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Least-Squares Approximation of Random Variables by Stochastic Integrals

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Dedicated to Professor K. Itô on the occasion of his 88th Birthday

Abstract.

This paper addresses the problem of approximating random variables in terms of sums consisting of a real constant and of a stochastic integral with respect to a given semimartingale X. The criterion is minimization of $\mathbf{L}^2$−distance, or "least-squares". This problem has a straightforward and well-known solution when X is a Brownian motion or, more generally, a square-integrable martingale, with respect to the underlying probability measure P. We address the general, semimartingale case by means of a *duality approach*; the adjoint variables in this duality are *signed measures*, absolutely continuous with respect to P, under which X behaves like a martingale. It is shown that this duality is useful, in that the value of an appropriately formulated dual problem can be computed fairly easily; that it "has no gap" (i.e., the values of the primal and dual problems coincide); that the signed measure which is optimal for the dual problem can be easily identified whenever it exists; and that the duality is also "strong", in the sense that one can then identify the optimal stochastic integral for the primal problem. In so doing, the theory presented here both simplifies and extends the extant work on the subject. It has also natural connections and interpretations in terms of the theory of "variance-optimal" and "mean-variance efficient" portfolios in Mathematical Finance, pioneered by H. Markowitz and then greatly extended by H. Föllmer, D. Sondermann and most notably M. Schweizer.

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§1. Introduction

Suppose we are given a square-integrable, d–dimensional process $X = \{X(t);\ 0 \le t \le T\}$ defined on the finite time-horizon $[0, T]$, which is a semimartingale on the filtered probability space $(\Omega, \mathcal{F}, P)$, $\mathbf{F} = \{\mathcal{F}(t)\}_{0 \le t \le T}$. *How closely can we approximate in the sense of least-squares a given, square-integrable and $\mathcal{F}(T)$–measurable random variable H, by a linear combination of the form $c + \int_0^T \vartheta' dX$?* Here c is a real number and ϑ a predictable d–dimensional process for which the stochastic integral $\int_0^\cdot \vartheta' dX \equiv \sum_{i=1}^d \int_0^\cdot \vartheta_i dX_i$ is well-defined and is itself a square-integrable semimartingale.

In other words, if we denote by Θ the space of all such processes ϑ, how do we compute

$$(1.1) \qquad V(c) \triangleq \inf_{\vartheta \in \Theta} E\left(H - c - \int_0^T \vartheta' dX\right)^2$$

if $c \in \mathbb{R}$ is given and we have the freedom to choose ϑ over the class Θ as above? How do we find

$$(1.2) \qquad V \triangleq \inf_{(c,\vartheta) \in \mathbb{R} \times \Theta} E\left(H - c - \int_0^T \vartheta' dX\right)^2 = \inf_{c \in \mathbb{R}} V(c)$$

when we have the freedom to select both c and ϑ? And how do we characterize, or even compute, the process $\vartheta^{(c)}$ and the pair $(\hat{c}, \hat{\vartheta})$ that attain the infimum in (1.1) and (1.2), respectively, whenever these exist? To go one step further: How does one

$$(1.3) \qquad \text{minimize the variance} \quad \mathrm{Var}\left(H - \int_0^T \vartheta' dX\right)$$

over all $\vartheta \in \Theta$ as above? Or even more interestingly, how does one

$$(1.4) \qquad \left\{ \begin{array}{c} \text{minimize the variance } \mathrm{Var}\left(H - \int_0^T \vartheta' dX\right) \\ \text{over } \ \vartheta \in \Theta \ \text{ with } \ E\left[\int_0^T \vartheta' dX\right] = \mu \end{array} \right\}$$

for some given $\mu \in \mathbb{R}$?

Questions such as (1.3) and (1.4) can be traced back to the pioneering work of H. Markowitz (1952, 1959), and have been studied more recently by Föllmer & Sondermann (1986), Föllmer & Schweizer

(1991), Duffie & Richardson (1991), Schäl (1992), in the modern context of Mathematical Finance. Most importantly, problems (1.1)-(1.4) have received an exhaustive and magisterial treatment in a series of papers by Schweizer (1992, 1994, 1995.a,b, 1996) and his collaborators (cf. Rheinländer & Schweizer (1997), [Ph.R.S.] (1998), [DMSSS] (1997), as well as Hipp (1993), [G.L.Ph.] (1996), Laurent & Pham (1999), Grandits (1999), Arai (2002)). In this context, the components $X_i(\cdot)$, $i = 1, \ldots, d$ of the semimartingale X are interpreted as the (discounted) stock-prices in a financial market, and H as a contingent claim, or liability, that one is trying to replicate as faithfully as possible at time T, starting with initial capital c and trading in this market. Such trading is modelled by the predictable *portfolio process* ϑ, whose component $\vartheta_i(t)$ represents the number of shares being held at time t in the i^{th} stock, for $i = 1, \ldots, d$. Then $\int_0^T \vartheta' dX \equiv \sum_{i=1}^d \int_0^T \vartheta_i(s) dX_i(s)$ corresponds to the (discounted) *gains from trading* accrued by the terminal time T, with which one tries to approximate the contingent claim H, and one might be interested in minimizing the variance of this approximation over all admissible portfolio choices (problem of (1.3)), or just over those portfolios that guarantee a given mean-rate-of-return (problem of (1.4)).

It turns out that solving the problem of (1.1) provides the key to answering all these questions. For instance, if $\vartheta^{(c)}$ attains the infimum in (1.1) and $\hat{c} \equiv \text{argmin}_{c \in \mathbb{R}} V(c)$, then $\hat{\vartheta} \equiv \vartheta^{(\hat{c})}$ is optimal for the problem of (1.3); the pair $(\hat{c}, \hat{\vartheta})$ is optimal for the problem of (1.2); and the process $\vartheta^{(c_\mu)}$ with $c_\mu = \big(E[\pi(H)] - E(H) + \mu\big) \,/\, \big(E[\pi(1)] - 1\big)$ is optimal for the problem of (1.4). Here π denotes the projection operator from the Hilbert space $\mathbf{L}^2(P)$ onto the orthogonal complement of its linear subspace $\left\{\int_0^T \vartheta' dX \,/\, \vartheta \in \Theta\right\}$.

The problem of (1.1) has a very simple solution, if X is a (square-integrable) *martingale*; then every H as above has the so-called Kunita-Watanabe decomposition

$$H = E(H) + \int_0^T (\zeta^H)' dX + L^H(T),$$

where $\zeta^H \in \Theta$ and $L^H(\cdot)$ is a square-integrable martingale strongly orthogonal to $\int_0^{\cdot} \vartheta' dX$ for every $\vartheta \in \Theta$. Then the infimum in (1.1) is computed as $V(c) = (E[H] - c)^2 + E\left(L^H(T)\right)^2$ and is attained by $\zeta^H \in \Theta$, which also attains the infimum $V = E\left(L^H(T)\right)^2$ in (1.2).

In order to deal with a general semimartingale X we develop a simple *duality approach*, which in a sense tries to reduce the problem to the "easy" martingale case just described. This approach is the main contribution of the present paper. The dual or "adjoint" variables in this duality are signed measures Q, absolutely continuous with respect to P and with $dQ/dP \in \mathbf{L}^2(P)$, under which X behaves like a martingale (Definition 2.1 and Remark 2.2). A simple observation, described in (3.1)-(3.7), leads to a dual *maximization problem.* The resulting duality is useful because, as it turns out, the dual problem is relatively straightforward to solve (Proposition 3.1); its value is easily computed as $E[\pi^2(H-c)]$ and coincides with the value $V(c)$ of the original problem (1.1), so *there is no "duality gap"*; and furthermore the duality is "strong", in that one can identify the optimal integrand $\vartheta^{(c)}$ of (1.1) rather easily, under suitable conditions (Theorem 4.1 and Remark 4.1). Several examples are presented in Section 5.

We follow closely the notation and the setting of Schweizer (1996), our great debt to which should be clear to anyone familiar with this excellent work. Indeed, the present paper can be considered as complementing and extending the results of this work, by means of our simple duality approach.

