Advanced Studies in Pure Mathematics 43, 2006
Singularity Theory and Its Applications
pp. 369–382

On the classification of 7th degree real decomposable curves

Grigory M. Polotovskiy

Abstract.

A survey of recent results in the problem of the topological classification of 7th degree decomposable curves in the real projective plane is given.

Let (x_0, x_1, x_2) be point coordinates in the real projective plane $\mathbb{R}P^2$. An algebraic curve of degree m is a homogeneous polynomial $F_m(x_0, x_1, x_2)$ over $\mathbb{R}$ of degree m considered up to a constant nonzero factor. The set

$$\mathbb{R}F_m = \{(x_0, x_1, x_2) \in \mathbb{R}P^2 | F_m(x_0, x_1, x_2) = 0\} \subset \mathbb{R}P^2$$

is called the set of real points of the curve. The algebraic curve F_m is called an M-curve if the set $\mathbb{R}F_m$ consists of $(m-1)(m-2)/2+1$ connected components.

The polynomial F_m is *decomposable* (in the product of two factors) if

$$F_m(x_0, x_1, x_2) = A_k(x_0, x_1, x_2) \cdot B_{m-k}(x_0, x_1, x_2),$$

where $k \leq [m/2]$, and the polynomials $A_k(x_0, x_1, x_2)$ of degree k and $B_{m-k}(x_0, x_1, x_2)$ of degree $m-k$ are irreducible over $\mathbb{R}$. Our problem is to obtain the topological classification of triples $(\mathbb{R}P^2, \mathbb{R}F_m, \mathbb{R}A_k)$, which satisfy the following conditions of maximality and general position:

(i) the curves A_k and B_{m-k} are M-curves ;

(ii) the set $\mathbb{R}A_k \bigcap \mathbb{R}B_{m-k}$ consists of $k(m-k)$ distinguish points;

Received March 30, 2004.
Revised May 31, 2005.
Supported by grant E02-1.0-199 from Department of Education of Russian Federation.

(iii) all points of the set $\mathbb{R}A_k \bigcap \mathbb{R}B_{m-k}$ are situated on the same connected component of $\mathbb{R}A_k$ and on the same connected component of $\mathbb{R}B_{m-k}$.

In case $m = 6$ this problem was solved by the author (under weaker conditions) – see [P1]–[P4]. In particular, the following theorem provides the classification for the case $m = 6, k = 1$.

Theorem 1. *Under conditions* (i)-(iii), *the classification of triples* $(\mathbb{R}P^2, \mathbb{R}F_6, \mathbb{R}A_1)$ *consists of 4 types shown in Figure 1.*

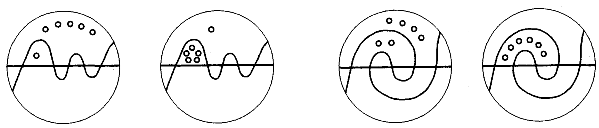

Fig. 1. Line and M-quintic (authors of the first constructions are marked).

Here and below we use the Poincaré disk (i.e. disk where every two diametrically opposite points of its boundary circle are identified) as model of the projective plane $\mathbb{R}P^2$.

The classification of arrangements of a quintic and a line in general position has important application: it gives classification of smoothings of generic five-fold point (singularity N_{16} in Arnold's notations). O.Viro showed (see [V1], [V2]) that from topological point of view, smoothing such a singular point is a result of gluing an affine quintic instead of a neighborhood of the point under condition of coincidence of asymptotic directions of the quintic with tangents to the branches at the singular point. E.Shustin proved [Sh1] that it is always possible to obtain this coincidence. We would like to point out that Theorem 1 provides also the classification of affine M-quintics: it is sufficient to consider the line $\mathbb{R}A_1$ as the line at infinity for the affine plane (in Figure 2 the line $\mathbb{R}A_1$ is shown as the boundary of the Poincaré disc).

In one's turn, smoothing five-fold points has been used by many authors for the constructions of nonsingular algebraic curves.

The classification of triples $(\mathbb{R}P^2, \mathbb{R}F_m, \mathbb{R}A_k)$ for $m = 6$ has many different applications, therefore it is naturally to consider the problem for $m = 7$. Below we give a survey of results in this direction.

The classification of affine M-sextics has been recently completed. One can find the proof in series of papers [O-Sh1], [K1], [K2], [Sh-K], [Sh2], [O1], [O2], [O-Sh2], [F-O]. The result is formulated in the following theorem.

