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An Approximation for Exponential Hedging

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Abstract.

An optimization problem in mathematical finance, called the exponential hedging problem is addressed. First, the relations between the problem and the backward stochastic differential equation (abbreviated to BSDE) having a quadratic growth term in the drift are reviewed. Next, the asymptotic analysis by Davis (2000) for the problem and the motivation of this paper are stated. Further, with some extensions, his analysis is reinterpreted by using the asymptotic expansion of the BSDE with respect to a small parameter, which suggests an alternative approach to the analysis, and the result on an approximated optimizer is obtained.

§1. Introduction

In [7], Rouge and El Karoui treated the following optimization problem of mathematical finance. For a fixed $T > 0$, let $S := (S_t)_{t\in[0,T]}$, $S_t := \left(S_t^1, \dots S_t^n\right)'$ be the price process of n-risky assets defined by the stochastic differential equation:

$$\begin{aligned} dS_t &= \operatorname{diag}(S_t)\left(\sigma_t dw_t + \mu_t dt\right), \quad S_0 \in \mathbf{R}_+^n, \\ \operatorname{diag}(S_t) &:= \begin{pmatrix} S_t^1 & 0 & \cdots & 0 \\ 0 & S_t^2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & S_t^n \end{pmatrix} \end{aligned}$$

on the probability space $(\Omega, \mathcal{F}, P)$ with a $d(\geq n)$-dimensional Brownian motion $w := (w_t)_{t\in[0,T]}$ on it and the augmented Brownian filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Here, σ is an $n \times d$-matrix-valued left-continuous adapted process such that $\sigma\sigma' \in L^\infty([0,T] \times \Omega, \mathbf{R}^{n\times n})$ and that $\sigma_t\sigma_t'$ is invertible

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for all $t \in [0,T]$ ($(\cdot)'$ denotes the transpose of a matrix or a vector), μ is an n-dimensional predictable process, and $\lambda := \sigma'(\sigma\sigma')^{-1}(\mu - r\mathbf{1})$ is an element of $L^\infty([0,T]\times\Omega, \mathbf{R}^d)$, where $r(>0)$ is the constant interest rate and $\mathbf{1} := (1,\ldots,1)' \in \mathbf{R}^n$. On the other hand, let $F \in L^\infty(\Omega, \mathcal{F}_T)$ be the payoff of a derivative security maturing at time T and consider a seller of the derivative security, who trades the assets continuously in self-financing way on the time-interval $[0,T]$ to control the terminal wealth. The value process of the self-financing portfolio is given by

$$dX_t^{x,\pi} = \pi_t'(\mathrm{diag}(S_t))^{-1}dS_t + (X_t^{x,\pi} - \pi_t'\mathbf{1})\,rdt, \quad X_0^{x,\pi} = x,$$

or equivalently,

$$X_t^{x,\pi} := e^{rt}\left\{x + \int_0^t \pi_u'\sigma_u\,(dw_t + \lambda_t dt)\right\}$$

where x is the initial capital and an n-dimensional predictable process π is the asset holding strategy. To optimize the terminal wealth $-F+X_T^{x,\pi}$ of the seller, the utility maximization problem (called the exponential hedging problem in this paper, following Delbaen et. al; 2002, [2])

$$\text{(P)} \qquad V(x) := \sup_{\pi\in\mathcal{A}} E\left[U_\gamma\left(-F + X_T^{x,\pi}\right)\right]$$

with respect to the exponential utility function:

$$U_\gamma(x) := -\frac{e^{-\gamma x}}{\gamma}, \quad (\gamma > 0)$$

over an appropriately chosen space $\mathcal{A}$ of admissible strategies is considered.

The importance of this problem is, from a viewpoint of mathematical finance, that it relates to the pricing and hedging problems of derivative securities in incomplete markets: the quantity called utility indifference price,

$$(1) \qquad p(x,F) := \inf_{y\in\mathbf{R}}\left\{V(x+y) \geq \sup_{\pi\in\mathcal{A}} E\left[U_\gamma\left(X_T^{x,\pi}\right)\right]\right\},$$

is proposed as a coherent price of the derivative security in Davis (2000), [1] and [7], and the optimizer of the problem (P) is focused and studied to control (hedge) the "risk" of the seller in [1], [2] and [7].

Duality argument is well established for utility maximization (cf., Karatzas and Shreve; 1998, [5], for example) and is often used to attack this problem, as follows. For example, let us employ the space

$$\mathcal{A}_2 := \left\{ \pi \in \mathcal{L}_T^{2,n};\ \pi_t \in C \text{ for all } t \in [0,T],\ E\left[\int_0^T |\pi_t|^2 dt\right] < \infty \right\},$$

as the set of admissible strategies, $\mathcal{A}$, where $C \subset \mathbf{R}^n$ is a fixed closed convex cone and $\mathcal{L}_T^{2,n}$ is the totality of the n-dimensional predictable processes π on the time-interval $[0,T]$ such that $\int_0^T |\pi_t|^2 dt < \infty$, a.s. For $f, x \in \mathbf{R}$, and $y > 0$, denote

$$u_\gamma(x; y, f) := U_\gamma(-f + x) - yx \quad \text{and} \quad I_\gamma(y) := \left(U'_\gamma\right)^{-1}(y) = -\frac{1}{\gamma}\log(y)$$

to see the relation

$$\sup_{x\in\mathbf{R}} u_\gamma(x; y, f) = u_\gamma\left(f + I_\gamma(y); y, f\right) = -y\left(f - \frac{1+\log y}{\gamma}\right).$$

Define

$$Z^\nu := \mathcal{E}\left(-(\lambda - \nu)\cdot w\right) \quad \text{and} \quad \widetilde{Z}_t^\nu := e^{-rt} Z_t^\nu,$$

where ν is an element of

$$\mathcal{D} := \left\{ \nu \in \mathcal{L}_T^{2,d};\ \text{bounded and } \ \nu_t \in \widetilde{(\sigma_t' C)} \text{ for all } t \in [0,T] \right\}$$

and $\widetilde{(\sigma_t' C)}$ is the notation for the negative polar cone of $\sigma_t' C$, i.e.,

$$\widetilde{(\sigma_t' C)}(\omega) := \left\{ y \in \mathbf{R}^d;\ xy \le 0 \text{ for all } x \in \sigma_t'(\omega) C \right\}.$$

For $\pi \in \mathcal{A} := \mathcal{A}_2$ and $\nu \in \mathcal{D}$, observe that

$$e^{-rt} X_t^{x,\pi} \le x + \int_0^t \pi_u' \sigma_u \left\{ dw_u + (\lambda_u - \nu_u) du \right\}$$

and that $Z^\nu \int \pi' \sigma \{dw + (\lambda - \nu) du\}$ is a martingale since $E\left[\sup_{t\in[0,T]} |Z_t^\nu|^2\right] < \infty$ and since

$$E\left[\sup_{t\in[0,T]} \left| \int_0^t \pi_u' \sigma_u \left\{ dw_u + (\lambda_u - \nu_u) du \right\} \right|^2 \right] \le C_1 E\left[\int_0^T |\pi_u|^2 du\right] < \infty$$

from Doob's inequality and the boundedness assumptions of $\sigma\sigma'$, λ and ν. Therefore, the relation

$$E\left[\widetilde{Z}_T^\nu X_T^{x,\pi}\right] \le x$$

follows. Based on the relation, for $\pi \in \mathcal{A}$ and $x \in \mathbf{R}, y > 0$, we observe the inequalities

$$
\begin{aligned}
(2) \qquad & E\left[U_\gamma\left(-F + X_T^{x,\pi}\right)\right] - yx \\
\leq\ & \inf_{\nu \in \mathcal{D}} E\left[U_\gamma\left(-F + X_T^{x,\pi}\right) - y\widetilde{Z}_T^\nu X_T^{x,\pi}\right] \\
\leq\ & \inf_{\nu \in \mathcal{D}} \sup_{\pi \in \mathcal{A}} E\left[u_\gamma\left(X_T^{x,\pi}; y\widetilde{Z}_T^\nu, F\right)\right] \\
\leq\ & \inf_{\nu \in \mathcal{D}} E\left[u_\gamma\left(F + I_\gamma\left(y\widetilde{Z}_T^\nu\right); y\widetilde{Z}_T^\nu, F\right)\right].
\end{aligned}
$$

The minimization problem

$$
\text{(D)} \qquad \widehat{V}(y) := \inf_{\nu \in \mathcal{D}} E\left[u_\gamma\left(F + I_\gamma\left(y\widetilde{Z}_T^\nu\right); y\widetilde{Z}_T^\nu, F\right)\right]
$$

is called the dual problem of the primal problem (P), and the inequality

$$
\text{(3)} \qquad V(x) \leq \inf_{y>0}\left(\widehat{V}(y) + yx\right)
$$

is deduced from (2). Indeed, the equality can be established in (3) (i.e., there is no "duality-gap") and the following expression is obtained.

