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On the decomposition of holomorphic functions by integrals and the local CR extension theorem

R. Michael Range

Among the classical and far reaching applications of integrals to the decomposition of holomorphic functions is P. Cousin's use of the Cauchy Integral Formula to obtain the most basic version which underlies the solution of what is now known as the additive Cousin problem. Concretely, if L is an (oriented) line segment in the complex plane $\mathbb{C}$, and if f is holomorphic in a neighborhood of L, then

$$F^{\pm}(z) = \frac{1}{2\pi i}\int_L \frac{f(\zeta)d\zeta}{\zeta - z}, \quad z \notin L$$

defines F^+ on the left side of L (resp. F^- on the right), both F^+ and F^- extend holomorphically across L, and

$$f(z) = F^+(z) - F^-(z) \text{ on } L.$$

In 1942, K. Oka [O] used a version of this principle with a Bergman-Weil type integral formula for polyhedra in his solution of the Levi problem.

Another well known application of this principle arises in the classical proof of the Hartogs extension theorem by means of the Bochner-Martinelli formula, discovered independently by E. Martinelli and S. Bochner in the early 1940s. Suitably modified, this principle allows also a simple natural proof of the corresponding *global CR* extension theorem of Severi and Fichera[1].

In this note I shall discuss an application of this principle to a proof of a version of the *local CR* extension theorem, valid under minimal regularity hypotheses. The main step reduces the question of CR extension to the classical problem of extension of holomorphic functions. While versions of this reduction have been known for a long time (see,

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[1]See [R1] for details.

for example, [AH]), the techniques used require stronger differentiability hypotheses, and typically involve some loss of regularity. In contrast, the Bochner-Martinelli kernel provides a simple mechanism to carry out the reduction in an optimal way. More precisely, I shall prove the following result.

Theorem 1. *Let S be a closed C^1 hypersurface in an open set $\Omega \subset \mathbb{C}^n$. Let $U \subset\subset \Omega$ be a neighborhood of $p \in S$ with $U \setminus S = U^+ \cup U^-$, where U^+ and U^- are disjoint and connected. Suppose U^- is not a domain of holomorphy at p, i.e., every $f \in \mathcal{O}(U^-)$ extends holomorphically to p. Then there exists a neighborhood W of p, such that every CR function $f \in C(U \cap S)$ extends holomorphically to $U^+ \cap W$.*

The hypothesis at the point p is satisfied, for example, in the classical situation where U^+ is strictly pseudoconvex at p. More generally, if S is of class C^2, it holds whenever the Levi form of S, viewed as part of the boundary of U^+, has at least one positive eigenvalue. Of course, the hypothesis on the given function f has to be interpreted in the weak sense, i.e., $\int_S f\bar{\partial}\varphi = 0$ for all $C^\infty_{(n,n-2)}$ forms with compact support in U.

Interest in this phenomenon was rekindled by the recent discovery of a long forgotten 1936 paper by Hellmuth Kneser [K], in which this theorem was proved for strictly pseudoconvex boundary points in $\mathbb{C}^2$, fully 20 years *before* Hans Lewy's famous 1956 theorem [H], which for a long time had been viewed as the first result of this sort[2].

Let us briefly recall some basic results about the Bochner-Martinelli kernel

$$K_{BM} = \frac{(n-1)!}{(2\pi i)^n} \frac{\sum_{j=1}^n (\overline{\zeta_j - z_j}) d\zeta_j \wedge (\Lambda_{k\neq j} d\overline{\zeta_k} \wedge d\zeta_k)}{|\zeta - z|^{2n}}.$$

(Complete proofs may be found, for example, in [R1].)

K_{BM} is real analytic in z. So, if S is an oriented C^1 hypersurface and $f \in C(S)$ has compact support, the Bochner-Martinelli transform

$$T_S f(z) = \int_S f(\zeta) K_{BM}(\zeta, z)$$

defines a real analytic function on $\mathbb{C}^n \setminus S$. With $U \setminus S = U^+ \cup U^-$ as in the theorem, one may consider the restrictions $T_S^+ f = T_S f \mid U^+$ and $T_S^- f = T_S f \mid U^-$. If f is Hölder continuous of some positive order,

[2]The reader may find a comprehensive account of the history of the local and global CR extension phenomena in the author's article [R2].

then $T_S^+ f$ and $T_S^- f$ extend continuously to S, and if the orientation of S agrees with the one it carries as part of the boundary of U^+, one has the "jump formula"

$$(1) \qquad f(z) = T_S^+ f(z) - T_S^- f(z) \text{ for } z \in U \cap S.$$

More generally, this formula remains valid for continuous f, whenever one can prove the continuous extension from at least one of the sides (continuity from the other side then follows as well).