Theorem 2. *Under conditions* (i)-(iii), *the classification of triples* $(\mathbb{R}P^2, \mathbb{R}F_7, \mathbb{R}A_1)$ *consists of 35 types shown in Figure 2.*

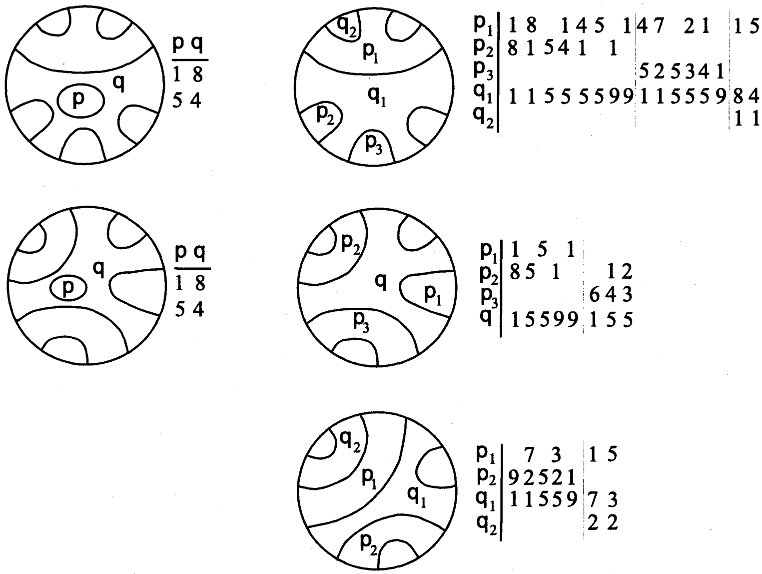

Fig. 2. Line and M-sextic (letters denote the number of ovals in domains of the same names).

Further it is natural to consider separately the cases when points of set $\mathbb{R}A_k \bigcap \mathbb{R}B_{7-k}$

a) are situated on ovals[1] of curves-factors and

b) lie on the odd branch of the factor of odd degree.

Note, that in case a) there always exists a pseudo-line which has no intersections with ovals[2]. Below for pictures we assume that this line is the boundary of the Poincaré disk and we do not draw it in the Figures. In case a) we also do not draw the odd branch.

One can find the proof of the classification for the case $k = 2$ under condition a) in papers [O3], [P5], [P6] and complete answer is:

[1]By definition, an oval and the odd branch are respectively a two-sided and one-sided circles embedded in $\mathbb{R}P^2$.

[2]In the opposite case there exists a pseudo-line consisting from arcs of ovals therefore the odd branch will intersect an oval, but it contradicts to the assumption a).

Theorem 3. *Under conditions* (i)-(iii) *in case* a) *the classification of triples* $(\mathbb{R}P^2, \mathbb{R}F_7, \mathbb{R}A_2)$ *consists of 42 types shown in Figure 3.*

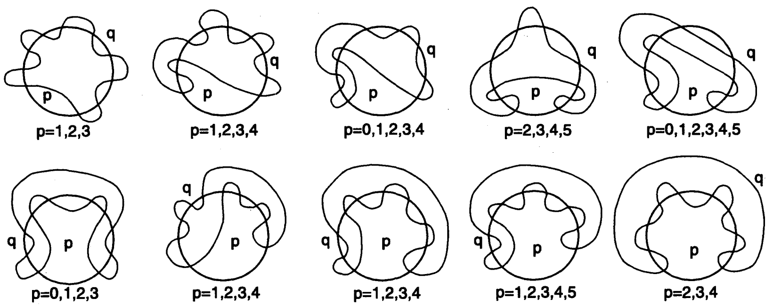

Fig. 3. Conic and M-quintic with common points on ovals; $p + q = 5$.

In the case b) for $k = 2$ the classification is in progress. In particular, at the present time about 60 types are constructed and for the same number of types the question about realizability is still open. Some details can be found in [P5], [G1], [G2]. In [K-P] was considered a problem intimately connected with case b) for $k = 2$: the classification of arrangements of a M-quintic and pair of lines. Namely, in [K-P] the following theorem was proved.