§2. The Problem

On a given complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbf{F} = \{\mathcal{F}(t)\,;\, 0 \le t \le T\}$ that satisfies the usual conditions, consider a process

$$X(t) = X(0) + M(t) + B(t), \qquad 0 \le t \le T, \tag{2.1}$$

defined on the finite time-horizon $[0,T]$ and belonging to the space $\mathcal{S}^2 \equiv \mathcal{S}^2(P)$ of square-integrable d-dimensional semimartingales. This means that each $X_i(0)$ is in $\mathbf{L}^2(\Omega, \mathcal{F}(0), P)$; that each $M_i(\cdot)$ belongs to the space $\mathcal{M}_0^2(P)$ of square-integrable $\mathbf{F}$-martingales with $M_i(0) = 0$ and RCLL paths; and that we have $B_i(\cdot) = A_i^+(\cdot) - A_i^-(\cdot)$, where $A_i^{\pm}(\cdot)$ are increasing, right-continuous and predictable processes with $A_i^{\pm}(0) = 0$ and $E\left(A_i^{\pm}(T)\right)^2 < \infty$, for every $i = 1, \cdots, d$. We denote by Θ the space of "good integrands" for the square-integrable semimartingale $X = \{X(t), 0 \le t \le T\}$, namely, those $\mathbf{F}-$predictable processes whose stochastic integrals with respect to X are themselves square-integrable semimartingales:

$$\Theta \triangleq \left\{ \vartheta \in \mathcal{L}(X) \;/\; G_{\cdot}(\vartheta) \equiv \int_0^{\cdot} \vartheta' dX \;\in\; \mathcal{S}^2(P) \right\}. \tag{2.2}$$

Here $\mathcal{L}(X)$ stands for the space of all $\mathbb{R}^d$-valued and predictable processes, whose stochastic integrals

$$(2.3)\qquad G_t(\vartheta) \triangleq \int_0^t \vartheta'(s)\,dX(s) = \sum_{i=1}^{d} \int_0^t \vartheta_i(s)\,dX_i(s), \quad 0 \le t \le T$$

with respect to X are well-defined.

Suppose now we are given a random variable H in the space $\mathbf{L}^2(P) \equiv \mathbf{L}^2(\Omega, \mathcal{F}(T), P)$. The following problem will occupy us in this paper.

Problem 2.1. *Given $H \in \mathbf{L}^2(P)$, compute*

$$(2.4)\qquad V \triangleq \inf_{(c,\vartheta)\in\mathbb{R}\times\Theta} E\big(H - c - G_T(\vartheta)\big)^2$$

and try to find a pair $(\hat{c}, \hat{\vartheta}) \in \mathbb{R} \times \Theta$ that attains the infimum, if such a pair exists. □

In other words, we are looking to find the *least-squares approximation* of H, as the sum of a constant $c \in \mathbb{R}$ and of the stochastic integral $G_T(\vartheta)$, for some process $\vartheta \in \Theta$.

This problem has a rather obvious solution, if it is known that the random variable H is of the form

$$(2.5)\qquad H = h + G_T(\zeta^H)$$

for some $h \in \mathbb{R}$ and $\zeta^H \in \Theta$; because then we can take $\hat{c} \equiv h$, $\hat{\vartheta} \equiv \zeta^H$, and deduce that $V = 0$ in (2.4). Now it is a classical result (e.g. Karatzas & Shreve (1991), pp. 181-185 for a proof) that *every* $H \in \mathbf{L}^2(P)$ can be written in the form (2.5), in fact with $h = E(H)$, *if* $X(\cdot)$ *is Brownian motion* and if $\mathbf{F}$ is the (augmentation of the) filtration $\mathbf{F}^X$ generated by $X(\cdot)$ itself. One can then also describe the integrand ζ^H in terms of the famous Clark (1970) formula, under suitable conditions on the random variable $H \in \mathbf{L}^2(P) \equiv \mathbf{L}^2(\Omega, \mathcal{F}^X(T), P)$. Thus, in this special case, we can take $\hat{c} = E(H)$, $\hat{\vartheta} = \zeta^H$, and have $V = 0$ in (2.4).

A little more generally, suppose that $X(\cdot) \in \mathcal{M}_0^2(P)$ is a square-integrable *martingale* (i.e., $X(0) = 0$ and $A(\cdot) \equiv 0$ in (2.1)). Then again it is well-known that every $H \in \mathbf{L}^2(P)$ admits the so-called *Kunita-Watanabe (1967) decomposition*

$$(2.6)\qquad H = h + G_T(\zeta^H) + L^H(T)$$

for $h = E(H)$, some $\zeta^H \in \Theta$, and some square-integrable martingale $L^H(\cdot) \in \mathcal{M}_0^2(P)$ which is strongly orthogonal to $G_{.}(\vartheta)$ for every $\vartheta \in \Theta$;

in particular,

$$(2.7) \qquad E\left[L^H(T)\cdot G_T(\vartheta)\right] \;=\; 0, \qquad \forall\, \vartheta \in \Theta.$$

Then

$$E\,(H-c-G_T(\vartheta))^2 \;=\; (h-c)^2 + E\left(G_T(\zeta^H-\vartheta)\right)^2 + E\left(L^H(T)\right)^2,$$

and it is clear that Problem 2.1 admits again the solution $\hat{c} = E(H)$, $\hat{\vartheta} = \zeta^H$, but now with $V = E\left(L^H(T)\right)^2$ in (2.4).

What happens for a *general, square-integrable semimartingale* $X(\cdot) \in \mathcal{S}^2(P)$? In view of the above discussion it is tempting to try and "reduce" this general problem to the case where $X(\cdot)$ is a martingale. This can be accomplished by absolutely continuous change of the probability measure P. We formalize this idea as in Schweizer (1996).

Definition 2.1. A signed measure Q on $(\Omega,\mathcal{F})$ is called a *signed* Θ-*martingale measure*, if $Q(\Omega)=1$, $Q \ll P$ with $(dQ/dP) \in \mathbf{L}^2(P)$, and

$$(2.8) \qquad E\left[\,\frac{dQ}{dP}\cdot G_T(\vartheta)\right] \;=\; 0, \qquad \forall\, \vartheta \in \Theta.$$

We shall denote by $\mathbf{P}_s(\Theta)$ the set of all such signed Θ-martingale measures, and introduce the closed, convex set

$$(2.9) \quad \mathcal{D} \triangleq \{D \in \mathbf{L}^2(P)\,/\,D = (dQ/dP), \text{ some } Q \in \mathbf{P}_s(\Theta)\}$$
$$= \{D \in \mathbf{L}^2(P)\,/\,E(D)=1 \text{ and } E(DG_T(\vartheta)) = 0, \quad \forall\, \vartheta \in \Theta\}.$$

□

We shall assume from now onwards, that

$$(2.10) \qquad \mathbf{P}_s(\Theta) \;\neq\; \emptyset \qquad (\text{equivalently, } \mathcal{D} \neq \emptyset).$$

Remark 2.1 : The linear subspace $G_T(\Theta) \triangleq \{\int_0^T \vartheta'(s)\,dX(s)\;/\;\vartheta \in \Theta\}$ of the Hilbert space $\mathbf{L}^2(P)$ is not necessarily closed for a general semimartingale $X(\cdot)$ (it is, if $B(\cdot) \equiv 0$ in (2.1), or equivalently if $X(\cdot)$ is a square-integrable martingale, since then the stochastic-integral of (2.3) is an isometry). For a semimartingale $X(\cdot)$ with continuous paths, necessary and sufficient conditions for the closedness of $G_T(\Theta)$ have been provided by Delbaen, Monat, Schachermayer, Schweizer & Stricker [DMSSS] (1997); see also Theorem 2 in Rheinländer & Schweizer (1997), as well as Corollary 4 in Pham, Rheinländer & Schweizer [Ph.R.S.] (1998) and section 5, equation (5.11) of the present paper, for sufficient conditions. As noted by W. Schachermayer (see Schweizer (1996), p. 210;

Lemma 4.1 in Schweizer (2001)), the assumption (2.10) is equivalent to the requirement that

$$(2.10)' \qquad \left\{ \begin{array}{c} \text{the closure in } \mathbf{L}^2(P) \text{ of } G_T(\Theta) \\ \text{does not contain the constant } 1 \end{array} \right\}.$$