Since K_{BM} is not holomorphic if $n > 1$, $T_S f$ will not be holomorphic on $\mathbb{C}^n \setminus S$ in general. Instead, one has

$$(2) \qquad \overline{\partial_z} K_{BM} = -\overline{\partial_\zeta} K_1 \text{ on } \mathbb{C}^n \times \mathbb{C}^n \setminus \{\zeta = z\},$$

where K_1 is an explicit double form of type $(0,1)$ in z and $(n, n-2)$ in ζ. A simple application of Stokes' theorem then implies that if S is compact without boundary, say if $S = bD$ for $D \subset\subset \mathbb{C}^n$, and if $f \in \mathcal{O}(S)$, then $T_S f$ is holomorphic on $\mathbb{C}^n \setminus S$. In fact, only the weaker hypothesis $\overline{\partial_b} f = 0$ on S is needed for this conclusion.

When S has nonempty boundary, $T_S f$ is no longer holomorphic in general. Instead, one has the following weaker result.

Lemma 2. *Suppose S is a C^1 hypersurface in $\mathbb{C}^n$, and $f \in C(S)$ has compact support in S. Let U be an open set, such that f is weakly CR on $S \cap U$. Then $\overline{\partial}(T_S f)$ extends to a C^∞ $(0,1)$ form on U.*

The important fact is that application of $\overline{\partial}$ eliminates the discontinuity of $T_S f$ across S.

Proof. $T_S f$ is clearly C^∞ outside S. We need to show that $T_S f$ extends C^∞ to any point $p \in S \cap U$. Fix p, and choose a neighborhood $V(p) \subset\subset U$ and $\chi \in C_0^\infty(U)$ with $\chi \equiv 1$ on V. Then

$$T_S f = \int_S f \chi K_{BM} + \int_S f(1-\chi) K_{BM},$$

where the 2nd integral is clearly C^∞ on V (indicated by +..... in the following). On $V \setminus S$ one therefore has

$$\overline{\partial_z} T_S f(z) = \int_S f(\zeta)\chi(\zeta)\overline{\partial_z} K_{BM}(\zeta, z) + \ldots = (\text{by}(2)) - \int_S f\chi \overline{\partial_\zeta} K_1 + \ldots =$$

$$= - \int_S f \; \overline{\partial_\zeta}(\chi K_1) + \int_S f \; (\overline{\partial_\zeta}\chi) K_1 + \ldots .$$

In the last equation, the first integral is 0 by the hypothesis on f [3], and hence extends trivially across $S \cap V$, while the 2nd integral is C^∞ on V since $\overline{\partial_\zeta \chi} \equiv 0$ on V.

Corollary 3. *If U is Stein, there exists $u \in C^\infty(U)$, such that $H^\pm = T_S^\pm f - u$ is holomorphic on U^+ (resp. U^-).*

Proof. Let u be any solution of $\overline{\partial} u = \overline{\partial} T_S f$ on U. Note that if U is convex with smooth boundary (for example a ball), such solutions can be found by means of elementary integral formulas.

The proof of the theorem is now very easy. Without loss of generality we may assume that U is a ball centered at p, and that f has compact support in S. Consider $T_S f$, and choose u as in the Corollary. By the hypothesis on p, the function H^- extends holomorphically across p, say to a neighborhood W of p (which depends only on the complex geometry of S near p, i.e., W can actually be chosen independently of H^- and f). Since u is C^∞ on U, $T_S^- f = H^- + u$ extends continuously (in fact C^∞) across $S \cap W$. Hence $T_S^+ f$, and then $H^+ = T_S^+ f - u$, also extends continuously from $U^+ \cap W$ to $S \cap W$, and the jump formula (1) holds on $S \cap W$. It follows that

$$f(z) = (T_S^+ f(z) - u) - (T_S^- f(z) - u) = H^+(z) - H^-(z) \text{ for } z \in W \cap S,$$

and thus $H^+ - H^-$ yields the desired holomorphic extension of f to $U^+ \cap W$.

Remark. The proof shows that the extension $H^+ - H^-$ of f is continuous on $W \cap \overline{U^+}$. In case $f \in C^1(S)$, one easily shows that the extension is in $C^1(W \cap \overline{U^+})$, and that analogous results hold when S and f are differentiable of higher order. The proofs follow by the same techniques used in the corresponding regularity results for the global CR extension theorem (see [R1] for example).

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[3] To be precise, one needs to approximate $K_1(\cdot, z)$, which has a singularity at z, by forms which are C^∞ on U.

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Department of Mathematics and Statistics
State University of New York at Albany
Albany, NY 12222
USA