Theorem 1. *(Theorem 2.1 of Rouge and El Karoui, [7]) For $\mathcal{A} := \mathcal{A}_2$, it holds that*

$$
\text{(4)} \qquad V(x) = U_\gamma\left(e^{rT}x - \frac{1}{\gamma}\sup_{\nu \in \mathcal{D}}\left\{E^\nu[\gamma F] - H\left(P^\nu | P\right)\right\}\right),
$$

where $E^\nu[\cdot]$ denotes the expectation with respect to the probability measure P^ν on $(\Omega, \mathcal{F}_T)$ defined by

$$
\left.\frac{dP^\nu}{dP}\right|_{\mathcal{F}_t} := Z_t^\nu
$$

and

$$
H(Q|P) := \begin{cases} E\left[\frac{dQ}{dP}\log\frac{dQ}{dP}\right] & \text{if } Q \ll P, \\ +\infty & \text{otherwise} \end{cases}
$$

is the relative entropy of Q with respect to P.

Remark 1. The duality relations similar to (4) have been obtained for more general semimartingale S and for other choices of the set of admissible strategies $\mathcal{A}$ by Delbaen et. al. in [2]. Also, the work by Kabanov and Stricker (2002), [4], should be referred.

For the computations of the value $V(x)$ and the optimizer, one can solve the BSDE for the value process of the dual problem, described as follows.

Theorem 2. *(Theorem 4.1-2 of Rouge and El Karoui, [7]) Denote* $Z^\nu_{t,T} := Z^\nu_T / Z^\nu_t$, $\widetilde{Z}^\nu_{t,T} := \widetilde{Z}^\nu_T / \widetilde{Z}^\nu_t$, *and* $\tau := T - t$ *for* $0 \leq t \leq T$. *Let*

$$\begin{aligned} & \operatorname*{essinf}_{\nu \in \mathcal{D}} E\left[u_\gamma\left(F + I_\gamma\left(y\widetilde{Z}^\nu_{t,T}\right); y\widetilde{Z}^\nu_{t,T}, F\right) \mid \mathcal{F}_t\right] \\ = \quad & \frac{ye^{-r\tau}}{\gamma}\left\{-\operatorname*{esssup}_{\nu \in \mathcal{D}} E^\nu\left[\gamma F - \log Z^\nu_{t,T} | \mathcal{F}_t\right] + (1 + \log y - r\tau)\right\} \\ =: \quad & \frac{ye^{-r\tau}}{\gamma}\left\{-Y_t + (1 + \log y - r\tau)\right\}. \end{aligned}$$

There exists $\Xi \in \mathbf{H}^{2,d}_T := \left\{f \in \mathcal{L}^{2,d}_T;\ E\left[\int_0^T |f_t|^2 dt\right] < \infty\right\}$ *such that* (Y, Ξ) *satisfies*

$$\begin{aligned} (5) \qquad dY_t &= f(t, \Xi_t)\, dt + \Xi'_t dw_t, \quad Y_T = \gamma F, \\ \textit{where} \quad f(t,\xi) &:= \lambda'_t \Pi_{\sigma'_t C}(\xi + \lambda_t) - \frac{1}{2}\left|\xi - \Pi_{\sigma'_t C}(\xi + \lambda_t)\right|^2. \end{aligned}$$

and $\Pi_{\sigma'_t(\omega)C} : \mathbf{R}^d \ni x \mapsto \Pi_{\sigma'_t(\omega)C}\, x \in \sigma'_t(\omega)C (\subset \mathbf{R}^d)$ *is the projection operator onto the closed convex cone* $\sigma'_t(\omega)C$.

In particular, $\pi^* \in \mathcal{A}_2$ *satisfying*

$$(6) \qquad \sigma'_t \pi^*_t := \frac{e^{-rT}}{\gamma}\Pi_{\sigma'_t C}(\Xi_t + \lambda_t) \quad \textit{for all } t \in [0, T]$$

is an optimizer of the primal problem (P) with $\mathcal{A} := \mathcal{A}_2$, *and* $\nu^* := (I - \Pi_{\sigma' C})(\Xi + \lambda)$ *attains the infimum of the dual problem (D). Further,*

$$(7) \qquad V(x) = U_\gamma\left(e^{rT}x - \frac{Y_0}{\gamma}\right)$$

holds.

Remark 2. The existence and the uniqueness of the solution (Y, Ξ) of the quadratic BSDE (5) in the space $\mathbf{H}^\infty_T \times \mathbf{H}^{2,d}_T$, where $\mathbf{H}^\infty_T := \{f \in L^\infty([0,T] \times \Omega);\ \text{predictable}\}$ is ensured by the work of Kobylanski (2000), [6]. Further, utilizing the dynamic programming principle and the comparison theorems between linear BSDEs and between quadratic BSDEs in [6], the above theorem is established.

On the other hand, if the model has a Markovian structure, one can solve a dynamic programming equation to compute the value, which is suggested in Delbaen et al (2002), [2], and is employed and studied in Davis (2000), [1]. In particular, in [1], a special but a typical situation is addressed, which can be stated as follows in our setting.

(i) Let $d = n = 2$. σ is the following constant matrix

$$\sigma := \begin{pmatrix} \sigma_1 & 0 \\ \sigma_2\sqrt{1-\epsilon^2} & \sigma_2\epsilon \end{pmatrix} \tag{8}$$

with $\sigma_1, \sigma_2 > 0$, $\epsilon \in [-1, 1]$. $\mu = (\mu_1, \mu_2)'$ is also a constant vector. Further, $\epsilon \neq 0$, $\epsilon \ll 1$ is assumed, i.e., two assets S^1 and S^2 are *closely correlated*:

$$\frac{d\langle S^1, S^2\rangle}{\sqrt{d\langle S^1\rangle d\langle S^2\rangle}} = \sqrt{1-\epsilon^2} \approx 1.$$

(ii) $F := h(S^1_T)$ with continuous, piecewise linear $h : \mathbf{R}_+ \mapsto \mathbf{R}$ bounded from above.

(iii) The constraint of the asset-holding strategy π is given by $\pi_t \in C := \{0\} \times \mathbf{R}$: only S^2 is *tradable*, and the derivative security is written on the *untradable* asset S^1.

Recall, in the situation, that the expressions

$$\sigma' C = \{k d_\epsilon;\ k \in \mathbf{R}\}, \quad \mathcal{D} = \left\{\eta d_\epsilon^\perp;\ \eta \in \mathcal{L}_T^{2,1}, \text{bounded}\right\},$$

and

$$\lambda^\epsilon := \sigma^{-1}(\mu - r\mathbf{1}) = \frac{1}{\epsilon\sigma_1\sigma_2}\begin{pmatrix} \epsilon\sigma_2 & 0 \\ -\sigma_2\sqrt{1-\epsilon^2} & \sigma_1 \end{pmatrix}\begin{pmatrix} \mu_1 - r \\ \mu_2 - r \end{pmatrix}$$

hold, where we denote

$$d_\epsilon := \left(\sqrt{1-\epsilon^2}, \epsilon\right)' \quad \text{and} \quad d_\epsilon^\perp := \left(\epsilon, -\sqrt{1-\epsilon^2}\right)'.$$

The dual problem is now, rewritten as

$$\begin{aligned} &\inf_{\nu\in\mathcal{D}} E\left[\left(-y\widetilde{Z}^\nu_T\right)\left\{h\left(S^1_T\right) - \frac{1}{\gamma}\left(1 + \log y + \log \widetilde{Z}^\nu_T\right)\right\}\right] \\ = \ &\frac{ye^{-rT}}{\gamma}\left\{-\sup_{\nu\in\mathcal{D}} E^\nu\left[\gamma h\left(S^1_T\right) - \log Z^\nu_T\right] + (1 + \log y - rT)\right\}. \end{aligned}$$