On the other hand, the orthogonal complement

$$(2.11) \qquad (G_T(\Theta))^{\perp} \triangleq \{D \in \mathbf{L}^2(P) \ / \ E(DG_T(\vartheta)) = 0, \ \forall \, \vartheta \in \Theta\}$$

of $G_T(\Theta)$ is a *closed* linear subspace of $\mathbf{L}^2(P)$, and its orthogonal complement $(G_T(\Theta))^{\perp\perp}$ is the smallest closed, linear subspace of $\mathbf{L}^2(P)$ that contains $G_T(\Theta)$. Clearly, $(G_T(\Theta))^{\perp}$ includes the set $\mathcal{D}$ of (2.9), and the requirement (2.10)′ amounts to

$$(2.10)'' \qquad 1 \notin (G_T(\Theta))^{\perp\perp}.$$

Remark 2.2 : The notion of signed Θ-martingale measure in Definition 2.1 depends on the space Θ itself, as well as on the definition of the stochastic integral $G_T(\vartheta)$, $\vartheta \in \Theta$. In many cases of interest, though, every $Q \in \mathbf{P}_s(\Theta)$ belongs also to the space $\mathbf{P}_s^2(X)$ of signed *martingale measures for* X, namely those signed measures Q on $(\Omega, \mathcal{F})$ with $Q \ll P$, $dQ/dP \in \mathbf{L}^2(P)$, $Q(\Omega) = 1$ and

$$(2.12) \qquad E\left[\frac{dQ}{dP} \cdot \big(X(t) - X(s)\big) \,\Big|\, \mathcal{F}(s) \right] = 0, \quad \text{a.s.}$$

for any $0 \le s \le t \le T$. (If Q is a true probability measure, as opposed to a signed measure with $Q(\Omega) = 1$, then (2.12) amounts to the martingale property of X under Q.) See Müller (1985), Lemma 12(b) in Schweizer (1996), as well as conditions (5.1)-(5.4) and the paragraph immediately following them in the present paper.

In addition to Problem 2.1, it is useful to consider also the following question, which is interesting in its own right.

Problem 2.2. *Given* $H \in \mathbf{L}^2(P)$ *and* $c \in \mathbb{R}$, *compute*

$$(2.13) \qquad V(c) \triangleq \inf_{\vartheta \in \Theta} E\Big(H - c - G_T(\vartheta)\Big)^2$$

and try to find $\vartheta^{(c)} \in \Theta$ *that attains the infimum in* (2.13), *if such exists.*

Clearly, $\inf_{c \in \mathbb{R}} V(c)$ coincides with the quantity V of (2.4); and if this last infimum is attained by some $\hat{c} \in \mathbb{R}$, then the pair $\big(\hat{c}, \vartheta^{(\hat{c})}\big)$ attains the infimum in (2.4). In the next Sections we shall try to solve Problems 2.2

and 2.1 using very elementary *duality* ideas. In this effort, the elements of the set $\mathcal{D}$ in (2.9) will play the role of *adjoint* or *dual variables.* For the duality methodology to work in any generality it is critical to allow, as we did in Definition 2.1, for *signed* measures Q with $Q(\Omega) = 1$, as opposed to just standard probability measures.

§3. The Duality

The duality approach to Problem 2.2 is simple, and is based on the elementary observation

$$\text{(3.1)} \qquad \min_{x\in\mathbb{R}} \left[(H-x)^2 + yx\right] = (H-(H-y/2))^2 + y\,(H-y/2) = yH - y^2/4, \qquad \forall\, y \in \mathbb{R}.$$

The key idea now, is to read (3.1) with $x = c + G_T(\vartheta)$, $y = 2kD$ for given $c \in \mathbb{R}$ and arbitrary $\vartheta \in \Theta$, $D \in \mathcal{D}$ as in (2.2) and (2.9), and with arbitrary $k \in \mathbb{R}$, to obtain

$$\text{(3.2)} \qquad (H - c - G_T(\vartheta))^2 + 2kD\,(c + G_T(\vartheta)) \ \geq\ 2kDH - k^2 D^2.$$

Note also that (3.2) holds as equality for some $\vartheta \equiv \vartheta^{(c)} \in \Theta$, $D \equiv D^{(c)} \in \mathcal{D}$ and $k \equiv k^{(c)} \in \mathbb{R}$, if and only if

$$\text{(3.3)} \qquad c + G_T(\vartheta^{(c)}) = H - k^{(c)} D^{(c)}.$$

Now let us take expectations in (3.2) to obtain, in conjunction with the properties of (2.9):

$$\text{(3.4)} \qquad E\,(H - c - G_T(\vartheta))^2 \ \geq\ -k^2 E(D^2) + 2k\,[E(DH) - c]\,,$$

for every $k \in \mathbb{R}, D \in \mathcal{D}$ and $\vartheta \in \Theta$. Clearly,

$$\text{(3.5)} \qquad E(D^2) - 1 = \mathrm{Var}(D) \ \geq\ 0, \qquad \forall\, D \in \mathcal{D}$$

and the mapping $k \mapsto -k^2 E(D^2) + 2k\,[E(DH) - c]$ attains over $\mathbb{R}$ its maximal value $(E(DH) - c)^2 / E(D^2)$ at the point

$$\text{(3.6)} \qquad k_{D,c} \triangleq \frac{E(DH) - c}{E(D^2)}\,.$$

Thus, we obtain from (3.4) the inequality

$$
\begin{aligned}
V(c) &\triangleq \inf_{\vartheta\in\Theta} E\left(H-c-G_T(\vartheta)\right)^2 \\
&\geq \sup_{D\in\mathcal{D}}\sup_{k\in\mathbb{R}}\left[-k^2E(D^2)+2k\left(E(DH)-c\right)\right] \qquad (3.7)\\
&= \sup_{D\in\mathcal{D}} \frac{\left(E(DH)-c\right)^2}{E(D^2)} =: \tilde{V}(c),
\end{aligned}
$$

which is the basis of our *duality approach.* Here $V(c)$ is the value of our original ("primal") optimization Problem 2.2, whereas $\tilde{V}(c)$ is the value of an auxiliary ("dual") optimization problem.

This kind of duality is useful, only if the dual problem is easier to solve than the primal Problem 2.2 and if there is no "duality gap" (i.e., equality holds in (3.7)), so that by computing the value of the dual problem we also compute the value of the primal. Both these features hold for our setting, as we are about to show. Furthermore, the duality is "strong", in the sense that we can identify an optimal $\tilde{D}_c \in \mathcal{D}$ for the dual problem, namely

$$
\tilde{V}(c) = \frac{\left(E(\tilde{D}_cH)-c\right)^2}{E(\tilde{D}_c^2)} \qquad (3.8)
$$

for all but a critical value of the parameter c, and then obtain from this an optimal process $\vartheta^{(c)}$ for the primal problem via (3.3).

In order to make headway with this program, let us start by introducing the *projection operator* $\pi : \mathbf{L}^2(P) \to (G_T(\Theta))^{\perp}$ with the property

$$
E\left[(H-\pi(H))\cdot D\right]=0, \quad \forall\, H\in\mathbf{L}^2(P), \quad \forall\, D\in (G_T(\Theta))^{\perp}. \qquad (3.9)
$$

In particular,

$$
E\left[(H_1-\pi(H_1))\cdot\pi(H_2)\right] = 0, \qquad \forall\, H_1,H_2 \in \mathbf{L}^2(P), \qquad (3.9)'
$$

and from (3.9)′ and (2.10)′ we have

$$
E[\pi(1)] = E[\pi^2(1)] > 0. \qquad (3.10)
$$

Proposition 3.1. *The value of the dual problem in (3.7), namely*

$$
\tilde{V}(c) \triangleq \sup_{D\in\mathcal{D}} \frac{\left(E(DH)-c\right)^2}{E(D^2)}, \qquad (3.11)
$$

can be computed as

$$(3.12) \qquad \tilde{V}(c) \;=\; E\left[\pi^2(H-c)\right], \qquad \forall\, c \in \mathbb{R}.$$

The supremum in (3.11) is attained by

$$(3.13) \qquad \tilde{D}_c \;\triangleq\; \frac{\pi(H-c)}{E\left[\pi(H-c)\right]}, \qquad \forall\, c \neq \hat{c} \;\triangleq\; \frac{E[\pi(H)]}{E[\pi(1)]} \;;$$

it is not attained for $c = \hat{c}$.