Theorem 4. *Every arrangement of two lines and quintic in maximal general position, for which there are only two arcs of odd branch having ends in points of intersection lying on different lines, is homeomorphic to one of 20 model depicted in Figure 4.*

The most difficult case is case $k = 3$ of mutual arrangements of a M-cubic and a M-quartic. In [O-P] the answer for case a) was obtained:

Theorem 5. *Under conditions* (i)-(iii) *in case* a) *the classification of triples* $(\mathbb{R}P^2, \mathbb{R}F_7, \mathbb{R}A_3)$ *consists of 31 types shown in Figure 5.*

In the case b) for $k = 3$ the classification has not been completed. Below we describe some details for this case. Simultaneously it will give an illustration of a general approach to the classification which consists in the following.

We draw *topological models*, i.e., collections of smooth circles in $\mathbb{R}P^2$, which may pretend to represent a triple of the kind $(\mathbb{R}P^2, \mathbb{R}F_m, \mathbb{R}A_k)$ up to a homeomorphism, and for each such a model we try to find out, to which extent this pretention can be justified. In other words, our procedure consists of the following steps.

Step 1. *Enumeration* of all admissible models.

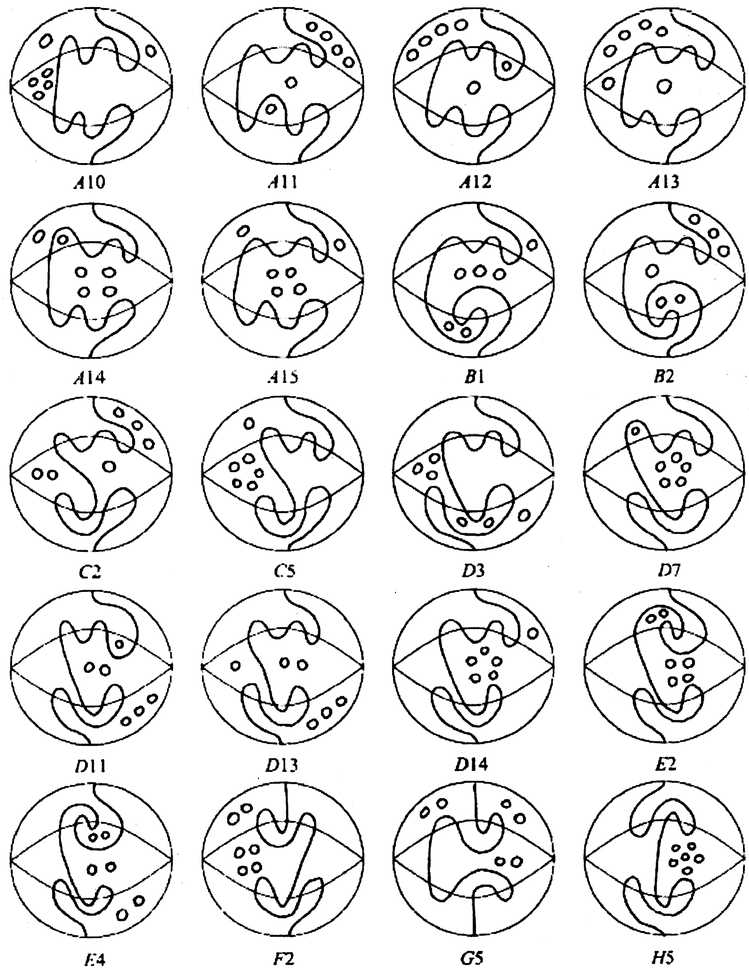

Fig. 4. M-quintic and pair of lines.

In essence, each time this is a special combinatorial problem, the algorithms for solution of the problems were described in [P6] (for $m = 6$ – in [P2], [P4]).

Step 2. *Constructions*, i.e., attempts to realize a given admissible model by a 7th degree curve.