Since

$$\log Z^\nu_T = -\int_0^T (\lambda^\epsilon - \nu_t)'\, dw^\nu_t + \frac{1}{2}\int_0^T |\lambda^\epsilon - \nu_t|^2\, dt,$$

where $w^\nu := (w^\nu_1, w^\nu_2)'$, $w^\nu_t := w_t + \int_0^t (\lambda^\epsilon - \nu_u) du$ is a 2-dimensional P^ν-Brownian motion, it is equivalent to solve the following:

$$\sup_{\nu\in\mathcal{D}} E^\nu\left[\gamma h\left(S^1_T\right) - \frac{1}{2}\int_0^T |\lambda^\epsilon - \nu_t|^2\, dt\right],$$

in which the process S^1 has the dynamics:

$$\begin{aligned} dS_t^1 &= S_t^1 \left[\sigma_1 dw_1^\nu(t) + \left\{ \mu_1 - \sigma_1(\overline{\lambda}^\epsilon - \nu_t^1) \right\} dt \right] \\ &= S_t^1 \left\{ \sigma_1 dw_1^\nu(t) + (r - \epsilon\sigma_1\eta_t)\, dt \right\}, \end{aligned}$$

where we denote $\nu := \eta d_\epsilon^\perp$ with some bounded predictable η. For the value function

$$v^\epsilon(t,y) := \operatorname*{esssup}_{\nu\in\mathcal{D}} E^\nu \left[\gamma h\left(S_T^1\right) - \frac{1}{2}\int_t^T |\lambda^\epsilon - \nu_t|^2\, dt \;\Big|\; S_t^1 = y \right],$$

a dynamic-programming equation is derived and the existence of its smooth solution is checked in the setting of [1]. Moreover, the following expressions are obtained.

Theorem 3. *(Theorem 6.1, 6.4 and 7.3 of Davis,* [1]*)*
1. *An optimal strategy of the problem (P) is given by*

$$\begin{aligned} \pi_t^* &= \frac{e^{-rT}}{\gamma}(\sigma')^{-1}\Pi_{\sigma' C}\left\{ \begin{pmatrix} -\partial_x v^\epsilon(t,S_t^1)S_t^1\sigma_1 \\ 0 \end{pmatrix} + \lambda_t^\epsilon \right\} \\ &= \begin{pmatrix} 0 \\ \frac{e^{-rT}}{\gamma}\left\{ \frac{\mu_2 - r}{\sigma_2^2} - \sqrt{1-\epsilon^2}\frac{\sigma_1}{\sigma_2}\partial_x v^\epsilon(t,S_t^1)S_t^1 \right\} \end{pmatrix}. \end{aligned}$$

2. *For the utility indifference price defined by (1),*

$$p(x,F) = \frac{e^{-rT}}{\gamma}\left\{ v^\epsilon(0,S_0^1) + \frac{T}{2}\left(\frac{\mu_2 - r}{\sigma_2}\right)^2 \right\}$$

holds for any $x \in \mathbf{R}$.
3. *As* $\epsilon \downarrow 0$, *the value function has the expansion*

$$\begin{aligned} v^\epsilon(t,y) &= \gamma E\left[h(A_T)|A_t = y\right] - \frac{T-t}{2}\left(\frac{\mu_2 - r}{\sigma_2}\right)^2 \\ &\quad + \epsilon^2\frac{\gamma^2}{2}\mathrm{Var}\left[h(A_T)|A_t = y\right] + O(\epsilon^4), \end{aligned} \tag{9}$$

where $\mathrm{Var}[*|\cdot] := E[(*)^2|\cdot] - (E[*|\cdot])^2$, $O(\epsilon^4)$ *depends on the value* (t,y), *and the process* A *is defined by*

$$dA_t = A_t\left[\sigma_1 dw_1(t) + \left\{ \mu_1 - \sqrt{1-\epsilon^2}\frac{\sigma_1(\mu_2 - r)}{\sigma_2} \right\} dt \right], \quad A_0 = S_0^1.$$

In particular, we are interested in the expansion (9). From a practical viewpoint, it is an effective and useful expansion: it gives nice approximations of the value of the problem (P) and the utility indifference price. By using the relation (7),

$$\log V^{\epsilon}(x) - \log U_{\gamma}\Bigg(e^{rT}x - \gamma E\left[h(A_T)\right] - \frac{T}{2}\left(\frac{\mu_2 - r}{\sigma_2}\right)^2 \\ -\epsilon^2\frac{\gamma^2}{2}\mathrm{Var}\left[h(A_T)\right]\Bigg) = O(\epsilon^4)$$

is observed, where we denote the value by $V^{\epsilon}(x)$ emphasizing ϵ, and

$$p(x,F) = e^{-rT}\left\{E\left[h(A_T)\right] + \epsilon^2\frac{\gamma}{2}\mathrm{Var}\left[h(A_T)\right]\right\} + O(\epsilon^4)$$

holds for any $x \in \mathbf{R}$. Also, both quantities $E\left[h(A_T)|A_t = y\right]$ and $\mathrm{Var}\left[h(A_T)|A_t = y\right]$ are fairly "computable". In [1], it is derived from a clever observation, however, the reason why the second term has $O(\epsilon^2)$ and the error term has $O(\epsilon^4)$ seems to be obscure. To see its intrinsic reason is one of our motivations.

Further, we are interested in the approximation of the optimal strategy (optimizer), which is not mentioned in [1]. It looks natural to deduce the strategy $\check{\pi} := \left(\check{\pi}^1, \check{\pi}^2\right)'$ defined by $\check{\pi}^1 \equiv 0$ and

$$\begin{aligned}\check{\pi}_t^2 \;:=\; & \frac{e^{-rT}}{\gamma}\Bigg[\frac{\mu_2 - r}{\sigma_2^2} - \sqrt{1-\epsilon^2}\frac{\sigma_1}{\sigma_2}S_t^1 \\ & \times\partial_y\left(\gamma E\left[h(A_T)|A_t = y\right] + \epsilon^2\frac{\gamma^2}{2}\mathrm{Var}\left[h(A_T)|A_t = y\right]\right)\Big|_{y=S_t^1}\Bigg]\end{aligned}$$

and expect the approximation such that

$$\log V^{\epsilon}(x) - \log E\left[U_{\gamma}\left(-F + X_T^{x,\check{\pi}}\right)\right] = O(\epsilon^4), \tag{10}$$

for example.

In the next section, using the BSDE in Theorem 2 and its asymptotic expansion with respect to ϵ (precisely saying, with respect to ϵ', cf., the BSDE (14)), we reconstruct the expansions (9-10), which yields an alternative approach to the above analysis. The main contribution of this paper is Theorem 4 in the next section, extensions of (9-10) under Assumption 1. It is also suggested that the i-th derivatives $\left(\partial_{\epsilon'}^i Y^{0,\epsilon}, \partial_{\epsilon'}^i \Xi^{0,\epsilon}\right) \equiv 0$ for odd numbers $i = 1, 3, 5, \cdots$ (cf., Remark 4).

§2. An approximated optimizer

In this section, the probability space is assumed to be the product of Wiener spaces: $(\Omega, \mathcal{F}, P) := \prod_{i=1}^{2}(\Omega_i, \mathcal{F}^i, P_i)$, where $\Omega_i := C_0([0,T], \mathbf{R})$, $\mathcal{F}^i := \mathcal{B}(\Omega_i)$ and P_i is the Wiener measure, the law of the i-th canonical Brownian motion $w^i := \left(w_t^i\right)_{t\in[0,T]}$. The filtration $(\mathcal{F}_t)_{t\in[0,T]} := (\mathcal{F}_t^1 \times \mathcal{F}_t^2)_{t\in[0,T]}$ is the augmented natural filtration. Sometimes a random variable X on $(\Omega_1, \mathcal{F}^1, P_1)$ is identified with $X \circ j_1$ on $(\Omega, \mathcal{F}, P)$, where $j_1 : \Omega \ni \omega := (\omega_1, \omega_2) \mapsto \omega_1 \in \Omega_1$ is the projection onto the first probability space.

We now impose the following conditions.

(i)' The volatility matrix of the process S is given by (8). On the other hand, $\mu = (\mu_1, \mu_2)'$ is a bounded $\mathcal{F}_t^1$-predictable process, i.e., $\mu : [0,T] \times \Omega_1 \ni (t, \omega_1) \mapsto \mu(t, \omega_1) \in \mathbf{R}^2$ is measurable with respect to the predictable σ-algebra on $[0,T] \times \Omega_1$.