For every $c \neq \hat{c}$, we shall call the random variable $\tilde{D}_c \in \mathcal{D}$ of (3.13) the "density of the *dual-optimal* signed martingale measure" in $\mathbf{P}_s(\Theta)$, namely

$$(3.14) \qquad \tilde{Q}_c(A) \;\triangleq\; \int_A \tilde{D}_c\, dP\,, \qquad A \in \mathcal{F}.$$

Remark 3.1 : Suppose that for some $h \in \mathbb{R}$ we have

$$(3.15) \qquad E(DH) \;=\; h, \qquad \forall\, D \in \mathcal{D}.$$

(For instance, this is the case when H is of the form (2.5).) Then the dual value function of (3.11) becomes

$$(3.16) \qquad \tilde{V}(c) \;=\; \frac{(h-c)^2}{\inf_{D\in\mathcal{D}} E(D^2)}$$

and, for $c \neq h$, the dual-optimal $\tilde{D}_c$ of (3.13) coincides with

$$(3.17) \qquad \tilde{D} \;\triangleq\; \arg\min_{D\in\mathcal{D}} E(D^2) \;=\; \frac{\pi(1)}{E[\pi(1)]} \;=\; E(\tilde{D}^2) + R \;\in\; \mathcal{D}$$

for some $R \in (G_T(\Theta))^{\perp\perp}$, and $E(\tilde{D}^2) \;=\; 1/E[\pi(1)] \;\geq\; 1$, as we shall establish below. Following Schweizer (1996), we shall call $\tilde{D}$ the "density of the *variance-optimal* signed martingale measure"

$$(3.18) \qquad \tilde{Q}(A) \;\triangleq\; \int_A \tilde{D}\, dP\,, \qquad A \in \mathcal{F}$$

in $\mathbf{P}_s(\Theta)$. This terminology should be clear from (3.5) and the definition in (3.17).

• *Proof of Proposition 3.1* : For every $D \in \mathcal{D}$, we have

$$E(DH) - c \;=\; E[D(H-c)] \;=\; E[D \cdot \pi(H-c)]$$

thanks to (3.9). Thus, from the Cauchy-Schwarz inequality,

$$(E(DH) - c)^2 = (E[D \cdot \pi(H-c)])^2 \leq E(D^2) \cdot E\left[\pi^2(H-c)\right],$$

which implies

$$\tilde{V}(c) \leq E\left[\pi^2(H-c)\right].$$

Now these last two inequalities are valid as equalities, if and only if we can find $\tilde{D}_c \in \mathcal{D}$ of the form

$$(3.13)' \qquad \tilde{D}_c = \text{const} \cdot \pi(H-c),$$

where the constant has to be chosen so that $E(\tilde{D}_c) = 1$. This is impossible to do if $E[\pi(H-c)] = E[\pi(H) - c \cdot \pi(1)]$ is equal to zero, i.e., if $c = \hat{c}$ as in (3.13); in other words, the supremum of (3.11) cannot be attained in this case. But for $c \neq \hat{c}$, the normalizing constant in (3.13)′ can be taken as $1/E[\pi(H-c)]$, leading to the expression of (3.13) and to (3.12) as well.

It remains to show that (3.12) holds even for $c = \hat{c}$. For this, let $c_n \triangleq c - 1/n$, $n \in \mathbb{N}$ and

$$\varphi_n \triangleq \pi(H - c_n), \quad \varphi \triangleq \pi(H-c), \quad \tilde{D}_{c_n} = \frac{\varphi_n}{E(\varphi_n)} \in \mathcal{D}$$

so that

$$\begin{aligned}
\frac{\left(E(\tilde{D}_{c_n}H) - c\right)^2}{E(\tilde{D}^2_{c_n})} &= \frac{\left(E(\tilde{D}_{c_n}H) - c_n - 1/n\right)^2}{E(\tilde{D}^2_{c_n})} \\
&= \frac{\left(E(\tilde{D}_{c_n}H) - c_n\right)^2}{E(\tilde{D}^2_{c_n})} + \frac{1/n^2}{E(\tilde{D}^2_{c_n})} - \frac{2}{n} \cdot \frac{E\left[\tilde{D}_{c_n}(H-c_n)\right]}{E(\tilde{D}^2_{c_n})} \\
&= E\left[\pi^2(H-c_n)\right] + \frac{1/n^2}{E(\tilde{D}^2_{c_n})} - \frac{2}{n} E[\pi(H-c_n)] \\
&= E(\varphi_n^2) + \frac{1/n^2}{E(\tilde{D}^2_{c_n})} - \frac{2}{n} E(\varphi_n) \\
&\to E\left(\varphi^2\right) = E\left[\pi^2(H-c)\right]
\end{aligned}$$

as $n \to \infty$. We have used the inequality $0 < 1/E\left(\tilde{D}^2_{c_n}\right) \leq 1$; the facts $\varphi_n - \varphi = \pi(1)/n \to 0$ a.s., $|\varphi_n| \leq |\varphi| + \pi(1) \in \mathbf{L}^2(P)$; the Dominated

Convergence Theorem; and the observation that, for $c \neq \hat{c}$, we have

(3.19)
$$\frac{E\left[\tilde{D}_c \cdot (H-c)\right]}{E(\tilde{D}_c^2)} = \frac{E\left[\tilde{D}_c \cdot \pi(H-c)\right]}{E(\tilde{D}_c^2)}$$
$$= \frac{E\left[\pi^2(H-c)\right]}{E\left[\pi(H-c)\right]} \cdot \frac{\left(E[\pi(H-c)]\right)^2}{E\left[\pi^2(H-c)\right]} = E[\pi(H-c)]$$

from (3.9)′, (3.13). □

• *Proof of (3.17)* : For any $D \in \mathcal{D}$, we have

$$1 = (E(D \cdot 1))^2 = (E(D \cdot \pi(1)))^2 \leq E\left(D^2\right) \cdot E\left[\pi^2(1)\right],$$

from (3.9) and the Cauchy-Schwarz inequality. Equality holds if and only if

$$D = \tilde{D} \triangleq \text{const} \cdot \pi(1),$$

and the normalizing constant has to be chosen so that $E(\tilde{D}) = 1$, namely, equal to $1/E[\pi(1)]$. We conclude that $\tilde{D} = \pi(1)/E[\pi(1)]$ satisfies

$$E\left(D^2\right) \geq \frac{1}{E\left[\pi^2(1)\right]} = \frac{1}{E\left[\pi(1)\right]} = E(\tilde{D}^2), \qquad \forall\, D \in \mathcal{D}. \tag{3.20}$$

On the other hand, since

$$1 = \pi(1) + \eta \qquad \text{for some} \quad \eta \in (G_T(\Theta))^{\perp\perp}, \tag{3.21}$$

we have $\tilde{D} = (1-\eta)/E[\pi(1)] = E(\tilde{D}^2) + R$, with $R = -\eta/E[\pi(1)]$. □

Remark 3.2 : If $G_T(\Theta)$ is closed in $\mathbf{L}^2(P)$, then (3.21) becomes

$$1 = \pi(1) + G_T(\xi^1), \qquad \text{for some} \quad \xi^1 \in \Theta \tag{3.21$'$}$$

and (3.17), (3.20) give

$$\tilde{D} = E(\tilde{D}^2) + G_T(\tilde{\xi}), \qquad \text{with} \quad \tilde{\xi} \triangleq -E(\tilde{D}^2) \cdot \xi^1 \in \Theta. \tag{3.22}$$

§4. Results

We are now in a position to use the duality developed in the previous section in order to provide solutions to Problems 2.1 and 2.2. First, a lemma from Schweizer (1996), pp. 230-231; we provide the proof for completeness.

Lemma 4.1. *Suppose that the infimum in (2.13) is attained by some $\vartheta^{(c)} \in \Theta$. Then this process satisfies*

$$(4.1) \qquad E\left[H - c - G_T\left(\vartheta^{(c)}\right)\right] = \frac{E(\tilde{D}H) - c}{E(\tilde{D}^2)} \qquad \textit{and}$$

$$(4.2) \qquad E\left[H - c - G_T\left(\vartheta^{(c)}\right)\right]^2 = \frac{c^2 - 2c \cdot E(\tilde{D}H)}{E(\tilde{D}^2)} + E\left[\pi^2(H)\right],$$

in the notation of (3.9) and (3.17).

