$ such that $\lim_{i \rightarrow \infty} x_i = x \in S_2$ and the limit of the tangent spaces $\lim_{i \rightarrow \infty} T_{x_i} S_1 = T$ then $dh(x) : T \rightarrow \mathbb{R}$ has maximal rank, that is, $h^{-1}(h(x))$ is transversal to T .

The following theorem is proved in [1]:

Theorem 6.5. ([1], Proposition 1.4) *Let \mathcal{T} a Whitney stratification of an open neighbourhood U of the origin in \mathbb{R}^n , with 0 being a stratum. Let $h : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}$ be a family of submersions, with $h(0, t) = 0$ and $h_t(x) = h(x, t)$. If the family h is generalised transverse with respect to \mathcal{T} , for all $t \in [0, 1]$, then there exists a germ of homeomorphism $G : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0$ preserving the strata of \mathcal{T} such that $h_0 \circ G = h_1$.*

Good examples of families satisfying the G.T. condition are the families of sections of an analytic variety defined by generic families of hyperplanes in \mathbb{R}^n . In this work, we substitute the G.T. condition by the finite determinacy of h_0 and the integral closure condition. Under these hypothesis we are able to obtain the topological triviality of families that do not satisfy the G.T. condition.

Example 6.6. *Let $V, 0 \subseteq \mathbb{C}^3, 0$ be the swallowtail parametrized by $(x, -4y^3 - 2xy, -3y^4 - xy^2)$. The module Θ_V is generated by $\eta_1 = (2x, 3y, 4z)$, $\eta_2 = (6y, -2x^2 - 8z, xy)$ and $\eta_3 = (-4x^2 - 16z, -8xy, y^2)$. The \mathcal{R}_V classification of germs $h : \mathbb{C}^3, 0 \rightarrow \mathbb{C}, 0$ given by Theorem 4.10 in [3], gives the normal form $z + ax^n + tx^{n+1}$, $n \geq 2$ which is finitely determined for $a \neq 0, n \neq 2$, and $a \neq 0, a \neq 1/12, n = 2$. Let $h_0(x, y, z) = z + ax^n$ from Theorem 4.3 we have that the family $h_t(x, y, z) = z + ax^n + tx^{n+1}$ is topologically \mathcal{R}_V -trivial. However this family h_t is not G.T. at 0 , since $dh_t(0, 0, 0) = (0, 0, 1)$ and the limit of tangent planes to the smooth part of V is the xy -plane.*

To prove Theorem 6.4, we first prove the following Lemma.

Lemma 6.7. *Let $\phi : \mathbb{C}^n, 0 \rightarrow \mathbb{C}, 0$ and $V = \phi^{-1}(0)$. Given $h : \mathbb{C}^{n+1}, 0 \rightarrow \mathbb{C}, 0$, define $G : \mathbb{C}^{n+1}, 0 \rightarrow \mathbb{C}^2, 0$, by $G(x, t) = (h(x, t), \phi(x))$. If $g \in \overline{dh_t(\Theta_V^0)}_{\mathcal{O}_{n+1}}$ then $(g, 0) \in \overline{\langle x_i \frac{\partial G}{\partial x_j} \rangle}_{\mathcal{O}_{G^{-1}(0)}}$.*

Proof. By hypothesis, for any analytic curve $\varphi : \mathbb{C}, 0 \rightarrow \mathbb{C}^{n+1}, 0$, it follows that $g \circ \varphi \in \langle dh_t(\alpha_i) \circ \varphi \rangle$ where α_i are generators of Θ_V^0 . Then for all $\varphi : \mathbb{C}, 0 \rightarrow V \times \mathbb{C}, 0$, we also have

$(g \circ \varphi, 0) \in \langle dh_t(\alpha_i) \circ \varphi, d\phi(\alpha_i) \circ \varphi \rangle$, since $d\phi(\alpha_i) \in \langle \phi \rangle$ and $\phi(V) = 0$. Therefore $(g \circ \varphi, 0) \in \overline{\langle (x_i \frac{\partial h}{\partial x_j}, x_i \frac{\partial \phi}{\partial x_j}) \circ \varphi \rangle}$. Thus,

$$(g, 0) \in \overline{\langle (x_i \frac{\partial h}{\partial x_j}, x_i \frac{\partial \phi}{\partial x_j}) \rangle}_{\mathcal{O}_{V \times \mathbb{C}}} = \overline{\langle x_i \frac{\partial G}{\partial x_j} \rangle}_{\mathcal{O}_{V \times \mathbb{C}}}. \text{ In particular, } (g, 0) \in \overline{\langle x_i \frac{\partial G}{\partial x_j} \rangle}_{\mathcal{O}_{G^{-1}(0)}}. \quad \text{Q.E.D.}$$