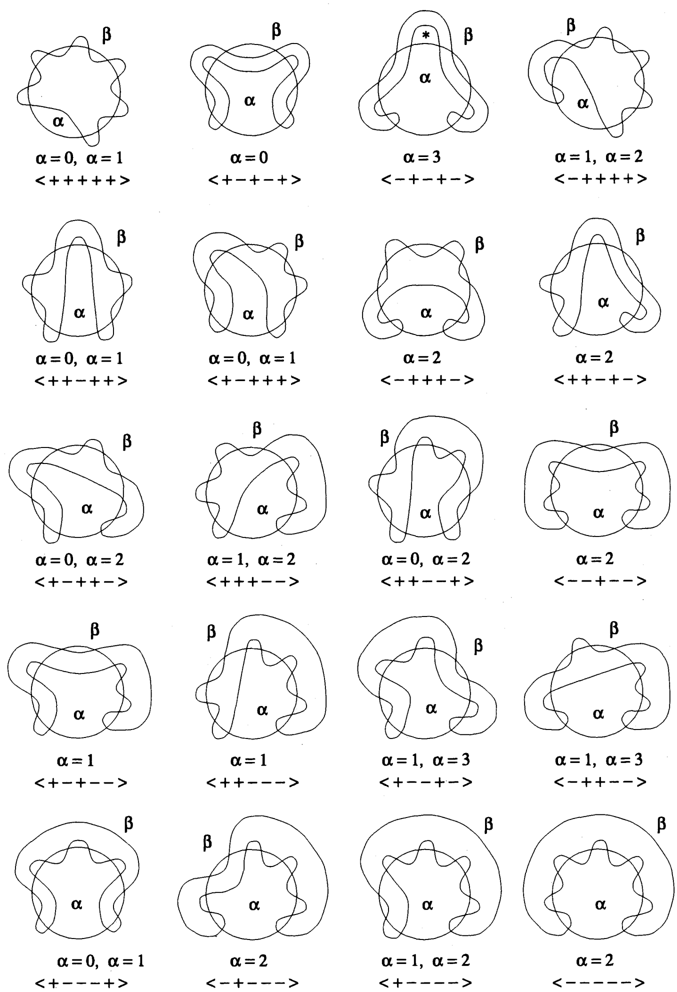

Fig. 5. M-cubic and M-quartic with common points on ovals; $\alpha + \beta = 3$.

For the constructions, different variants of the small parameter methods (including Viro's technique of gluing of charts of polynomials [V1], [V2]) and quadratic transformations were applied.

Step 3. *Prohibitions*, i.e., attempts to prove that a given admissible model cannot be realized by a 7th degree curve.

The main methods of prohibitions are the Orevkov method [O2] based on the link theory, and the Hilbert-Rohn-Gudkov method (see [O-Sh2]) based on the bifurcation theory.

Now we return to the case b) for $k = 3$: M-cubic and a M-quartic with 12 common points on an oval O_4 of the quartic and the odd branch J_3 of the cubic.

Let the system of coordinates in $\mathbb{R}P^2$ be such that the straight line $x_2 = 0$ does not intersect ovals of the quartic (to get this, it is sufficient to assume that $x_2 = 0$ is the result of small shifting a double tangent line to the quartic). To reduce the space of our paper, we consider here only the case when there exists a pseudo-line S such that the odd branch of the cubic intersect this pseudo-line at one point only. Let us consider S as "the line at infinity" (i.e. the boundary of the Poincaré disk). There are 12 arcs on the odd branch J_3 and 12 arcs on the oval O_4, which appear under intersection of the odd branch with the oval O_4. We assume that the endpoints of the arc of the odd branch, which intersect the line S, coincide with two endpoints of the same arc of the oval O_4 (series "A" in [P6]).

The admissible models of $(\mathbb{R}P^2, O_4 \bigcup J_3)$ are enumerated by codes which are lexicographically ordered in the second column of Table 1. To obtain the model, which corresponds to a code, it is sufficient
(i) to draw a circle in the interior of the Poincaré disk, which displays as a model of the oval O_4,
(ii)to mark on this circle 12 points and denote consecutively these points by symbols $1, 2, \ldots, 9, a, b, c$ successively, and
(iii)to draw the model of J_3 in the order given by the code so that the arc $(c, 1)$ of J_3 (with the endpoints c and 1) intersects the line at infinity (in our case the boundary of the Poincarè disk) at one point.

For each of 83 models of Table 1 the set $(\mathbb{R}P^2 \setminus O_4 \bigcap J_3)$ consists of 13 connected domains: the closures of 12 of them are homeomorphic to a disk and the closure of one domain is homeomorphic to a Möbius band. In all cases we denote the last domain by β. The set $\mathbb{R}F_m \setminus (O_4 \bigcup J_3)$ consists of four ovals, which are called "free". The quartic provides three free ovals and the cubic provides one free oval denoted by O_3. The free ovals are located in these domains. Simple arguments (topological corollaries of the Bézout theorem and so on, see for details [P6]) show that some of the domains can not contain free ovals, and free ovals can not surround each other in the domain different from the domain β [3].