(ii)' $F(\omega_1) = h(S^1(\omega_1))$ with a bounded measurable function h on $C([0,T], \mathbf{R}_+)$.

(iii) The constraint of the strategy π is given by $\pi_t \in C := \{0\} \times \mathbf{R}$.

Remark 3. The condition (i)' is considered as an extension of the constant μ case employed in (i) in the previous section, though Assumption 1.1 will be added later. On the other hand, the conditoin (ii)' does not include the conditon (ii) in the previous section.

Further, we consider the problem (P) over the extended space: $\mathcal{A} := \mathcal{A}_1$, where

$$\mathcal{A}_1 := \left\{ \pi \in \mathcal{L}_T^{2,2};\ \pi_t \in C \text{ for } \forall t \in [0,T], \right.$$
$$\left. E\left[\left(\int_0^T |\pi_t|^2 dt\right)^{q/2}\right] < \infty \text{ for } \exists q > 1 \right\}$$

and construct an approximated optimizer in $\mathcal{A}_1$, not in $\mathcal{A}_2$. We first remark the following.

Proposition 1. *Let π^* be the process defined by the formula (6) and by the solution $(Y, \Xi) \in \mathbf{H}_T^\infty \times \mathbf{H}_T^{2,2}$ of the BSDE (5). It is also an optimizer of the problem (P) with $\mathcal{A} := \mathcal{A}_1$.*

Proof. We first observe that $E\left[\widetilde{Z}_T^\nu X_T^{x,\pi}\right] \le x$ for all $(\pi, \nu) \in \mathcal{A}_1 \times \mathcal{D}$ and $x \in \mathbf{R}$. For the purpose, since

$$e^{-rt} X_t^{x,\pi} = x + \int_0^t \pi_u' \sigma_u dw_u^\nu$$

holds, to show the martingale property of the process $Z^\nu \int \pi' \sigma dw^\nu$ is sufficient, which can be verified by checking

$$\begin{aligned}
& E\left[\sup_{t\in[0,T]}\left|Z_t^\nu \int_0^t \pi_u'\sigma_u dw_t^\nu\right|\right] \\
\leq\ & E\left[\sup_{t\in[0,T]}\left|Z_t^\nu \int_0^t \pi_u'\sigma_u dw_t\right|\right] + E\left[\sup_{t\in[0,T]}\left|Z_t^\nu \int_0^t \pi_u'\sigma_u(\lambda_u - \nu_u)du\right|\right] \\
\leq\ & E\left[\sup_{t\in[0,T]}(Z_t^\nu)^p\right]^{1/p}\left\{E\left[\sup_{t\in[0,T]}\left|\int_0^t \pi_u'\sigma_u dw_u\right|^q\right]^{1/q}\right. \\
& \qquad \left. + E\left[\sup_{t\in[0,T]}\left|\int_0^t \pi_u'\sigma_u(\lambda_u - \nu_u)du\right|^q\right]^{1/q}\right\} \\
\leq\ & C_1 E\left[\langle Z^\nu\rangle_T^{p/2}\right]^{1/p} \\
& \times\left\{C_2 E\left[\left(\int_0^T |\pi_u|^2 du\right)^{q/2}\right]^{1/q} + C_3 E\left[\left(\int_0^T |\pi_u| du\right)^q\right]^{1/q}\right\} \\
\leq\ & C_4 E\left[\langle Z^\nu\rangle_T^{p/2}\right]^{1/p} E\left[\left(\int_0^T |\pi_u|^2 du\right)^{q/2}\right]^{1/q} < \infty
\end{aligned}$$

for $p, q > 1$ satisfying $1/p + 1/q = 1$ by using the Hölder inequality and the Burkholder-Davis-Gundy inequality. In particular, the inequalities (2) and

$$E\left[U_\gamma\left(-F + X_T^{x,\pi}\right)\right] - yx \leq \widehat{V}(y) \tag{11}$$

are deduced for any $\pi \in \mathcal{A}_1$ $x \in \mathbf{R}$ and $y > 0$.

Next, note that the pair (π^*, ν^*) defined by (6) and the formula $\nu_t^* := \left(I - \Pi_{\sigma_t' C}\right)\left(\Xi_t^\epsilon + \lambda_t^\epsilon\right)$ satisfies the relation

$$\left\{\begin{array}{l} F + I_\gamma\left(\mathcal{Y}(x)\widetilde{Z}_T^{\nu^*}\right) = X_T^{x,\pi^*} \\ \text{with } \ \mathcal{Y}(x) := \exp\left(Y_0 + rT - \gamma e^{rT} x\right). \end{array}\right. \tag{12}$$

In fact, the solution of the BSDE satisfies

$$\begin{aligned} F &= \frac{1}{\gamma}\left\{Y_0 + \int_0^T f(t,\Xi_t)\,dt + \int_0^T (\Xi_t)'\,dw_t\right\} \\ &= \frac{Y_0}{\gamma} + \int_0^T \left\{e^{rT}(\pi_t^*)'\sigma_t\lambda_t - \frac{1}{2\gamma}\left|\lambda_t - \nu_t^*\right|^2\right\} dt \\ &\quad + \int_0^T \left\{e^{rT}\sigma_t'\pi_t^* - \frac{1}{\gamma}(\lambda_t - \nu_t^*)\right\}' dw_t \\ &= \frac{Y_0}{\gamma} + X_T^{0,\pi^*} - I_\gamma\left(Z_T^{\nu^*}\right), \end{aligned}$$

which is equivalent to (12). Using (12) and Theorem 2, we observe that

$$\text{(13)} \qquad \begin{aligned} E\left[\widetilde{Z}_T^{\nu^*} X_T^{x,\pi^*}\right] &= E\left[\widetilde{Z}_T^{\nu^*}\left(F + I_\gamma\left(\mathcal{Y}(x)\widetilde{Z}_T^{\nu^*}\right)\right)\right] \\ &= \frac{e^{-rT}}{\gamma}\left\{Y_0 - \log\left(\mathcal{Y}(x)e^{-rT}\right)\right\} = x. \end{aligned}$$

Finally, replacing y by $\mathcal{Y}(x)$ in (11) and using Theorem 2 and (12-3), we deduce that

$$\begin{aligned} & E\left[U_\gamma\left(-F + X_T^{x,\pi}\right)\right] - \mathcal{Y}(x)x \\ \leq\; & E\left[u_\gamma\left(F + I_\gamma\left(\mathcal{Y}(x)\widetilde{Z}_T^{\nu^*}\right); \mathcal{Y}(x)\widetilde{Z}_T^{\nu^*}, F\right)\right] \\ =\; & E\left[U_\gamma\left(-F + X_T^{x,\pi^*}\right)\right] - \mathcal{Y}(x)x \end{aligned}$$

for all $\pi \in \mathcal{A}_1$, which implies the optimality of π^*. ∎

The BSDE (5) for the optimizer is now rewritten as, in the situation of this section,

$$\begin{aligned} dY_t^\epsilon &= f(t,\Xi_t^\epsilon,\epsilon)\,dt + (\Xi_t^\epsilon)'\,dw_t, \quad Y_T^\epsilon = \gamma F, \\ \text{where}\quad f(t,\xi,\epsilon) &:= \frac{1}{2}\left\{\overline{\lambda}_t^2 - \left(\xi, d_\epsilon^\perp\right)^2\right\} + \overline{\lambda}_t\,(\xi, d_\epsilon), \end{aligned}$$

$(\cdot,\cdot)$ denotes the standard inner-product in $\mathbf{R}^2$ and

$$\overline{\lambda} := (\lambda^\epsilon, d_\epsilon) = \frac{\mu_2 - r}{\sigma_2}.$$

Denote

$$\overline{w}_t^\epsilon = (\overline{w}_1^\epsilon(t), \overline{w}_2^\epsilon(t))' := w_t + \left(\int_0^t \overline{\lambda}_u du\right) d_\epsilon$$

to reexpress the solution $(Y^\epsilon, \Xi^\epsilon) = (Y^{\epsilon,\epsilon}, \Xi^{\epsilon,\epsilon})$ by using the BSDE:

$$
\begin{aligned}
(14) \qquad dY_t^{\epsilon',\epsilon} &= g\left(t, \Xi_t^{\epsilon',\epsilon}, \epsilon'\right) dt + \left(\Xi_t^{\epsilon',\epsilon}\right)' d\overline{w}_t^\epsilon, \quad Y_T^{\epsilon',\epsilon} = \gamma F, \\
\text{where} \quad g(t, \xi, \epsilon') &:= \frac{1}{2}\left\{\overline{\lambda}_t^2 - \left(\xi, d_{\epsilon'}^\perp\right)^2\right\}.
\end{aligned}
$$