Remark 6.8. *The above result remains true under the weaker hypothesis $g \in \overline{dh(\Theta_V^0)}_{\mathcal{O}_{V \times \mathbb{C}}}$.*

We now proceed to prove Theorem 6.4; our proof is analogous to the proof of Theorem 6.5 in [1]. As in [1], we divide the proof in steps:

Step 1. The stratification \tilde{S} is Whitney regular.

Proof. The Whitney regularity of a pair of strata (S_1, S_2) follows easily, with exception of the regularity condition of the strata over $\{0\} \times \mathbb{C}$. Clearly the strata of type $(S \times \mathbb{C}) - h^{-1}(0)$ are regular with respect to $\{0\} \times B_r$, since the original stratification satisfies the Whitney conditions. Then we only have to verify that $(S \times B_r) \cap h^{-1}(0)$ is regular over $\{0\} \times B_r$. From hypothesis, $\frac{\partial h}{\partial t} \in \overline{dh_t \Theta_V^0}$ and from Lemma 6.7 it follows that $(\frac{\partial h}{\partial t}, 0) \in \left\langle x_i \frac{\partial G}{\partial x_j} \right\rangle_{\mathcal{O}_{G^{-1}(0)}}$. Now, from Theorem 6.2, $(G^{-1}(0) - \Sigma_{G^{-1}(0)}, \{0\} \times B_r) = (h^{-1}(0) \cap \tilde{V} - \{0\} \times B_r, \{0\} \times B_r)$ is Whitney regular. Q.E.D.

Step 2. For some $\varepsilon' > 0$ and all $0 < \varepsilon \leq \varepsilon'$ the product of the boundary of the ε -ball, ∂B_ε , by B_r meets the strata of \tilde{S} transversally.

Proof. The argument is the same as in Theorem 6.5 in [1]. Let us suppose that the statement is false. Then we can find a sequence of points (x_i, t_i) in some stratum \tilde{S} with $x_i \rightarrow 0$ and $T_{(x_i, t_i)} \tilde{S} \subset T_{(x_i, t_i)}(\partial B_{\varepsilon_i} \times B_r)$ where $\varepsilon_i = \|x_i\|$. Then $(x_i, 0)$ is perpendicular to $T_{(x_i, t_i)} \tilde{S}$. This contradicts the Whitney condition B. Q.E.D.

We then have the first approximation to our stratified diagram, that is,

$$B_\varepsilon \times B_r \xrightarrow{F} \mathbb{C} \times B_r \xrightarrow{\pi} B_r$$

where B_ε is the closed ball in \mathbb{C}^n of radius ε , $\varepsilon \leq \varepsilon'$, $F(x, t) = (h(x, t), t)$ and π is the projection to the second factor. We stratify $\mathbb{C} \times B_r$ by $(\mathbb{C} - \{0\}) \times B_r \cup \{0\} \times B_r$ and we refine the stratification of $B_\varepsilon \times B_r$, taking the intersection of the strata in \tilde{S} with $\partial B_\varepsilon \times B_r$ and $\text{int} B_\varepsilon \times B_r$. We would like to show that this stratification satisfies Thom's condition, but h_t might have critical points on ∂B_ε . To get around this difficulty we need the following.

Step 3. For some $\delta > 0$, $B_\delta - \{0\}$ in \mathbb{C} consists only of regular values of h_t for every $t \in B_r$.

Proof. This follows from the fact that h is a good deformation of h_0 . Q.E.D.

In the above diagram we change \mathbb{C} by B_δ , where B_δ is the ball with radius δ , with the stratification $\partial B_\delta \cup \{0\} \cup \text{int} B_\delta - \{0\}$, and satisfying the

conditions in Step 3. We then get a new stratification of $F^{-1}(B_\delta \times B_r)$ pulling back the strata. We consider now

$$F^{-1}(B_\delta \times B_r) \xrightarrow{F} B_\delta \times B_r \xrightarrow{\pi} B_r$$

Step 4. The above diagram is Thom stratified.