[3]Sometimes (for example, in case no.1) such situation for free ovals in β is possible.

Table 1.

no	code	♯ of cases for ovals	realized (no in [O4])	no	code	♯ of cases for ovals	realized (no in [O4])
1	123456789abc	13	1 – 5	43	12543a9678bc	3	–
2	12345678ba9c	12	13	44	12543a9876bc	3	–
3	1234567a98bc	12	–	45	1256789ab43c	12	134
4	123456987abc	12	–	46	125678ba943c	3	–
5	1234569ab87c	12	29,30	47	12569ab8743c	3	–
6	123456ba789c	9	31	48	1256ba98743c	3	–
7	123456ba987c	13	32 – 34	49	1276589ab43c	12	136
8	123458769abc	12	35	50	1278965ab43c	12	138
9	12345876ba9c	12	37	51	12789ab6345c	9	139
10	1234589a76bc	9	–	52	12789ab6543c	13	140
11	12345a9678bc	12	43	53	1278ba96345c	3	–
12	12345a9876bc	13	44,45	54	1278ba96543c	3	–
13	1234765a98bc	3	–	55	1298765ab43c	12	142
14	123478965abc	12	61,62	56	12987ab6345c	8	143
15	1234789ab65c	12	65,66	57	12987ab6543c	12	144
16	123478ba965c	3	–	58	129ab834567c	8	–
17	123498567abc	9	–	59	129ab854367c	8	–
18	123498765abc	13	68 – 70	60	129ab876345c	9	145
19	1234987a65bc	12	74,75	61	129ab876543c	13	146
20	12349ab8567c	9	76	62	12ba3456789c	9	–
21	12349ab8765c	13	77 – 80	63	12ba5436789c	8	–
22	1234ba56789c	8	–	64	12ba7634589c	8	–
23	1234ba76589c	8	–	65	12ba7654389c	8	–
24	1234ba98567c	9	81	66	12ba9834567c	9	–
25	1234ba98765c	13	82 – 85	67	12ba9854367c	8	–
26	123654987abc	12	–	68	12ba9876345c	9	147
27	1236549ab87c	12	87	69	12ba9876543c	13	148
28	123654ba789c	8	88	70	1432789ab65c	12	149
29	123654ba987c	12	89	71	1432987ab65c	12	151
30	12367854ba9c	8	–	72	14329ab8567c	8	152
31	1236789a54bc	8	–	73	14329ab8765c	12	153
32	12367a9854bc	8	–	74	1432ba56789c	8	–
33	12387456ba9c	12	95	75	1432ba76589c	8	–
34	12387654ba9c	12	99	76	1432ba98567c	8	154
35	12389a7456bc	9	–	77	1432ba98765c	8	155
36	12389a7654bc	9	–	78	1456329ab87c	8	156
37	123a945678bc	12	–	79	145632ba789c	8	–
38	123a945876bc	12	–	80	145632ba987c	8	157
39	123a965478bc	3	–	81	1652349ab87c	12	158
40	123a987456bc	13	–	82	165234ba987c	12	159
41	123a987654bc	13	–	83	165432ba987c	8	160
42	1254389a76bc	3	–		Total	784	63

Note that domain β is always admissible for the free ovals. The number of admissible distributions of ovals is shown in the third column of table 1 [4].

Some constructions of arrangements of M-cubic and a M-quartic having 12 points of intersection of the oval O_4 and the odd branch J_3 were described in [P6]. Recently using some new approach, S.Orevkov [O4] obtained a list of 237 distinguish arrangements of such sort (his list includes results of all previous constructions). In the fourth column of Table 1, we indicate the numbers of realized models from the Orevkov list[5].

Now we give a short explanation of the application of the Orevkov prohibition method [O2] taking as an example case no.3 from the Table 1 (many details of the method can be found in [O2],[O3],[K-P],[O-P]). The topological model for this case is shown to the left picture of Figure 6, where each Greek letter denotes the numbers of free ovals in the domain and the same time the name of this domain. The right picture represents the same model in the more realistic view.

Suppose that this model with some distribution of free ovals is realized by some curve C_7 of degree 7. The enumeration of admissible distribution of free ovals is very simple: the oval O_3 can be in one of the domains α, β, δ (the domain γ is free of free ovals by virtue of the complex orientations formulas); for free ovals of quartic we have $\beta + \delta = 3$ for every position of O_3. Thus, the total number of distributions of free ovals is 12 (compare with Table 1).