We consider the asymptotic expansion of $\left(Y^{\epsilon',\epsilon}, \Xi^{\epsilon',\epsilon}\right)$ with respect to ϵ' at 0. Let $\left(\partial_{\epsilon'}^0 Y^{0,\epsilon}, \partial_{\epsilon'}^0 \Xi^{0,\epsilon}\right) := \left(Y^{0,\epsilon}, \Xi^{0,\epsilon}\right)$ and introduce the BSDEs:

$$
(15) \quad \begin{cases} d\left(\partial_{\epsilon'}^i Y_t^{0,\epsilon}\right) = g_i\left(t, \left(\partial_{\epsilon'}^j \Xi_t^{0,\epsilon}\right)_{j=0,\ldots,i}, 0\right) dt + \left(\partial_{\epsilon'}^i \Xi_t^{0,\epsilon}\right)' d\overline{w}_t^\epsilon, \\ \partial_{\epsilon'}^i Y_T^{0,\epsilon} = 0, \end{cases}
$$

using the functions g_i defined inductively

$$
\begin{aligned}
g_0\left(t, \xi^0, \epsilon'\right) &:= g\left(t, \xi^0, \epsilon'\right) \\
\text{and} \quad g_i\left(t, (\xi^j)_{j=0,\ldots,i}, \epsilon'\right) &:= \sum_{j=0}^{i-1} \left(\partial_{\xi^j} g_{i-1}\left(t, (\xi^k)_{k=0,\ldots,i-1}, \epsilon'\right), \xi^{j+1}\right) \\
&\quad + \partial_{\epsilon'} g_{i-1}\left(t, (\xi^k)_{k=0,\ldots,i-1}, \epsilon'\right).
\end{aligned}
$$

Formally, it is expected that $\left(\partial_{\epsilon'}^i Y^{0,\epsilon}, \partial_{\epsilon'}^i \Xi^{0,\epsilon}\right)$ is the i-th derivative of $\left(Y^{\epsilon',\epsilon}, \Xi^{\epsilon',\epsilon}\right)$ with respect to ϵ' at $\epsilon' = 0$, although we have not been able to show the property. The standard results on the differentiation of the solution of BSDE with respect to a parameter (cf., El Karoui et. al ; 1997, [3], for example) cannot be applied to our quadratic BSDE (14).

Define the probability measure $\overline{P}^\epsilon$ on $(\Omega, \mathcal{F}_T)$ by

$$
\begin{aligned}
\left.\frac{d\overline{P}^\epsilon}{dP}\right|_{\mathcal{F}_t} &:= \mathcal{E}_t\left(-\int \overline{\lambda} d_\epsilon' dw\right) =: \overline{Z}_t^\epsilon \\
&= \mathcal{E}_t\left(-\sqrt{1-\epsilon^2}\int \overline{\lambda} dw_1\right) \mathcal{E}_t\left(-\epsilon \int \overline{\lambda} dw_2\right) =: \overline{Z}_1^\epsilon(t) \overline{Z}_2^\epsilon(t)
\end{aligned}
$$

and the space $\mathbf{H}_T^{2,2,\epsilon} := \left\{f \in \mathcal{L}_T^{2,2};\ \int_0^T |f_t|^2 dt \in L^1(\overline{P}^\epsilon)\right\}$ to obtain the expressions for the solution of (15) for $i = 0, 1, 2, 3$, as follows.

Lemma 1. 1. *The solution* $(Y^{0,\epsilon}, \Xi^{0,\epsilon})$ *in the space* $\mathbf{H}_T^\infty \times \mathbf{H}_T^{2,2}$ *has the expressions:*

$$\begin{aligned} Y_t^{0,\epsilon} &= \overline{E}^\epsilon\left[\gamma F - \frac{1}{2}\int_t^T \overline{\lambda}_u^2 du \,\Big|\, \mathcal{F}_t\right], \\ Y_0^{0,\epsilon} + \int_0^t \Xi_1^{0,\epsilon}(u) d\overline{w}_1^\epsilon(u) &= \overline{E}^\epsilon\left[\gamma F - \frac{1}{2}\int_0^T \overline{\lambda}_u^2 du \,\Big|\, \mathcal{F}_t\right], \end{aligned}$$

and $\Xi_2^{0,\epsilon}(t) = 0$ *for all* $t \in [0,T]$, *where* $\overline{E}^\epsilon[\cdot]$ *denotes the expectation with respect to the probability measure* $\overline{P}^\epsilon$.
2. $\left(\partial_{\epsilon'}^i Y^{0,\epsilon}, \partial_{\epsilon'}^i \Xi^{0,\epsilon}\right) \equiv 0$ *for* $i = 1, 3$.
3. *A solution of (15) with* $i = 2$ *exists in* $\mathbf{H}_T^\infty \times \mathbf{H}_T^{2,2,\epsilon}$ *and is given by*

$$\begin{aligned} \partial_{\epsilon'}^2 Y_t^{0,\epsilon} &= \overline{E}^\epsilon\left[\int_t^T \left(\Xi_1^{0,\epsilon}(u)\right)^2 du \,\Big|\, \mathcal{F}_t\right] \\ &= \overline{\mathrm{Var}}^\epsilon\left[\gamma F - \frac{1}{2}\int_t^T \overline{\lambda}_u^2 du \,\Big|\, \mathcal{F}_t\right], \\ \partial_{\epsilon'}^2 Y_0^{0,\epsilon} + \int_0^t \partial_{\epsilon'}^2 \Xi_1^{0,\epsilon}(u) d\overline{w}_1^\epsilon(u) &= \overline{E}^\epsilon\left[\int_0^T \left(\Xi_1^{0,\epsilon}(u)\right)^2 du \,\Big|\, \mathcal{F}_t\right], \end{aligned}$$

and $\partial_{\epsilon'}^2 \Xi_2^{0,\epsilon}(t) = 0$ *for all* $t \in [0,T]$, *where we denote* $\overline{\mathrm{Var}}^\epsilon[\cdot|\mathcal{F}_t] := \overline{E}^\epsilon[(\cdot)^2|\mathcal{F}_t] - \left(\overline{E}^\epsilon[\cdot|\mathcal{F}_t]\right)^2$.

Proof. 1. Suppose $\Xi_2^{0,\epsilon} \equiv 0$, then

$$dY_t^{0,\epsilon} = \frac{1}{2}\overline{\lambda}_t^2 dt + \Xi_1^{0,\epsilon}(t) d\overline{w}_1^\epsilon(t), \quad Y_T^{0,\epsilon} = \gamma F$$

is observed. **1** is now a consequence of the standard result of linear BSDE (cf., El Karoui et. al, [3]) and the result on the uniqueness of the quadratic BSDE studied in Kobylanski (2000), [6].
2-3. Observe that

$$\begin{aligned} d_\epsilon^\perp &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \epsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\epsilon^2}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{\epsilon^3}{3!}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + O(\epsilon^4) \\ &=: d_0^\perp + \sum_{i=1}^3 \frac{\epsilon^i}{i!}\partial_{\epsilon'}^i d_0^\perp + O(\epsilon^4), \end{aligned}$$

where $O(\epsilon^4) \in \mathbf{R}^2$ is a vector with the norm $|O(\epsilon^4)| \sim \epsilon^4$.

