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Function Spaces and Symmetric Markov Processes

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Dedicated to Professor Kiyosi Itô on the occasion of his 88th birthday

Abstract.

We exhibit some mutual interactions between potential theory for concrete function spaces on $\mathbb{R}^n$ and the Dirichlet space theory associated with symmetric Markov processes. Our first concern is the role of the Dirichlet form version of the capacitary strong type inequality in the study of the ultracontractivity of the transition semigroup of time changed symmetric Markov processes. In particular, we study time changes of symmetric stable processes in relation to d-bounds of measures. We next show how the theory on capacity and the spectral synthesis for the Dirichlet space can be well inherited to a general function space with contractive p-norm. A link connecting those two topics is a contractive Besov space over a d-set of $\mathbb{R}^n$.

§1. Introduction

Since the publication of the seminal work of Beurling and Deny [5], their axiomatic potential theory of the Dirichlet space $(\mathcal{F}, \mathcal{E})$ has been unified under one roof with the theory of the symmetric Markov process $\mathbf{M}$. In particular, any σ-finite positive measure μ charging no set of zero capacity can now be studied in relation to the trace Dirichlet space $(\check{\mathcal{F}}, \check{\mathcal{E}})$ on the support F of μ and the time changed process $\check{\mathbf{M}}$ on F of $\mathbf{M}$ by means of the positive continuous additive functional associated with μ ([13]).

In §2, we shall see for $\kappa \in (0,1)$ that a simple *capacitary isoperimetric inequality*

$$(1) \qquad \mu(K)^\kappa \leq \Theta \ \mathrm{Cap}(K), \qquad \forall K(\text{compact}),$$

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is equivalent to the *ultracontractivity*

$$\check{p}_t(x,y) \le \left(\frac{H}{t}\right)^{\frac{1}{1-\kappa}}, \qquad t>0, \tag{2}$$

of the transition function $\check{p}_t$ of $\check{\mathbf{M}}$, with the isoperimetric constant Θ for the measure μ and the heat constant H for the process $\check{\mathbf{M}}$ controlling each other. When the (extended) Dirichlet space is the Riesz potential space $\dot{L}^{\alpha,2}(\mathbb{R}^n)$ and $\mathbf{M}$ is the symmetric 2α-stable process $(0<\alpha<1)$, we shall also see in §3 that the isoperimetric constant can be replaced by the *d-bound*

$$v_d(\mu) = \sup_{x\in\mathbb{R}^n, r>0} \frac{\mu(B(x,r))}{r^d}, \tag{3}$$

of the measure μ. Detailed proof of theorems in §2 and §3 can be found in [16].

An important ingredient in proving the above equivalence is the *capacitary strong type inequality*

$$\int_0^\infty \mathrm{Cap}(\{x\in X : |u(x)|\ge t\})d(t^2) \le 4\mathcal{E}(u,u) \quad \forall u\in\mathcal{F}\cap C_0(X), \tag{4}$$

which readily ensures the equivalence of (1) to a Sobolev type imbedding of the trace Dirichlet space $\check{\mathcal{F}}$. We can then invoke the works by Carlen,Kusuoka and Stroock[8] and Bakry,Coulhon,Ledoux and Saloff-Coste[2] to relate (1) and (2).

In the meantime, potential theory have advanced being modelled on concrete function spaces like Sobolev spaces $W^{r,p}$, Bessel potential spaces $L^{\alpha,p}$, Besov spaces $B^{p,q}_\alpha$ and so on. Imbedding theorems and spectral synthesis have been among important issues in potential theory ([1], [4], [23]).

Actually the capacitary strong type inequality was first established by Maz'ya[22] for the Sobolev space $W^{1,p}(\mathbb{R}^n)$, $1<p<\infty$. It was then extended to a large class of function spaces on $\mathbb{R}^n$ including the Riesz and Bessel potential spaces. It has been also proved in [21], [14] for a general function space with *contractive p-norm* $(1\le p<\infty)$

$$\begin{cases} |||u|||_p^p &= \int_{X\times X\setminus d} |u(x)-u(y)|^p N(x,dy)m(dx) \\ \mathcal{F}_p &= \{u\in L^p(X;m) : |||u|||_p^p<\infty\}, \end{cases} \tag{5}$$

which include as an important example the *contractive Besov space* $B^{p,p}_\alpha(F)$, $0<\alpha<1, 1\le p<\infty$, over a d-set $F\subset\mathbb{R}^n$ defined as

(6)
$$\begin{cases} \|u; B_\alpha^{p,p}(F)\| & = \|u\|_{L^p(F;\mu)} + \left(\iint_{F\times F} \frac{|u(x)-u(y)|^p}{|x-y|^{d+\alpha p}}\mu(dx)\mu(dy)\right)^{1/p} \\ B_\alpha^{p,p}(F) & = \{u \text{ is measurable } : \ \|u; B_\alpha^{p,p}(F)\| < \infty\}. \end{cases}$$

μ being taken to be the restriction to F of the d-dimensional Hausdorff measure.

Its Dirichlet space version (4) accompanied by the best constant 4 was proved rather recently by Vondraček [25]. [16] provides an alternative simple proof of (4).

When $p = 2$, the contractive Besov space on a d-set is a regular Dirichlet space on $L^2(F;\mu)$ and the properties of the associated jump type Markov process on F have been studied in [14], [6] and [9]. As we shall see in §3, this space is closely related to the Dirichlet space $(\check{\mathcal{F}}, \check{\mathcal{E}})$ on $L^2(F;\mu)$ of the time changed process of a symmetric stable process on $\mathbb{R}^n$ in the sense that the former is continuously imbedded into the latter, although these two spaces are generally different because the latter may involve a killing term in general.