1. To apply the Orevkov method [O2] we need in a pencil L_P of lines in $\mathbb{R}P^2$ with center at a point $P \in \mathbb{R}P^2 \setminus \mathbb{R}C_7$, which has a maximal general position with respect to the curve $\mathbb{R}C_7$. Here the *maximal general position* means that (i) for every line $l \in L_P$ the set $l \bigcap \mathbb{R}C_7$ consists of at least 5 points and there exists some such line having 7 common points with $\mathbb{R}C_7$, (ii)the multiplicity of intersection of every line $l \in L_P$ and the curve $\mathbb{R}C_7$ at every point is no more than two, and (iii) for every line $l \in L_P$ the number of such points with multiplicity two is no more than one. The points of intersection of l and $\mathbb{R}C_7$ with multiplicity two are called *critical* of the pencil L_P. They can be either points of tangency of l and $\mathbb{R}C_7$ or double points of $\mathbb{R}C_7$. A line l having critical points is called *critical.*

[4]Here corollaries of the Rokhlin and Mishachev formulas of complex orientations are taken into account; applications of these formulas in such situations are described in [P6], [K-P], [O-P].

[5]Pictures in the Orevkov list in [O4] are not numbered. We enumerate them along rows of his figures.

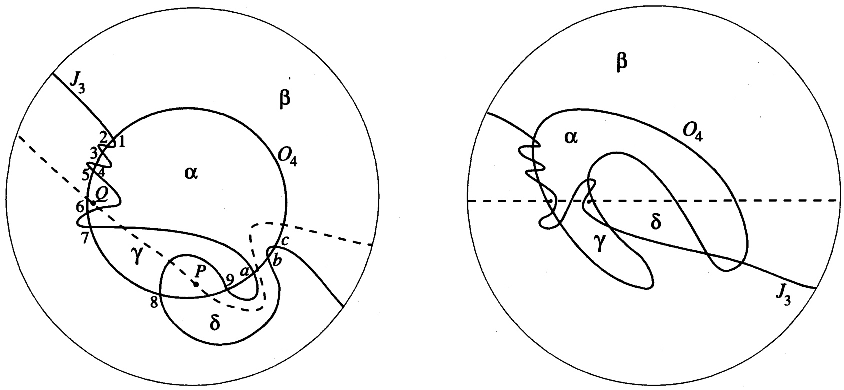

Fig. 6. Model for arrangement with code (12345678*ba*9*c*) (no.3 from Table 1).

Let a center P of the pencil be chosen by an appropriate way for a given topological model of $J_3 \bigcup O_4$. After that we need to consider all different admissible possibilities for mutual arrangement of the model of the pencil with respect to the model of $J_3 \bigcup O_4$. The Bézout theorem admits several (usually two or three) such essentially different arrangements[6].

2. Let us choose point P in the interior of the digon with vertices 8, 9 (see Figure 6). Let Q be some interior point in the digon with vertices 5, 6. The dotted line in Figure 6 represents one of admissible positions of the line PQ. It is convenient to redraw the picture such that line PQ becomes the boundary circle of the Poincaré disk, see Figure 7. If we draw the corresponding affine plane, where the center P of the pencil L_P is located on the line at infinity, then the pencil L_P in this affine plane constitutes a set of parallel lines. Free ovals may be only in vertical zones bounded by critical lines and filled by lines of the pencil, each line of which has 5 real points of intersection with $J_3 \bigcup O_4$. We must consider all admissible distributions of free ovals in these zones taking into account their mutual order.

3. Consider complexification of our construction. Let

$$\mathbb{C}C_7 = \{(x_0, x_1, x_2) \in \mathbb{C}P^2 | C_7 = 0\}$$

[6]"Essentially different" means that corresponding braids, which will be constructed below, are nonconjugate in the braid group.

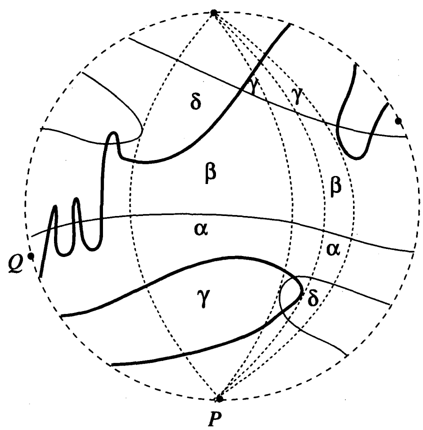

Fig. 7. Model no.3 and pencil L_P.

be the set of complex points of the curve C_7, $\mathbb{C}l$ be the set of complex points of the line l and $\mathbb{C}L_P = \bigcup \mathbb{C}l$ for $l \in L_P$. The intersection $\mathbb{C}C_7 \bigcap \mathbb{C}L_P$ can be described as a union of 7 circles. Every two circles either are disjoint or intersect at critical points of the pencil $\mathbb{C}L_P$; and the intersection of every three circles is empty.