(i) Noting that

$$g_1\left(t,(\xi^j)_{j=0,1},0\right) = -\left(\xi^0, d_0^\perp\right)\left\{\left(\xi^1, d_0^\perp\right) + \left(\xi^0, \partial_{\epsilon'} d_0^\perp\right)\right\}$$

and that $\Xi_2^0 \equiv 0$, we can deduce

$$d\left(\partial_{\epsilon'} Y_t^{0,\epsilon}\right) = \partial_{\epsilon'} \Xi_t^{0,\epsilon} d\overline{w}_t^\epsilon, \quad \partial_{\epsilon'} Y_T^{0,\epsilon} \equiv 0$$

and $\left(\partial_{\epsilon'} Y^{0,\epsilon}, \partial_{\epsilon'} \Xi^{0,\epsilon}\right) \equiv 0$.
(ii) Observing that

$$\begin{aligned} & g_2\left(t,(\xi^j)_{j=0,1,2},0\right) \\ = & -\left(\xi^1, d_0^\perp\right)\left\{\left(\xi^1, d_0^\perp\right) + \left(\xi^0, \partial_{\epsilon'} d_0^\perp\right)\right\} \\ & -\left(\xi^0, d_0^\perp\right)\left\{\left(\xi^2, d_0^\perp\right) + \left(\xi^1, \partial_{\epsilon'} d_0^\perp\right)\right\} \\ & -\left(\xi^0, \partial_{\epsilon'} d_0^\perp\right)\left\{\left(\xi^1, d_0^\perp\right) + \left(\xi^0, \partial_{\epsilon'} d_0^\perp\right)\right\} \\ & -\left(\xi^0, d_0^\perp\right)\left\{\left(\xi^1, \partial_{\epsilon'} d_0^\perp\right) + \left(\xi^0, \partial_{\epsilon'}^2 d_0^\perp\right)\right\}, \end{aligned}$$

we rewrite the BSDE for $\left(\partial_{\epsilon'}^2 Y^{0,\epsilon}, \partial_{\epsilon'}^2 \Xi^{0,\epsilon}\right)$ as

$$d\left(\partial_{\epsilon'}^2 Y_t^{0,\epsilon}\right) = -\left(\Xi_1^{0,\epsilon}(t)\right)^2 dt + \left(\partial_{\epsilon'}^2 \Xi_t^{0,\epsilon}\right)' d\overline{w}_t^\epsilon, \quad \partial_{\epsilon'}^2 Y_T^{0,\epsilon} \equiv 0$$

since $\Xi_2^{0,\epsilon} \equiv 0$ and $\partial_{\epsilon'} \Xi^{0,\epsilon} \equiv 0$. Define $\overline{P}^\epsilon$-martingales M, N by the formulas

$$M_t := \int_0^t \Xi_1^{0,\epsilon}(u) d\overline{w}_1^\epsilon(u) := -Y_0^{0,\epsilon} + \overline{E}^\epsilon\left[\gamma F - \frac{1}{2}\int_0^T \overline{\lambda}_u^2 du \;\middle|\; \mathcal{F}_t\right]$$

and $N_t := \overline{E}^\epsilon\left[\langle M\rangle_T | \mathcal{F}_t\right]$ for $t \in [0,T]$, respectively. Note that M is bounded and that N is $\overline{P}^\epsilon$-square integrable:

$$\overline{E}^\epsilon\left[N_t^2\right] \le \overline{E}^\epsilon\left[\langle M\rangle_t^2\right] = \overline{E}^\epsilon\left[\left(M_t^2 - M_0^2 - 2\int_0^t M_u dM_u\right)^2\right] < \infty.$$

The martingale representation theorem implies that $H_t := E[\overline{Z}_1^\epsilon(T) N_T | \mathcal{F}_t]$ $= N_0 + \int_0^t \phi_u dw_1(u)$ holds for all $t \in [0,T]$ and for some $\mathcal{F}_t^1$-predictable ϕ such that $\int_0^T \phi_u^2 du < \infty$. Therefore,

$$N_t - N_0 = \int_0^t d\left(\frac{\overline{Z}_1^\epsilon(u) N_u}{\overline{Z}_1^\epsilon(u)}\right) = \int_0^t \frac{\phi_u + \sqrt{1-\epsilon^2} H_u \overline{\lambda}_u}{\overline{Z}_1^\epsilon(u)} d\overline{w}_1^\epsilon(u)$$

is observed from the Itô-formula. The solution is now constructed by setting

$$\partial^2_{\epsilon'} Y^{0,\epsilon}_t := N_t - \int_0^t \left(\Xi^{0,\epsilon}_1(u)\right)^2 du, \quad \partial^2_{\epsilon'}\Xi^{0,\epsilon}_1(t) := \frac{\phi_t + \sqrt{1-\epsilon^2} H_t \overline{\lambda}_t}{\overline{Z}^{\epsilon}_1(t)}$$

and $\partial^2_{\epsilon'}\Xi^{0,\epsilon}_2 \equiv 0$.

(iii) For $\left(\xi^j\right)_{j=0,1,2,3}$ such that $\xi^0_2 = \xi^2_2 = 0$ and $\xi^1 = 0$, we can check that

$$g_3\left(t, (\xi^j)_{j=0,1,2,3}, 0\right) = 0,$$

so the equation

$$d\left(\partial^3_{\epsilon'} Y^{0,\epsilon}_t\right) = \partial^3_{\epsilon'}\Xi^{0,\epsilon}_t d\overline{w}^{\epsilon}_t, \quad \partial^3_{\epsilon'} Y^{0,\epsilon}_T \equiv 0$$

and $\left(\partial^3_{\epsilon'} Y^{0,\epsilon}, \partial^3_{\epsilon'}\Xi^{0,\epsilon}\right) \equiv 0$ are deduced. ∎

We are now in the position to state our last theorem, an extension of Theorem 3.3. We require the conditions:

Assumption 1. 1. *The process μ_2 is bounded and deterministic.* **2.** *There is a kernel ∂F, finite measures $\partial F(\omega_1, \cdot)$ on $\mathcal{B}([0,T])$ for each $\omega_1 \in \Omega_1$, satisfying*

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon}\{F(\omega_1 + \epsilon\phi) - F(\omega_1)\} = \int_0^T \phi(t)\partial F(\omega_1, dt)$$

for all $\phi \in C^1([0,T])$ and the Clark formula:

$$F = E[F] + \int_0^T E[\partial F(\cdot, (t,T])|\mathcal{F}_t] dw_1(t).$$

(For sufficient conditions on F and ∂F to ensure the formula, cf., Appendix E of [5], for example). Moreover, $\partial F(\cdot, (\cdot, T]) \in L^\infty(\Omega_1 \times [0,T])$ holds.

For $n = 1, 2, \ldots$, define

$$\overline{Y}^{\epsilon,n} := \sum_{i=0}^{n} \partial^{2i}_{\epsilon'} Y^{0,\epsilon}_t \frac{\epsilon^{2i}}{(2i)!} \quad \text{and} \quad \overline{\Xi}^{\epsilon,n} := \sum_{i=0}^{n} \partial^{2i}_{\epsilon'}\Xi^{0,\epsilon}_t \frac{\epsilon^{2i}}{(2i)!}$$

and introduce the approximated strategy $\overline{\pi}^{\epsilon,n} := (\overline{\pi}^{\epsilon,n}_t)_{t\in[0,T]}$ by the formula

$$(16)\ \overline{\pi}^{\epsilon,n}_t := \frac{e^{-rT}}{\gamma}(\sigma')^{-1}\Pi_{\sigma' C}\left(\overline{\Xi}^{\epsilon,n}_t + \lambda^{\epsilon}(t)\right) = \begin{pmatrix} 0 \\ \frac{e^{-rT}}{\gamma}\left\{\sigma_2^{-2}(\mu_2(t) - r) + \sqrt{1-\epsilon^2}\sigma_2^{-1}\overline{\Xi}^{\epsilon,n}_1(t)\right\} \end{pmatrix}.$$

Note that $\overline{\pi}^{\epsilon,1} \in \mathcal{A} := \mathcal{A}_1$ since $E\left[\int_0^T \left|\Xi_t^{0,\epsilon}\right|^2 dt\right] < \infty$ and since

$$\begin{aligned} & E\left[\left(\int_0^T \left|\partial^2_{\epsilon'}\Xi_t^{0,\epsilon}\right|^2 dt\right)^{q/2}\right] \\ = & \overline{E}^\epsilon\left[\left(\overline{Z}_T^\epsilon\right)^{-1}\left(\int_0^T \left|\partial^2_{\epsilon'}\Xi_t^{0,\epsilon}\right|^2 dt\right)^{q/2}\right] \\ \le & \overline{E}^\epsilon\left[\left(\overline{Z}_T^\epsilon\right)^{-\frac{2}{2-q}}\right]^{\frac{2-q}{2}} \overline{E}^\epsilon\left[\int_0^T \left|\partial^2_{\epsilon'}\Xi_t^{0,\epsilon}\right|^2 dt\right]^{q/2} < \infty \end{aligned}$$

for $0 < q < 2$. We obtain the following.