Proof of (4.1) : The assumption implies that, for any given $\xi \in \Theta$, the function

$$\begin{aligned} f(\varepsilon) &\triangleq E\left[H - c - G_T\left(\vartheta^{(c)} + \varepsilon\xi\right)\right]^2 \\ &= \varepsilon^2 \cdot E\left[G_T^2(\xi)\right] - 2\varepsilon \cdot E\left[\left(H - c - G_T\left(\vartheta^{(c)}\right)\right) \cdot G_T(\xi)\right] + V(c) \end{aligned}$$

attains its minimum over $\mathbb{R}$ at $\varepsilon = 0$. This gives $f'(0) = 0$, or equivalently

$$(4.3) \qquad E\left[\left(H - c - G_T\left(\vartheta^{(c)}\right)\right) \cdot G_T(\xi)\right] = 0, \qquad \forall\, \xi \in \Theta.$$

Let us also notice that the mapping

$$(4.4) \qquad D \mapsto E(\tilde{D}D) \text{ is constant on } \mathcal{D},$$

since we have

$$\begin{aligned} E(\tilde{D}^2) - E(\tilde{D}D) = E\left[\tilde{D}(\tilde{D} - D)\right] &= E\left[\pi(1)(\tilde{D} - D)\right]/E\left[\pi(1)\right] \\ &= E(\tilde{D} - D)/E\left[\pi(1)\right] = 0 \end{aligned}$$

thanks to (3.9).

Now denote $\gamma \triangleq E\left[H - c - G_T\left(\vartheta^{(c)}\right)\right]$. If $\gamma = 0$, then the random variable

$$D_1 \triangleq \tilde{D} + \left(H - c - G_T\left(\vartheta^{(c)}\right)\right)$$

belongs to $\mathcal{D}$ by virtue of (4.3), and (4.4) implies

$$0 = E\left[\tilde{D}(D_1 - \tilde{D})\right] = E\left[\tilde{D}\left(H - c - G_T\left(\vartheta^{(c)}\right)\right)\right] = E(\tilde{D}H) - c,$$

so that (4.1) holds. If $\gamma \neq 0$, then $D_2 \triangleq \left[H - c - G_T\left(\vartheta^{(c)}\right)\right]/\gamma$ is in $\mathcal{D}$, and by (4.4) once again:

$$E(\tilde{D}^2) \;=\; E(\tilde{D}D_2) \;=\; \frac{1}{\gamma}\left(E(\tilde{D}H) - c\right),$$

and so (4.1) holds in this case too. □

• *Proof of (4.2)* : From (4.3), the random variable $H - c - G_T\left(\vartheta^{(c)}\right)$ belongs to the closed subspace $(G_T(\Theta))^{\perp}$ of (2.11), so we have

$$\begin{aligned} &E\left[H - c - G_T\left(\vartheta^{(c)}\right)\right]^2 \\ &\quad= E\left(\left[H - c - G_T\left(\vartheta^{(c)}\right)\right] \cdot \left[H - \pi(H) + \pi(H) - c - G_T\left(\vartheta^{(c)}\right)\right]\right) \\ &\quad= E\left(\left[H - c - G_T\left(\vartheta^{(c)}\right)\right] \cdot \left[\pi(H) - c\right]\right) \\ &\quad= E\left[\pi^2(H)\right] - c\,E\left[\pi(H)\right] - c\,\frac{E(\tilde{D}H) - c}{E(\tilde{D}^2)} \end{aligned}$$

thanks to (4.3), (3.9) and (4.1). The equation (4.2) now follows from

$$E(\tilde{D}H) \;=\; E(\tilde{D}^2) \cdot E\left[\pi(H)\right]. \tag{4.5}$$

In order to check (4.5), recall (3.17), (3.20) and use (3.9)′ repeatedly, to wit:

$$\frac{E(\tilde{D}H)}{E(\tilde{D}^2)} \;=\; E\left[H\pi(1)\right] \;=\; E\left[\pi(H)\pi(1)\right] \;=\; E\left[\pi(H)\right]. \tag{4.6}$$

□

Theorem 4.1. *(i) Suppose that there exists some $\vartheta^{(c)} \in \Theta$ which attains the infimum in (2.13). Then this $\vartheta^{(c)}$ satisfies*

$$H - c - G_T\left(\vartheta^{(c)}\right) \;=\; \pi(H - c), \tag{4.7}$$

and there is no duality gap in (3.7), namely

(4.8)

$$\begin{aligned} V(c) \;=\; \tilde{V}(c) \;&=\; E\left[\pi^2(H - c)\right] \\ &=\; \frac{\left(E(\tilde{D}H) - c\right)^2}{E(\tilde{D}^2)} + E\left[\pi^2(H)\right] - \frac{\left(E\pi(H)\right)^2}{E\left[\pi(1)\right]}. \end{aligned}$$

(ii) Conversely, suppose there exists some $\vartheta^{(c)} \in \Theta$ that satisfies (4.7); then this $\vartheta^{(c)}$ attains the infimum in (2.13), and the equalities of (4.8) hold.

• *Proof of (4.8), Part (i)* : Under the assumption of (i), we claim that

$$
\begin{aligned}
V(c) &= E\left(H - c - G_T\left(\vartheta^{(c)}\right)\right)^2 \\
&= \frac{\left(E(\tilde{D}H) - c\right)^2}{E(\tilde{D}^2)} + E\left[\pi^2(H)\right] - \left(E\pi(H)\right)^2 \cdot E(\tilde{D}^2) \qquad (4.9)\\
&= E\left[\pi^2(H-c)\right] \;=\; \tilde{V}(c),
\end{aligned}
$$

which clearly proves (4.8) in light of the last equality in (3.20). Indeed, the first equality in (4.9) holds by assumption, whereas the second is a consequence of (4.2), (4.5). The third equality is a consequence of (4.6), (3.20) and (4.2), thanks to the simple computation

$$
\begin{aligned}
E\left[\pi^2(H-c)\right] &= E\left(\pi(H) - c \cdot \pi(1)\right)^2 \\
&= E\left[\pi^2(H)\right] + c^2 \cdot E\left[\pi^2(1)\right] - 2c \cdot E\left[\pi(1)\pi(H)\right] \\
&= E\left[\pi^2(H)\right] + \frac{c^2 - 2c \cdot E(\tilde{D}H)}{E(\tilde{D}^2)} .
\end{aligned}
$$

Finally, the last equality in (4.9) is just (3.12).

• *Proof of (4.7), Part (i); $c \neq \hat{c}$* : Let us write (3.2) with $\vartheta \equiv \vartheta^{(c)}$, $D \equiv \tilde{D}_c$ as in (3.13), and

$$
k \equiv k_c \triangleq k_{\tilde{D}_c, c} = \frac{E(\tilde{D}_c H) - c}{E(\tilde{D}_c^2)}
$$

as in (3.6): namely,

$$
\begin{aligned}
&\left(H - c - G_T\left(\vartheta^{(c)}\right)\right)^2 + 2k_c\tilde{D}_c\left(c + G_T\left(\vartheta^{(c)}\right)\right) \qquad (4.10)\\
&\qquad\qquad \geq 2k_c\tilde{D}_c H - \left(k_c\tilde{D}_c\right)^2, \text{ a.s.}
\end{aligned}
$$

Taking expectations in (4.10), and recalling the optimality of $\vartheta^{(c)}$ as well as Proposition 3.1, we obtain

$$
\begin{aligned}
V(c) &= E\left(H-c-G_T\left(\vartheta^{(c)}\right)\right)^2 \\
&\geq -k_c^2\, E(\tilde{D}_c^2)+2k_c\left(E(\tilde{D}_c H)-c\right) \\
&= \frac{\left(E(\tilde{D}_c H)-c\right)^2}{E(\tilde{D}_c^2)} = \tilde{V}(c).
\end{aligned}
\tag{4.11}
$$

But from (4.8) we know that (4.11) actually holds as equality, which means that the left-hand side and the right-hand side of (4.10) have the same expectation. In other words, (4.10) must hold as equality, which we know happens only if (3.3) holds, namely only if

$$
H-c-G_T\left(\vartheta^{(c)}\right) = k_c\,\tilde{D}_c = \frac{E(\tilde{D}_c H)-c}{E(\tilde{D}_c^2)}\,\frac{\pi(H-c)}{E\left[\pi(H-c)\right]} = \pi(H-c)\,,
$$

holds a.s., thanks to (3.19).