Some standard perturbation (see details in [O2]) of the union of circles turns it into a link K of disjoint circles. Let b be a braid in the group B_7 of braids of 7 strings, whose closure $\bar{b}$ coincides with the link K. It is clear that the braid b is defined up to conjugation in the group B_7. The fact that the pencil L_P is in maximal general position with respect to $\mathbb{R}C_7$ implies that the braid b is uniquely defined (up to conjugation) by visible mutual arrangement of the model of $\mathbb{R}C_7$ and the pencil L_P in $\mathbb{R}P^2$.

The construction implies that the link $K = \bar{b}$ is the boudary link for a part of a surface $\mathbb{C}C_7 \in \mathbb{C}P^2$. It is well known (see, for example, [R]) that it is possible only if the braid b is a so called quasi-positive braid. As a necessary condition of quasipositivity, as in [O2],[K-P],[O-P], we apply the Murasugi-Tristram Inequality, which for our case can be written in the form

$$h = |\sigma(\bar{b})| + n - e(b) - \mathrm{null}(\bar{b}) \leq 0,$$

where $\sigma(\bar{b})$ is the signature, $\mathrm{null}(\bar{b})$ is the nullity of the link $\bar{b}$, and $e(b) = \sum k_i$ for $b = \prod \sigma_i^{k_i}$, where σ_i, $i = 1, 2, \ldots, 6$, are standard generators of the B_7.

4. One can check that for every position of the pencil with respect to the model no. 3 and for every distribution of free ovals, the value of h is always positive. Thus, the model no. 3 from Table 1 is unrealizable by an algebraic curve of degree 7.

For all of other considered cases, including all cases of the Table 1, we have obtain $h = 0$ only for the arrangements which were realized by S.Orevkov. This leads to the following

Conjecture. *Under conditions* (i)-(iii) *in the case* b) *every union of an M-cubic and M-quartic is homeomorphic to some disposition from the Orevkov's list* [O4] *of realized models.*

Remarks. 1. The most difficult step in the application of the Orevkov method is the choice of the point P and enumeration of admissible arrangements of the pencil L_P, i.e., items 1 and 2 above. These steps were made "by hand". All other steps were made on a computer by using of a number of programs written by M.Guschin. Other variant of programs was created by S.Orevkov.

2. The prohibitions for cases of the Table 1, which satisfy the assumption that the oval O_3 lies outside of ovals of the quartic and outside of β (for the example, in δ of the model No. 3 above), were independently considered by S.Orevkov (see Proposition 6.2 in [O4]). In these cases, one can choose the center P of the pencil L_P inside the oval O_3; and the disposition of L_P with respect to the model is easily determined.

Acknowledgement. My thanks to A. B. Korchagin from Texas Tech University for his friendly support and helpful discussions and to the referee for very useful remarks to the initial text.

References

[G1] M. A. Guschin, A conic and an M-quintic with one point in infinity, Zapiski nauchn. seminarov POMI, T. **329** (2005), 14–27.

[G2] M. A. Guschin, Construction of some mutual arrangements of conic and M-quintic with a single point at infinity, Vestnik Nizhegorodskogo universiteta – Matematika, **1** (2004), 43–52.

[F-O] S. Fiedler-LeTouze and S. Yu. Orevkov, A flexible affine M-sextic which is algebraically unrealizable, J. of Algebraic Geometry, **11** (2002), 293–310.

[K1] A. B. Korchagin, On the reduction of singularities and on the classification of nonsingular affine sextics, Deposited in VINITI, **1107-B86** (1986), 1–18 (Russian).

[K2] A. B. Korchagin, Smoothing of 6-Fold Singular Points and Constructions of 9th Degree M-Curves, AMS Translations (2), **173** (1996), 141–155.

[K-P] A. B. Korchagin and G. M. Polotovskii, On arrangements of a plane real quintic curve with respect to a pair of lines, Communications in Contemporary Math., **5** (2003), 1–24.