Theorem 4. *Under Assumption 1, the relations*

$$\begin{aligned} \left\|Y^\epsilon - \overline{Y}^{\epsilon,1}\right\|_{L^\infty([0,T]\times\Omega)} &= O(\epsilon^4) \\ \textit{and} \quad \log V^\epsilon(x) - \log E\left[U_\gamma\left(-F + X_T^{x,\overline{\pi}^{\epsilon,1}}\right)\right] &= O(\epsilon^4) \end{aligned}$$

hold as $\epsilon \downarrow 0$.

Proof. Denote $\Lambda^\epsilon(t) := (\int_0^t \overline{\lambda}_u du)d_\epsilon$ and define the Wiener functional and the kernel:

$$G(\omega_1) := \gamma F(\omega_1 - \Lambda_1^\epsilon) - \frac{1}{2}\int_0^T \overline{\lambda}_u^2 du, \quad \partial G(\omega_1, dt) := \gamma \partial F(\omega_1 - \Lambda_1^\epsilon, dt).$$

First, we observe the following:

$$\begin{aligned} \Xi_1^{0,\epsilon}(t,\omega_1) &= E[\partial G(\cdot,(t,T])|\mathcal{F}_t](\omega_1 + \Lambda_1^\epsilon), \\ \partial^2_{\epsilon'}\Xi_1^{0,\epsilon}(t,\omega_1) &= 2\left\{E\left[\partial G(\cdot,(t,T])G|\mathcal{F}_t\right]\right. \\ & \quad \left. - E\left[\partial G(\cdot,(t,T])|\mathcal{F}_t\right]E\left[G|\mathcal{F}_t\right]\right\}(\omega_1 + \Lambda_1^\epsilon). \end{aligned}$$

In fact, the first expression is a consequence from the Clark formula,

$$G(\omega_1) = E[G] + \int_0^T E[\partial G(\cdot,(t,T])|\mathcal{F}_t](\omega_1) dw_1(t,\omega_1),$$

the Cameron-Martin formula, $\overline{P}^\epsilon(\cdot) = P(\cdot + \Lambda^\epsilon)$, and the relation $w_t(\omega + \Lambda^\epsilon) = \overline{w}_t^\epsilon(\omega)$,

$$\begin{aligned} G(\omega_1 + \Lambda_1^\epsilon) &= \overline{E}^\epsilon\left[G(\omega_1 + \Lambda_1^\epsilon)\right] \\ & \quad + \int_0^T E[\partial G(\cdot,(t,T])|\mathcal{F}_t](\omega_1 + \Lambda_1^\epsilon) d\overline{w}_1^\epsilon(t,\omega_1). \end{aligned}$$

The second expression is deduced from the relation

$$\begin{aligned}
&\int_0^T \left(\Xi_1^{0,\epsilon}(t)\right)^2 dt \\
= &\left(\int_0^T \Xi_1^{0,\epsilon}(t) d\overline{w}_1^\epsilon(t)\right)^2 - 2\int_0^T \left(\int_0^t \Xi_1^{0,\epsilon}(u) d\overline{w}_1^\epsilon(u)\right) \Xi_1^{0,\epsilon}(t) d\overline{w}_1^\epsilon(t) \\
= &\left(G - \overline{E}^\epsilon[G]\right)^2 - 2\int_0^T \{E[G|\mathcal{F}_t](\cdot + \Lambda^\epsilon) - E[G]\} \, \Xi_1^{0,\epsilon}(t) d\overline{w}_1^\epsilon(t) \\
= &\, G^2 - \left(\overline{E}^\epsilon[G]\right)^2 - 2\int_0^T \left(E[G|\mathcal{F}_t] E[\partial G(\cdot, (t,T])|\mathcal{F}_t]\right)(\cdot + \Lambda_1^\epsilon) d\overline{w}_1^\epsilon(t),
\end{aligned}$$

the Clark formula, and the chain rule for differentiation. In particular, it holds that $\Xi_1^{0,\epsilon}, \partial_{\epsilon'}^2 \Xi_1^{0,\epsilon} \in \mathbf{H}_T^\infty$. Therefore, in the BSDE for $\left(\overline{Y}^{\epsilon,n}, \overline{\Xi}^{n,\epsilon}\right)$:

$$\text{(17)} \qquad \begin{aligned}
d\overline{Y}_t^{\epsilon,n} &= \left\{ g\left(t, \overline{\Xi}_t^{\epsilon,n}, \epsilon\right) + R_t^{\epsilon,n} \right\} dt + \overline{\Xi}_t^{\epsilon,n} d\overline{w}_t^\epsilon, \quad \overline{Y}_T^{\epsilon,n} = \gamma F, \\
\text{where } \; R_t^{\epsilon,n} &:= \sum_{i=0}^{2n+1} g_i \left(t, \left(\partial_{\epsilon'}^j \Xi_t^0\right)_{j=0,\ldots,i}, 0\right) \frac{\epsilon^i}{i!} - g\left(t, \overline{\Xi}_t^{\epsilon,n}, \epsilon\right),
\end{aligned}$$

$\|R^{\epsilon,1}\|_{L^\infty([0,T],\Omega)} = O(\epsilon^4)$ is satisfied because of the boundedness of λ^ϵ, $\partial_{\epsilon'}^i d_0^\perp$, and $\partial_{\epsilon'}^i \Xi^{0,\epsilon}$ $(i = 0, \ldots, 3)$.

Next, we introduce the linear BSDE for $(\Delta Y^{\epsilon,n}, \Delta\Xi^{\epsilon,n}) := \left(Y^\epsilon - \overline{Y}^{\epsilon,n}, \Xi^\epsilon - \overline{\Xi}^{\epsilon,n}\right)$, described as

$$\begin{cases}
d\Delta Y_t^{\epsilon,n} = \left\{ -\frac{1}{2}\left(\Xi_t^\epsilon + \overline{\Xi}_t^{\epsilon,n}, d_\epsilon^\perp\right)\left(\Delta\Xi_t^{\epsilon,n}, d_\epsilon^\perp\right) - R_t^{\epsilon,n} \right\} dt + \Delta\Xi_t^{\epsilon,n} d\overline{w}_t^\epsilon, \\
\Delta Y_T^{\epsilon,n} \equiv 0
\end{cases}$$

to observe the expression:

$$\text{(18)} \qquad -\Gamma_s \Delta Y_s^{\epsilon,n} = -\Gamma_t \Delta Y_t^{\epsilon,n} - \int_s^t \Gamma_u R_u^{\epsilon,n} du + M_t - M_s$$

for $0 \le s \le t \le T$, where $\Gamma := (\Gamma_t)_{t\in[0,T]}$ is the solution of the SDE:

$$d\Gamma_t = \Gamma_t \left\{ \frac{1}{2}\left(\Xi_t^\epsilon + \overline{\Xi}_t^{\epsilon,n}, d_\epsilon^\perp\right)\left(d_\epsilon^\perp\right)' d\overline{w}_t^\epsilon \right\}, \quad \Gamma_0 = 1$$

and $M := (M_t)_{t\in[0,T]}$ is the $\overline{P}^\epsilon$-local-martingale defined by

$$M_t := \int_0^t \Gamma_u \left\{ \Delta\Xi_u^{\epsilon,n} + \frac{1}{2}\Delta Y_u^{\epsilon,n}\left(\Xi_u^\epsilon + \overline{\Xi}_u^{\epsilon,n}, d_\epsilon^\perp\right) d_\epsilon^\perp \right\}' d\overline{w}_u^\epsilon.$$