• *Proof of (4.7), part (i); $c=\hat{c}$* : In this case we shall need a new kind of duality, namely with

$$
\mathcal{L} \triangleq \left\{L \in (G_T(\vartheta))^{\perp} \;/\; E(L)=0\right\}
\tag{4.12}
$$

replacing the space $\mathcal{D}$ of (2.9); the elements of $\mathcal{L}$ will be the dual (adjoint) variables in this new duality. We begin by writing (3.1) with $x=c+G_T(\vartheta)$, $y=2L$ for arbitrary $\vartheta\in\Theta$, $L\in\mathcal{L}$:

$$
(H-c-G_T(\vartheta))^2+2L\,(c+G_T(\vartheta)) \;\geq\; 2LH-L^2,
\tag{4.13}
$$

with equality if and only if

$$
H-c-G_T(\vartheta) \;=\; L
\tag{4.14}
$$

holds a.s. Taking expectations in (4.13), we obtain

$$
\begin{aligned}
E\,(H-c-G_T(\vartheta))^2 &\geq E\left[2L(H-c)-L^2\right] \\
&= E\left[2L\cdot\pi(H-c)-L^2\right] \\
&= E\left[\pi^2(H-c)\right]-E\,(L-\pi(H-c))^2 .
\end{aligned}
\tag{4.15}
$$

This suggests that we should read (4.13)-(4.15) with $\vartheta\equiv\vartheta^{(c)}$, the element of Θ that attains the infimum in (2.13) and is thus optimal for

Problem 2.2, and $L \equiv \tilde{L} \triangleq \pi(H-c)$, noting that $E(\tilde{L}) = 0$ since $c = \hat{c} = E[\pi(H)]/E[\pi(1)]$. With these choices, the left-most member of (4.15) becomes

$$E\left(H - c - G_T\left(\vartheta^{(c)}\right)\right)^2 = V(c),$$

whilst its right-most member is $E\left[\pi^2(H-c)\right] = \tilde{V}(c)$. From (4.8) we know that these two quantities are equal, so the two sides of (4.13) have the same expectation. This means that (4.13) must holds as equality, which happens only if (4.14) is valid, namely

$$H - c - G_T\left(\vartheta^{(c)}\right) = \pi(H-c), \quad \text{a.s.}$$

• *Proof of Part (ii)* : Suppose there exists some $\vartheta^{(c)} \in \Theta$ that satisfies (4.7); then

$$E\left(H - c - G_T\left(\vartheta^{(c)}\right)\right)^2 = E\left[\pi^2(H-c)\right] = \tilde{V}(c)$$

from Proposition 3.1. But we also have

$$\tilde{V}(c) \leq V(c) \leq E\left(H - c - G_T\left(\vartheta^{(c)}\right)\right)^2,$$

thanks to (3.7) and (2.13). In other words, these last two inequalities are valid as equalities, $\vartheta^{(c)}$ attains the infimum in (2.13), and (4.8) holds.

□

Remark 4.1 : The case of $G_T(\Theta)$ closed.
If $G_T(\Theta)$ is closed in $\mathbf{L}^2(P)$, then the infimum in (2.13) is attained, as was assumed in Lemma 4.1 and in Theorem 4.1(i). In this case we have of course $G_T(\Theta) = (G_T(\Theta))^{\perp\perp}$, and every $H \in \mathbf{L}^2(P)$ admits a decomposition of the form

$$H = \pi(H) + G_T(\xi^H) \qquad \text{for some } \xi^H \in \Theta. \tag{4.16}$$

In particular, there exists $\xi^1 \in \Theta$ so that (3.21) holds, and thus

$$H - c - \pi(H-c) = [H - \pi(H)] - c\,[1 - \pi(1)] = G_T\left(\xi^H - c\xi^1\right).$$

Comparing this expression with (4.7), we conclude that (4.7) is satisfied with the choice

$$\vartheta^{(c)} = \xi^H - c\xi^1. \tag{4.17}$$

According to Theorem 4.1(ii), *the process $\vartheta^{(c)} \in \Theta$ of (4.17) is then optimal for Problem 2.2, and (4.8) holds.*

Example 4.1. Föllmer-Schweizer decomposition. The assertion at the end of Remark 4.1 remains valid even if $G_T(\Theta)$ is not closed in $\mathbf{L}^2(P)$, provided that (3.21) and (4.16) still hold. Consider, for example, the case of a semimartingale $X(\cdot) \in \mathcal{S}^2(P)$ and of a random variable $H \in \mathbf{L}^2(P)$ which admits the so-called "Föllmer-Schweizer decomposition"; this means that H can be written in the form $H = h + G_T(\zeta^H) + L^H(T)$ of (2.6), for $h = E(H)$, some $\zeta^H \in \Theta$, and some martingale $L^H(\cdot) \in \mathcal{M}_0^2(P)$ that satisfies the property (2.7).

Suppose that (3.21) is also satisfied; then $\pi(H) = h\pi(1) + L^H(T)$ and we have $H - \pi(H) = h\big(1 - \pi(1)\big) + G_T\big(\zeta^H\big) = G_T\big(h\xi^1 + \zeta^H\big)$, so we may take $\xi^H \equiv h\xi^1 + \zeta^H$ in (4.16) and

$$\vartheta^{(c)} \equiv (h-c)\cdot\xi^1 + \zeta^H \tag{4.17$'$}$$

in (4.7). The process $\vartheta^{(c)}$ of (4.17)$'$ is then optimal for the Problem 2.2, i.e., attains the infimum in (2.13), which can be readily computed as

$$V(c) = (h-c)^2 \cdot E[\pi(1)] + E\left(L^H(T)\right)^2 .$$

This simple result may be compared with Theorem 3 and Proposition 18 in Schweizer (1994).

We are now in a position to discuss the solution of Problem 2.1 as well.

Theorem 4.2. *Suppose that $G_T(\Theta)$ is closed in $\mathbf{L}^2(P)$. Then the value of Problem 2.1 is given as*

$$V = V(\hat{c}) = E\left[\pi^2(H)\right] - \frac{\left(E\left[\pi(H)\right]\right)^2}{E\left[\pi(1)\right]} \tag{4.18}$$

with the notation

$$\hat{c} = \frac{E\left[\pi(H)\right]}{E\left[\pi(1)\right]} = E(\tilde{D}H) = E\left(\frac{d\tilde{Q}}{dP}H\right) \tag{4.19}$$

of (3.13). Furthermore, the infimum in (2.4) is attained by the pair $(\hat{c}, \hat{\vartheta})$ with $\hat{c}$ as in (4.19) and with

$$\hat{\vartheta} \triangleq \vartheta^{(\hat{c})} = \xi^H - \hat{c}\xi^1 . \tag{4.20}$$

Proof : Immediate from Theorem 4.1 and Remark 4.1, when it is observed that the number $\hat{c}$ of (4.19) minimizes the expression of (4.8) over $c \in \mathbb{R}$. □

Note that when the signed measure $\tilde{Q}$ of (3.8) is a probability measure (i.e., when $P[\pi(1) \geq 0] = 1$), the quantity of (4.19) is just *the expectation of the random variable H under the dual-optimal measure $\tilde{Q}$*. Sufficient conditions are spelled out in the next section.

Remark 4.2 : Variance Minimization. If $G_T(\Theta)$ is closed in $\mathbf{L}^2(P)$, then the process $\hat{\vartheta}$ of (4.20) also

$$\text{(4.21)} \qquad \text{minimizes} \quad \operatorname{Var}(H - G_T(\vartheta)), \quad \text{over all } \vartheta \in \Theta.$$

This is because for any $\vartheta \in \Theta$, and with $c_\vartheta \triangleq E[H - G_T(\vartheta)]$, we have:

$$\begin{aligned} \operatorname{Var}(H - G_T(\vartheta)) &= E(H - c_\vartheta - G_T(\vartheta))^2 \\ &\geq E\left(H - \hat{c} - G_T(\hat{\vartheta})\right)^2 = \operatorname{Var}\left(H - G_T(\hat{\vartheta})\right), \end{aligned}$$

from Theorem 4.2.