[O1] S. Yu. Orevkov, A new affine M-sextic, Funk. analiz i ego priloj, **32** (1998), 91–94; English transl. in Funct. Analysis and Appl., **32** (1998), 141–143; II. Uspekhi Mat. Nauk, **53** (1999), 243–244; English transl. in Russian Math. Surveys, **53** (1999), 1099–1101.

[O2] S. Yu. Orevkov, Link theory and oval arrangements of real algebraic curves, Topology, **38** (1999), 779–810.

[O3] S. Yu. Orevkov, Projective conics and M-quintics in general position with a maximally intersecting pair of ovals, Math. Notes, **65** (1999), 528–532 (Russian).

[O4] S. Yu. Orevkov, Construction of arrangements of an M-quartics and an M-cubics with a maximal intersections of an oval and the odd branch, Vestnik Nizhegorodskogo universiteta – Matem. modelirovanie i optimalnoe upravlenie, **1** (2002), 12–48; English transl. is available in http://picard.ups-tlse.fr/ orevkov/ .

[O-P] S. Yu. Orevkov and G. M. Polotovskii, Projective M-cubics and M-quartics in general position with maximal intersecting pairs of ovals, Algebra i analiz, **11** (1999), 166–184; English transl. in St. Petersburg Math. J., **11** (2000), 837–852.

[O-Sh1] S. Yu. Orevkov and E. I. Shustin, Pseudoholomorphic, algebraically unrealizable curves, Moscow Math. J., **3** (2003), 1053–1083.

[O-Sh2] S. Yu. Orevkov and E. I. Shustin, Flexible, algebraically unrealizable curves: rehabilitation of Hilbert-Rohn-Gudkov approach, J. Reine Angew. Math., **551** (2002), 145–172.

[P1] G. M. Polotovskii, A catalogue of M-decomposing curves of 6-th degree, Dokl. Akad. Nauk SSSR, **236** (1977), 548–551; English transl. in Soviet Math. Dokl., **18** (1977), 1241–1245.

[P2] G. M. Polotovskii, Complete classification of M-decomposable curves of 6th degree in real projective plane, Deposited in VINITI, **1349-78** (1978), 1–103 (Russian).

[P3] G. M. Polotovskii, $(M-1)$- and $(M-2)$-decomposable curves of degree 6, In: Metody kachestvennoi teorii differentsial'nykh uravnenii, Gorky, Gos. Universitet, 1978, 130–148 (Russian).

[P4] G. M. Polotovskii, Topological classification of decomposable curves of degree 6, PhD Thesis, Gorky, 1979, 1–260 (Russian).

[P5] G. M. Polotovskii, On the classification of decomposing plane algebraic curves, Lect. Notes in Math., **1524** (1992), 52–74.

[P6] G. M. Polotovskii, On the Classification of Decomposable 7-th Degree Curves, Contemporary Mathematics, **253** (2000), 219–234.

[R] L. Rudolph, Algebraic functions and closed braids, Topology, **22** (1983), 191–202.

[Sh1] E. I. Shustin, The Hilbert-Rohn method and bifurcations of complicated singular points of 8th degree curves, Uspekhi Mat. Nauk, **38** (1983), 157–158 (Russian).

[Sh2] E. I. Shustin, Toward isotopic classificatiom of affine M-curves of degree 6, In: Metody kachestvennoi teorii differentsial'nykh uravnenii, Gorky, Gos. Universitet, 1988, 97–105 (Russian).

[Sh-K] E. I. Shustin and A. B. Korchagin, Affine curves of degree 6 and smoothings of ordinary 6th order singular point, Izvestiya AN SSSR, ser. Math., **52** (1988), 1181–1199; English transl. in Math. USSR Izvestiya, **33** (1989), 501–520.

[V1] O. Ya. Viro, Gluing of algebraic hypersurfaces, smoothing of singularities, and construction of curves, Proc. Leningrad Intern. Topological Conf., Leningrad, Aug., 1983, "Nauka", Leningrad, 1983, 149–197 (Russian).

[V2] O. Ya. Viro, Plane real algebraic curves: constructions with controlled topology, Algebra i analys, **1** (1989), 1–73; English transl. in Leningrad Math. J., **1** (1989), 1059–1134.

Department of Mathematics and Mechanics
Nizhny Novgorod University
Russia