Let $n = 1$. For a sequence of increasing stopping times $(\tau_m)_{m\in\mathbf{N}}$, which localizes the local martingale M, we deduce the relation

$$\Gamma_{t\wedge\tau_m}\left|\Delta Y^{\epsilon,1}_{t\wedge\tau_m}\right| \le \overline{E}^{\epsilon}\left[\Gamma_{T\wedge\tau_m}\left|\Delta Y^{\epsilon,1}_{T\wedge\tau_m}\right| + \epsilon^4 C_1\int_{t\wedge\tau_m}^{T\wedge\tau_m}\Gamma_u du \;\Big|\; \mathcal{F}_{t\wedge\tau_m}\right]$$

with some constant $C_1 > 0$ from (18). The first term of the right-hand-side is

$$\le \overline{E}^{\epsilon}\left[\Gamma_{T\wedge\tau_m}\mid\mathcal{F}_{t\wedge\tau_m}\right]\left\|\Delta Y^{\epsilon,1}_{T\wedge\tau_m}\right\|_{L^\infty(\Omega)} \le \Gamma_{t\wedge\tau_m}\left\|\Delta Y^{\epsilon,1}_{T\wedge\tau_m}\right\|_{L^\infty(\Omega)} \to 0$$

as $m\to\infty$ by using the optional stopping theorem, and the second term of the right-hand-side is

$$= \epsilon^4 C_1\overline{E}^{\epsilon}\left[\int_{t\wedge\tau_m}^{T\wedge\tau_m}\Gamma_u du \;\Big|\; \mathcal{F}_{t\wedge\tau_m}\right] \to \epsilon^4 C_1\overline{E}^{\epsilon}\left[\int_t^T\Gamma_u du \;\Big|\; \mathcal{F}_t\right] \le \epsilon^4 C_1 T\Gamma_t$$

as $m\to\infty$ for a continuous version of $\overline{E}^{\epsilon}[\int_\cdot^T\Gamma_u du|\mathcal{F}_\cdot]$ by using the monotone convergence theorem. Therefore, $\|\Delta Y^{\epsilon,1}\|_{L^\infty([0,T]\times\Omega)} = O(\epsilon^4)$ follows.

Finally, define the process $\overline{\nu}^{\epsilon,n} := (\overline{\nu}^{\epsilon,n})_{t\in[0,T]}$ by

$$\overline{\nu}^{\epsilon,n}_t := (I - \Pi_{\sigma' C})\left(\overline{\Xi}^{\epsilon,n}_t + \lambda^{\epsilon}_t\right) \tag{19}$$

to deduce

$$\begin{aligned}\gamma F &= \overline{Y}^{\epsilon,n}_0 + \int_0^T\left(e^{rT}\gamma\sigma'\overline{\pi}^{\epsilon,n}_t - \lambda^{\epsilon}_t + \overline{\nu}^{\epsilon,n}_t\right)' d\overline{w}^{\epsilon}_t,\\ &\quad + \int_0^T\left(\frac{|\lambda^{\epsilon}_t|^2 - |\overline{\nu}^{\epsilon,n}_t|^2}{2} + R^{\epsilon,n}_t\right)dt\end{aligned}$$

from (16-7) and (19). Therefore, for $x\in\mathbf{R}$, we obtain that

$$\begin{aligned}F + I_\gamma\left(\overline{\mathcal{Y}}^{\epsilon,n}(x)Z^{\overline{\nu}^{\epsilon,n}}_T\right) &= X^{x,\overline{\pi}^{\epsilon,n}}_T + \int_0^T R^{\epsilon,n}_t dt,\\ \text{where}\quad \overline{\mathcal{Y}}^{\epsilon,n}(x) &= \exp\left(\overline{Y}^{\epsilon,n}_0 - \gamma e^{rT}x\right),\end{aligned}$$

which implies

$$\begin{aligned}
& \log E\left[U_\gamma\left(-F+X_T^{x,\overline{\pi}^{\epsilon,1}}\right)\right] \\
= & \log E\left[U_\gamma\left(I_\gamma\left(\overline{\mathcal{Y}}^{\epsilon,1}(x)Z_T^{\overline{\nu}^{\epsilon,1}}\right)-\int_0^T R_t^{\epsilon,1}dt\right)\right] \\
= & -\frac{1}{\gamma}\overline{\mathcal{Y}}^{\epsilon,1}(x)+O(\epsilon^4) \\
= & \log U_\gamma\left(e^{rT}x-\frac{\overline{Y}_0^{\epsilon,1}}{\gamma}\right)+O(\epsilon^4) \\
= & \log U_\gamma\left(e^{rT}x-\frac{Y_0^{\epsilon}}{\gamma}\right)+O(\epsilon^4) \quad \text{as } \epsilon \downarrow 0. \blacksquare
\end{aligned}$$

Remark 4. For the higher order terms, the following is observed, for example:

$$\begin{aligned}
\partial_\epsilon^4 Y_t^{0,\epsilon} &= -12\overline{E}^\epsilon\left[\int_t^T \Xi_1^{0,\epsilon}(u)\partial_{\epsilon'}^2\Xi_1^{0,\epsilon}(u)du \,\middle|\, \mathcal{F}_t\right], \\
\partial_{\epsilon'}^4 Y_0^{0,\epsilon} &- \int_0^t \partial_{\epsilon'}^4\Xi_1^{0,\epsilon}(u)d\overline{w}_1^\epsilon(u) \\
&= -12\overline{E}^\epsilon\left[\int_0^T \Xi_1^{0,\epsilon}(u)\partial_{\epsilon'}^2\Xi_1^{0,\epsilon}(u)du \,\middle|\, \mathcal{F}_t\right],
\end{aligned}$$

$\partial_{\epsilon'}^4\Xi_2^{0,\epsilon}(t) = 0$, and $\partial_{\epsilon'}^5\Xi^{0,\epsilon}(t) = 0$ for all $t \in [0,T]$. So, if we assume $\partial_{\epsilon'}^4\Xi^{0,\epsilon} \in \mathbf{H}_T^\infty$, then

$$\begin{aligned}
\left\|Y^\epsilon - \overline{Y}^{\epsilon,2}\right\|_{L^\infty([0,T]\times\Omega)} &= O(\epsilon^6) \\
\text{and} \quad \log V^\epsilon(x) - \log E\left[U_\gamma\left(-F+X_T^{x,\overline{\pi}^{\epsilon,2}}\right)\right] &= O(\epsilon^6)
\end{aligned}$$

are deduced as $\epsilon \downarrow 0$.

Example: European put option case. Let μ_1, μ_2 be constant and set $F = (k - S_T^1)^+$ $(k > 0)$. Then, Assumption 1 is satisfied, and we have that

$$\begin{aligned}
Y_t^{0,\epsilon} &= \gamma\left\{k\Phi(-c_t^-) - e^{\eta^\epsilon(T-t)}S_t^1\Phi(-c_t^+)\right\} - \frac{\overline{\lambda}^2(T-t)}{2}, \\
\partial_{\epsilon'}^2 Y_t^{0,\epsilon} &= \gamma^2\left\{k^2\Phi(-c_t^-) - 2ke^{\eta^\epsilon(T-t)}S_t^1\Phi(-c_t^+)\right. \\
&\qquad \left. + e^{(2\eta^\epsilon+\sigma_1^2)(T-t)}(S_t^1)^2\Phi(-c_t^{++})\right\}, \\
\Xi_1^{0,\epsilon}(t) &= -\gamma\sigma_1 e^{\eta^\epsilon(T-t)}\Phi(-c_t^+)S_t^1, \\
\partial_{\epsilon'}^2 \Xi_1^{0,\epsilon}(t) &= 2\gamma^2\sigma_1\left\{-ke^{\eta^\epsilon(T-t)}S_t^1\Phi(-c_t^+)\right. \\
&\qquad \left. + e^{(2\eta^\epsilon+\sigma_1^2)(T-t)}(S_t^1)^2\Phi(-c_t^{++})\right\} \\
&\quad + 2\gamma\sigma_1 e^{\eta^\epsilon(T-t)}\Phi(-c_t^+)S_t^1 \\
&\quad \times\left[\gamma\left\{k\Phi(-c_t^-) - e^{\eta^\epsilon(T-t)}S_t^1\Phi(-c_t^+)\right\} - \frac{\overline{\lambda}^2(T-t)}{2}\right],
\end{aligned}$$

where

$$\begin{aligned}
\eta^\epsilon &:= \mu_1 - \sqrt{1-\epsilon^2}\sigma_1\sigma_2^{-1}(\mu_2 - r), \\
\Phi(d) &:= \int_{-\infty}^d \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx, \\
c_t^- &:= \frac{1}{\sigma_1\sqrt{T-t}}\left\{\log\left(\frac{S_t^1}{k}\right) + \left(\eta^\epsilon - \frac{\sigma_1^2}{2}\right)(T-t)\right\}, \\
c_t^+ &:= c_t^- + \sigma_1\sqrt{T-t}, \quad \text{and} \quad c_t^{++} := c_t^- + 2\sigma_1\sqrt{T-t}.
\end{aligned}$$

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