Proof. We have to show that the diagram satisfies the condition A_{h_t} . Given two strata \tilde{S}_1, \tilde{S}_2 with $(x_i, t_i) \in \tilde{S}_1$, and $(x_i, t_i) \rightarrow (x, t) \in \tilde{S}_2$, the restriction of the kernel of $dF(x_i, t_i)$ to $T_{(x_i, t_i)}\tilde{S}_1$, say K_i , is $T_{(x_i, t_i)}\tilde{S}_1 \cap (\ker dh_{t_i}(x_i) \times \{0\})$. The limit of this sequence of spaces is contained in $T \cap (\ker dh_t(x) \times \{0\})$ where $T = \lim_{i \rightarrow \infty} T_{(x_i, t_i)}\tilde{S}_1$. If $x \neq 0$ then $\ker dh_t(x) \times \{0\}$ is transversal to T , hence $\lim_{i \rightarrow \infty} K_i = T \cap (\ker dh_t(x) \times \{0\})$. Since $T \supset T_{(x, t)}\tilde{S}_2$ (Whitney condition A), then $\lim_{i \rightarrow \infty} K_i$ contains the restriction of the kernel of $dF(x, t)$ to $T_{(x, t)}\tilde{S}_2$. If $x = 0$ then $\tilde{S}_2 = \{0\} \times B_r$, and Thom condition follows trivially. Q.E.D.

Remark 6.9. The V -equisingularity of a family h as above implies that h is topologically \mathcal{R}_V -trivial.

In fact, from Thom's second isotopy lemma ([12], p.62), there exist homeomorphisms

$$H : F^{-1}(B_\delta \times \{0\}) \times B_r \rightarrow F^{-1}(B_\delta \times B_r)$$

$$H' : B_\delta \times \{0\} \times B_r \rightarrow B_\delta \times B_r,$$

preserving the stratifications, such that the following diagram commutes:

$$\begin{array}{ccccc} F^{-1}(B_\delta \times \{0\}) \times B_r & \xrightarrow{F \times id} & B_\delta \times \{0\} \times B_r & \xrightarrow{\pi_3} & B_r \\ \downarrow H & & \downarrow H' & & \downarrow id \\ F^{-1}(B_\delta \times B_r) & \xrightarrow{F} & B_\delta \times B_r & \xrightarrow{\pi_2} & B_r \end{array}$$

Then $H(x, 0, t) = (\bar{h}(x, t), t)$, $F(\bar{h}(x, t), t) = (h(x, 0), t)$ and it follows that $h(\bar{h}(x, t), t) = h_0(x)$ for all t and x . Therefore h is topologically \mathcal{R}_V -trivial.

The example below shows that the condition $g \in \overline{dh(\Theta_V^0)}_{\mathcal{O}_{V \times C}}$ is stronger than the condition $(g, 0) \in \overline{\langle x_i \frac{\partial G}{\partial x_j} \rangle}_{\mathcal{O}_{G^{-1}(0)}}$.

Example 6.10. Let $V, 0 \subset k^3, 0$ be defined by $\phi(x, y, z) = 2x^2y^2 + y^3 - z^2 + x^4y = 0$ and $h : \mathbb{C}^4, 0 \rightarrow \mathbb{C}, 0$, $h(x, y, z, t) = y + (a + t)x^2$ and $G : \mathbb{C}^4, 0 \rightarrow \mathbb{C}^2, 0$ given by $G(x, y, z, t) = (y + (a + t)x^2, 2x^2y^2 + y^3 - z^2 + x^4y)$. The module Θ_V is generated by $\eta_1 = (2x, 4y, 6z)$, $\eta_2 = (0, 2z, x^4 + 4x^2y + 3y^2)$, $\eta_3 = (x^2 + 3y, -4xy, 0)$ and $\eta_4 = (z, 0, 2x^3y + 2xy^2)$. The element $\frac{\partial h}{\partial t} = x^2$ is not in the integral closure of the ideal $dh_t(\Theta_V^0)$ (it also follows that $x^2 \notin \overline{\langle dh(\eta_i) \rangle}_{\mathcal{O}_{V \times \mathbb{C}}}$). In fact, given $\phi : k, 0 \rightarrow k^4, 0$, $\phi(s) = (s, -as^2, 0, 0)$, it follows that $\frac{\partial h}{\partial t} \circ \phi$ is not in $(\phi^*(dh_t(\Theta_V^0)))\mathcal{O}_1$, then by Theorem 3.1, $\frac{\partial h}{\partial t} = x^2 \notin \overline{dh_t(\Theta_V^0)}$. We can verify that $(x^2, 0) \in \overline{\langle x_i \frac{\partial G}{\partial x_j} \rangle}_{\mathcal{O}_{G^{-1}(0)}}$. In fact, we will show that $(x^2, 0) \in \overline{\langle x_i \frac{\partial G}{\partial x_j}, e_i G_j \rangle}_{\mathcal{O}_4}$ and the result will follow from this. We have