More generally, for any given $c \in \mathbb{R}$, the process $\vartheta^{(c)} \in \Theta$ of (4.17) has the *"mean-variance efficiency" property*

(4.22)
$$\left\{ \begin{array}{c} \operatorname{Var}\left(H - G_T(\vartheta^{(c)})\right) \leq \operatorname{Var}(H - G_T(\vartheta)), \quad \text{for any} \\ \vartheta \in \Theta \ \text{ that satisfies } \ E[H - G_T(\vartheta)] = E\left[H - G_T\left(\vartheta^{(c)}\right)\right] \end{array} \right\}.$$

Indeed, let $\mu_c \triangleq E\left[H - G_T\left(\vartheta^{(c)}\right)\right]$ and observe that, for any $\vartheta \in \Theta$ with $E[H - G_T(\vartheta)] = \mu_c$, we have

$$\begin{aligned} & \operatorname{Var}(H - G_T(\vartheta)) \\ = \ & \operatorname{Var}(H - c - G_T(\vartheta)) = E(H - c - G_T(\vartheta))^2 - (\mu_c - c)^2 \\ \geq \ & E\left(H - c - G_T(\vartheta^{(c)})\right)^2 - \left(E\left(H - c - G_T(\vartheta^{(c)})\right)\right)^2 \\ = \ & \operatorname{Var}\left(H - c - G_T\left(\vartheta^{(c)}\right)\right) = \operatorname{Var}\left(H - G_T\left(\vartheta^{(c)}\right)\right). \end{aligned}$$

Remark 4.3 : Mean-Variance Frontier. Suppose that $G_T(\Theta)$ is closed in $\mathbf{L}^2(P)$, that we have $P\left[\pi(1) \neq E(\pi(1))\right] > 0$; this implies $E(\tilde{D}^2) > 1$ in (3.5), hence $E[\pi(1)] < 1$. For some given $m \in \mathbb{R}$, consider the following problem:

$$\text{(4.23)} \qquad \left\{ \begin{array}{c} \text{To minimize the variance } \operatorname{Var}(H - G_T(\vartheta)), \\ \text{over } \vartheta \in \Theta \ \text{ with } \ E[H - G_T(\vartheta)] = m. \end{array} \right\}$$

In view of the property (4.22), it suffices to show that we can find $c \equiv c_m$ such that

$$\text{(4.24)} \qquad E\left[H - G_T\left(\vartheta^{(c)}\right)\right] = m.$$

Then the solution of the problem (4.23) will be given by $\vartheta^{(c)} \in \Theta$ as in (4.17), with $c \equiv c_m$. Now from (4.1) we have $E\left[H - G_T\left(\vartheta^{(c)}\right)\right] = c + \frac{E(\tilde{D}H) - c}{E(\tilde{D}^2)}$, so that (4.24) amounts to

$$c = c_m \triangleq \frac{m \cdot E(\tilde{D}^2) - E(\tilde{D}H)}{E(\tilde{D}^2) - 1} = \frac{m - E\left[\pi(H)\right]}{1 - E\left[\pi(1)\right]},$$

thanks to (4.6) and (3.20). We take these two Remarks 4.2, 4.3 from Schweizer (1994, 1996).

§5. A Mathematical Finance Interpretation

The Problems 2.1, 2.2 have an interesting interpretation in the context of Mathematical Finance, when one interprets the components $X_i(\cdot)$ of the semimartingale in (2.1) as the (discounted) prices of several risky assets in a financial market. In this context, $\vartheta_i(t)$ represents the number of shares in the corresponding i^{th} asset, held by an investor at time $t \in [0, T]$. The resulting process $\vartheta \in \Theta$ stands then for the investor's (self-financing) *trading strategy*, and $G_t(\vartheta) = \int_0^t \vartheta'(s)\, dX(s) = \sum_{i=1}^d \int_0^t \vartheta_i(s)\, dX_i(s)$ for the (discounted) *gains-from-trade* associated with the strategy ϑ by time t.

Suppose now that the investor faces a contingent claim (liability) H at the end T of the time-horizon $[0, T]$. Starting with a given initial capital c, and using a trading strategy $\vartheta \in \Theta$, the investor seeks to replicate this contingent claim H *as faithfully as possible*, in the sense of minimizing the expected squared-error loss $E\left(H - c - G_T(\vartheta)\right)^2$. This leads us to Problem 2.2. When the determination of the "right" initial capital c is also part of the problem, one is led naturally to the formulation of Problem 2.1. Similarly, one may consider minimizing the variance of the discrepancy $H - c - G_T(\vartheta)$ over all trading strategies $\vartheta \in \Theta$ (problem of (4.21)), or just over those strategies that guarantee a given "mean-gains-from-trade" level $E\left[G_T(\vartheta)\right] = E(H) - m$ (problem of (4.23)).

If one decides to stick with this interpretation, it makes sense to ask whether the model for the financial assets represented by (2.1) admits *arbitrage opportunities*; these are trading strategies $\vartheta \in \Theta$ with $P[G_T(\vartheta) \geq 0] = 1$ and $P[G_T(\vartheta) > 0] > 0$. To this effect, let A be an increasing, predictable and RCLL process with values in $[0, \infty)$ and $A(0) = 0$, $\langle M \rangle_i \ll A$ for $i = 1, \cdots, d$, and suppose that the processes $M = (M_1, \ldots, M_d)$ and $B = (B_1, \ldots, B_d)$, in the decomposition

$X = X(0) + M + B$ of (2.1) for the semimartingale $X = (X_1, \ldots, X_d)$, satisfy

$$B_i(\cdot) \ll \langle M \rangle_i(\cdot) \tag{5.1}$$

$$B_i(t) = \int_0^t \gamma_i(s)\, dA(s), \qquad 0 \le t \le T \tag{5.2}$$

$$\langle M_i, M_j \rangle(t) = \int_0^t \sigma_{ij}(s)\, dA(s), \qquad 0 \le t \le T \tag{5.3}$$

for $i = 1, \cdots, d$ and $j = 1, \cdots, d$. Here $\gamma(\cdot) = (\gamma_1(\cdot), \cdots, \gamma_d(\cdot))'$ and $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1 \le i,j \le d}$ are suitable predictable processes that satisfy

$$\sigma(t)\lambda(t) = \gamma(t), \qquad \text{a.e. } t \in [0, T] \tag{5.4}$$

almost surely, for some $\lambda(\cdot) = (\lambda_1(\cdot), \cdots, \lambda_d(\cdot))'$ in the space

(5.5)

$$\mathcal{L}^2(M) \triangleq \Bigg\{ \vartheta : [0, T] \to \mathbb{R}^d \text{ predictable } \Big/ \\ E \int_0^T \vartheta'(s)\sigma(s)\vartheta(s)\, dA(s) = E \left(\int_0^T \vartheta'(s)\, dM(s) \right)^2 < \infty \Bigg\}.$$

Following Schweizer (1996), we shall refer to (5.1)-(5.4) as the *structure conditions* on the semimartingale $X(\cdot)$.

Under these conditions, it can be shown that the semimartingale $X(\cdot)$ does not admit arbitrage opportunities, and that we have equality $\mathbf{P}_s(\Theta) = \mathbf{P}_s^2(X)$ in Remark 2.2 (cf. Ansel & Stricker (1992); Schweizer (1995); and Schweizer (1996), Lemma 12). If, in addition, $X(\cdot)$ has continuous paths, then it can be shown that the variance-optimal martingale measure $\tilde{Q}$ of (3.18) is nonnegative, namely a probability measure:

$$P[\tilde{D} \ge 0] = 1 \quad \text{and} \quad \tilde{Q}(\Omega) = E(\tilde{D}) = 1 \tag{5.6}$$

in (3.17), (3.18). This $\tilde{Q}$ is in fact equivalent to P (i.e., $P[\pi(1) > 0] = 1$), under the extra assumption

$$\left\{ \begin{array}{c} X(\cdot) \text{ is a } Q\text{-local martingale under some} \\ \text{probability measure } Q \sim P \text{ with } (dQ/dP) \in \mathbf{L}^2(P) \end{array} \right\} \tag{5.7}$$

(cf. Schweizer (1996), Theorem 13; Delbaen & Schachermayer (1996), Theorem 1.3). This condition (5.7) also implies that

$$\text{the mapping } \vartheta \mapsto G_T(\vartheta) \text{ is injective;} \tag{5.8}$$

cf. [DMSSS], Lemma 3.5.

Always under the structure conditions (5.1)-(5.4) on the semimartingale $X(\cdot)$ of (2.1), consider now the so-called *mean-variance tradeoff* process

$$K(t) \triangleq \int_0^t \lambda'(s)\, dB(s) = \int_0^t \lambda'(s)\sigma(s)\lambda(s)\, dA(s) = \left\langle \int \lambda'\, dM \right\rangle(t), \quad 0 \le t \le T.$$

If this process is P-a.s. *bounded*, then

$$\Theta = \mathcal{L}^2(M) \tag{5.9}$$

in the notation of (2.2), (5.5) and

$$G_T(\Theta) \text{ is closed in } \mathbf{L}^2(P) \tag{5.10}$$

([Ph.R.S.], Corollary 4 and below; Schweizer (1996), Lemma 12). See also [DMSSS] where conditions both necessary and sufficient for (5.10) are presented.