- (a) $zG_z = (0, -2z^2)$
- (b) $e_1G_1 = (y + (a + t)x^2, 0)$
- (c) $x^2G_y = (x^2, 4x^4y + 3x^2y^2 + x^6)$
- (d) $e_2G_2 + \frac{1}{2}zG_z = (0, 2x^2y^2 + y^3 + x^4y)$

Let $\varphi : \mathbb{C}, 0 \rightarrow \mathbb{C}^4, 0$ be given by $\varphi(u) = (\varphi_1(u), \varphi_2(u), \varphi_3(u), \varphi_4(u))$. We shall see that $(\varphi_1^2, 0) \in \overline{\langle (x_i \frac{\partial G}{\partial x_j}, e_i G_j) \circ \varphi \rangle}_{\mathcal{O}_1}$. Let $r = \text{ord}(\varphi_1)$ and $s = \text{ord}(\varphi_2)$, if $s \leq r$ or $2s = r$ then it follows from (b) that $(\varphi_1^2, 0) \in \overline{\langle (x_i \frac{\partial G}{\partial x_j}, e_i G_j) \circ \varphi \rangle}_{\mathcal{O}_1}$.

If $s > r$ then it follows from (c) that

$$x^2G_y \circ \varphi = (\varphi_1^2, 4\varphi_1^4\varphi_2 + 3\varphi_1^2\varphi_2^2 + \varphi_1^6) = (\varphi_1^2, 0) + (0, 4\varphi_1^4\varphi_2 + 3\varphi_1^2\varphi_2^2 + \varphi_1^6)$$

and from (d) we get that $(0, 4\varphi_1^4\varphi_2 + 3\varphi_1^2\varphi_2^2 + \varphi_1^6) \in \overline{\langle (x_i \frac{\partial G}{\partial x_j}, e_i G_j) \circ \varphi \rangle}_{\mathcal{O}_1}$, hence,

$$(\varphi_1^2, 0) \in \overline{\langle (x_i \frac{\partial G}{\partial x_j}, e_i G_j) \circ \varphi \rangle}_{\mathcal{O}_1} \text{ or } (x^2, 0) \in \overline{\langle x_i \frac{\partial G}{\partial x_j}, e_i G_j \rangle}_{\mathcal{O}_4}.$$

Remark 6.11. When the variety V reduces to 0, Tessier in [17] proved that the set

$$\{t \in \mathbb{C}, 0 : \frac{\partial h}{\partial t} \in \overline{\langle x_i \frac{\partial h}{\partial x_j} \rangle}\}$$

is open and dense. In the relative case, we can obtain a similar result as a consequence of Gaffney in [11], that is :

$$\{t \in \mathbb{C}, 0 : (\frac{\partial h}{\partial t}, 0) \in \overline{\langle x_i \frac{\partial G}{\partial x_j} \rangle}_{\mathcal{O}_{G^{-1}(0)}}\}$$

is open and dense. However, the corresponding statement does not hold for $\overline{dh_t(\Theta_V^0)}$. In fact, with a slight modification of the arguments in the above example, we see that the set:

$$\{t \in \mathbb{C}, 0 : \frac{\partial h}{\partial t} \in \overline{dh_t(\Theta_V^0)}\}$$

is empty.

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