Remark 5.1 : In the one-dimensional case $d = 1$, the structure conditions (5.1)-(5.4) are satisfied if there exists a process $\lambda \in \mathcal{L}^2(M)$ with

$$X(t) = X(0) + M(t) + \int_0^t \lambda(s)\, d\langle M\rangle(s), \qquad 0 \le t \le T. \tag{5.11}$$

In this case, the mean-variance tradeoff process of (5.9) takes the form

$$K(t) = \int_0^t \lambda^2(s)\, d\langle M\rangle(s). \tag{5.12}$$

Remark 5.2 : Suppose that $X(\cdot)$ has continuous paths, and that (5.11) and (5.7) hold. If the random variable $H \in \mathbf{L}^2(P)$ is of the form $H = h + G_T(\zeta^H)$ in (2.5), then

$$H - \pi(H) = G_T(h\xi^1 + \zeta^H)$$

and the injectivity property (5.8) allows us to make the identifications

$$\xi^H = h\xi^1 + \zeta^H, \qquad \vartheta^{(c)} = (h - c)\cdot \xi^1 + \zeta^H \tag{5.13}$$

in (4.16) and (4.17), respectively.

More generally, under the assumptions of Remark 5.2 but now for *any* $H \in \mathbf{L}^2(P)$, we have the Kunita-Watanabe decomposition under the variance-optimal probability measure $\tilde{Q}$ of (3.18), namely

$$\tilde{E}[H \mid \mathcal{F}(t)] = \tilde{h} + G_t(\tilde{\zeta}^H) + \tilde{L}^H(t), \quad 0 \le t \le T \tag{5.14}$$

(cf. Ansel & Stricker (1993), or Theorem 3 in Rheinländer & Schweizer (1997)). Here $\tilde{E}$ denotes expectation with respect to the probability measure $\tilde{Q}$, $\tilde{h} \triangleq \tilde{E}(H) = E(\tilde{D}H)$, $\tilde{\zeta}^H \in \Theta$, and $\tilde{L}^H \in \mathcal{S}^2(P)$ is a $\tilde{Q}$-martingale with $\tilde{L}^H(0) = 0$ and

$$(5.15) \qquad \left\langle \tilde{L}^H, X_i \right\rangle (\cdot) \equiv 0, \qquad \forall\, i = 1, \cdots, d.$$

On the other hand, since $\tilde{L}^H(T)$ belongs to the space $\mathbf{L}^2(P)$, we also have its decomposition

$$(5.16) \qquad \tilde{L}^H(T) = G_T(\tilde{\vartheta}^H) + \pi(\tilde{L}^H(T))$$

from the closedness of $G_T(\Theta)$, where $\tilde{\vartheta}^H \in \Theta$ is such that

$$(5.17) \qquad E\left[\left(\tilde{L}^H(T) - G_T(\tilde{\vartheta}^H)\right) \cdot G_T(\vartheta)\right] = 0, \qquad \forall\, \vartheta \in \Theta.$$

Back in (5.14), this gives $H = \tilde{h} + G_T(\tilde{\zeta}^H + \tilde{\vartheta}^H) + \pi(\tilde{L}^H(T))$, so that

$$\begin{aligned} (H-c) - \pi(H-c) &= (\tilde{h}-c)(1-\pi(1)) + G_T(\tilde{\zeta}^H + \tilde{\vartheta}^H) \\ &= G_T\left((\tilde{h}-c)\xi^1 + \tilde{\zeta}^H + \tilde{\vartheta}^H\right) \end{aligned}$$

from (3.21)′. Again, the injectivity property (5.8) allows us to make the identification

$$(5.18) \qquad \vartheta^{(c)} = (\tilde{h}-c)\xi^1 + \tilde{\zeta}^H + \tilde{\vartheta}^H$$

in (4.17). Finally, let us consider the positive $\tilde{Q}$-martingale

$$\begin{aligned} (5.19) \qquad \tilde{D}(t) \triangleq \tilde{E}[\tilde{D} \,|\, \mathcal{F}(t)] &= E(\tilde{D}^2) + G_t(\tilde{\xi}) \\ &= E(\tilde{D}^2)\left[1 - G_t(\xi^1)\right], \quad 0 \le t \le T \end{aligned}$$

obtained by taking conditional expectations in (3.22) under $\tilde{Q}$.

We are now in a position to identify the process $\tilde{\vartheta}^H$ appearing in (5.16), (5.18) and state the following result, which simplifies and generalizes Theorems 5, 6 of Rheinländer & Schweizer (1997).

Theorem 5.1. *Suppose that the semimartingale $X(\cdot) \in \mathcal{S}^2(P)$ has continuous paths, and that (5.7), (5.11) hold. Then the optimal process $\vartheta^{(c)} \in \Theta$ for Problem 2.2 takes the form*

$$(5.20) \qquad \vartheta^{(c)} = \tilde{\zeta}^H + \left[(E(\tilde{D}H) - c) + E(\tilde{D}^2)\int_0^{\cdot} \frac{d\tilde{L}^H(s)}{\tilde{D}(s)}\right] \cdot \xi^1$$

in the notation of (5.14), (5.19).

Sketch of Proof: The $\tilde{Q}$-martingales $\tilde{N}(t) \triangleq \int_0^t (1/\tilde{D}(s))\, d\tilde{L}^H(s)$, $0 \le t \le T$ and $\tilde{D}(\cdot)$ of (5.19) are orthogonal, since

$$\langle \tilde{N}, \tilde{D} \rangle(\cdot) = \sum_{i=1}^{d} \int_0^{\cdot} \frac{\tilde{\xi}_i(s)}{\tilde{D}(s)}\, d\langle \tilde{L}^H, X_i \rangle(s) \equiv 0$$

from (5.15) and (5.19). Thus, integration by parts gives

(5.21)

$$\begin{aligned} \tilde{L}^H(T) &= \int_0^T \tilde{D}(s)\, d\tilde{N}(s) = \tilde{D}(T)\, \tilde{N}(T) - \int_0^T \tilde{N}(s) \tilde{\xi}'(s)\, dX(s) \\ &= \tilde{D}\tilde{N}(T) - G_T(\tilde{N}\tilde{\xi}), \end{aligned}$$

and transforms (5.17) into

$$E\left[G_T(\tilde{\vartheta}^H + \tilde{N}\tilde{\xi}) \cdot G_T(\vartheta) \right] = \tilde{E}\left[\tilde{N}(T) G_T(\vartheta) \right], \qquad \forall\, \vartheta \in \Theta. \tag{5.22}$$

But the right-hand-side of (5.22) vanishes, since

$$\begin{aligned} \tilde{E}\left[\tilde{N}(T) G_T(\vartheta) \right] &= \tilde{E}\left[\int_0^T \frac{d\tilde{L}^H(s)}{\tilde{D}(s)} \cdot \sum_{i=1}^{d} \int_0^T \vartheta_i(s)\, dX_i(s) \right] \\ &= \tilde{E} \sum_{i=1}^{d} \int_0^T \frac{\vartheta_i(s)}{\tilde{D}(s)}\, d\langle \tilde{L}^H, X_i \rangle(s) = 0 \end{aligned}$$

thanks to (5.15). Thus the left-hand-side of (5.22) also vanishes for every $\vartheta \in \Theta$, which suggests

$$\tilde{\vartheta}^H = -\tilde{N}\tilde{\xi} = E(\tilde{D}^2)\, \xi^1 \int_0^{\cdot} \frac{d\tilde{L}^H(s)}{\tilde{D}(s)}$$

and leads to (5.20) after substitution into (5.18). □

In order to justify the legitimacy of the above argument, particularly the steps leading to (5.22), one needs to show that the random variable $\sup_{0 \le t \le T} \left| \tilde{D}(t) \int_0^t (1/\tilde{D}(s))\, d\tilde{L}^H(s) \right|$ belongs to $\mathbf{L}^2(P)$; this is carried out on pp. 1820-1823 of Rheinländer & Schweizer (1997).

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