Even when $p \neq 2$, the function space (5) with contractive p-norm shares with the Dirichlet space a common feature that every normal contraction operates on it and deserves to be studied on its own light. We shall see in §4 that the well known theory on capacity and spectral synthesis for the Dirichlet space ([5], [10], [13]) can be well inherited to the function space (5).

In particular, the spectral synthesis is possible for the contractive Besov space on a d-set $F \subset \mathbb{R}^n$ for $1 < p < \infty$. As an application, we shall get in §4 the following criterion for an relatively open set $H \subset F$ such that $F \setminus H$ has a locally finite positive $\tilde{d}$-dimensional Hausdorff measure with $\tilde{d} < d$:

$$B_{\alpha,0}^{p,p}(H) = B_\alpha^{p,p}(F) \Longleftrightarrow \alpha \le \frac{d-\tilde{d}}{p}, \tag{7}$$

$B_{\alpha,0}^{p,p}(H)$ being the closure of $B_\alpha^{p,p}(F) \cap C_0(H)$ in the space $B_\alpha^{p,p}(F)$.

This completes and extends the corresponding results by Caetano [7] and Farkas and Jacob [11]. When $p = 2$, $d = n$, $F = \overline{D}$, $H = D$, for an open set $D \subset \mathbb{R}^n$, (7) has been shown by Bogdan,Burdzy and Chen [6] giving a complete characterization for almost no sample path of the censored 2α-stable process on D to approach the boundary ∂D in finite time. Detailed proof of theorems in §4 can be found [15].

§2. Capacitary bounds of measures and time changed processes

Let $(X, m, \mathcal{E}, \mathcal{F})$ be a regular transient Dirichlet space. By this, we mean that X is a locally compact separable metric space, m is an everywhere dense positive Radon measure on X, and that $(\mathcal{E}, \mathcal{F})$ is a regular transient Dirichlet form on $L^2(X; m)$. The 0-order capacity of a compact set $K \subset X$ is then defined by

$$\text{(8)} \qquad \mathrm{Cap}(K) = \inf\{\mathcal{E}(u,u) : u \in \mathcal{F} \cap C_0(X),\ u(x) \geq 1,\ x \in K\}$$

and extended to any subsets of X as a Choquet capacity. $\mathcal{F}_e$ denotes the extended Dirichlet space. In what follows, any function $u \in \mathcal{F}_e$ will be always taken to be quasi-continuous (cf. [13]).

Owing to Vondraček [25], we then have the capacitary strong type inequality (4), which in turn implies the following (cf. [1, §7.2]):

Theorem 1. *Let μ be a Borel measure on X and $\kappa \in (0,1]$.*
(i) *If the capacitary isoperimetric inequality* (1) *holds for some positive constant Θ, then μ is a smooth Radon measure and*

$$\text{(9)} \qquad ||u||^2_{L^{2/\kappa}(X;\mu)} \leq S\mathcal{E}(u,u), \quad \forall\, u \in \mathcal{F}_e,$$

for some positive constant $S \leq (4/\kappa)^\kappa \Theta$.
(ii) *Conversely, if* (9) *holds for any $u \in \mathcal{F} \cap C_0(X)$ and for some positive constant S, then* (1) *holds for some positive constant $\Theta \leq S$.*

For a measure μ on X, we introduce its *isoperimetric constant* and *Sobolev constant* respectively by

$$\text{(10)} \qquad \Theta_\kappa(\mu) = \sup_K \frac{\mu(K)^\kappa}{\mathrm{Cap}(K)} \qquad \kappa \in (0,1],$$

$$\text{(11)} \qquad S_\eta(\mu) = \sup_{u \in \mathcal{F} \cap C_0(X)} \frac{||u||^2_{L^\eta(\mu)}}{\mathcal{E}(u,u)} \qquad \eta \in [2,\infty).$$

The supremum in (11) can be taken for all $u \in \mathcal{F}_e$. $S_2(\mu)$ may be called the *Poincaré constant* of μ. Theorem 1 can be rephrased as follows:

Corollary 1. *For a measure μ on X and for $\kappa \in (0,1]$, $0 < \Theta_\kappa(\mu) < \infty$ if and only if $0 < S_{2/\kappa}(\mu) < \infty$. Moreover,*

$$\text{(12)} \qquad \Theta_\kappa(\mu) \leq S_{2/\kappa}(\mu) \leq (4/\kappa)^\kappa\, \Theta_\kappa(\mu), \qquad \kappa \in (0,1].$$

Let $\mathbf{M} = \{X_t, P_x\}$ be an m-symmetric Hunt process on X associated with the Dirichlet form $\mathcal{E}$ and $A = A_t$ be a PCAF of $\mathbf{M}$ whose Revuz measure is a given smooth Radon measure μ. Denote by F and $\tilde{F}$ the support of μ and A respectively. Then $\tilde{F} \subset F$ q.e., $\mu(F \setminus \tilde{F}) = 0$ and further $\tilde{F}$ is a quasi-support of μ, namely, if quasi-continuous functions coincide μ-a.e., then they coincide q.e. on $\tilde{F}$. Recall that each element $u \in \mathcal{F}_e$ is taken to be quasi-continuous.

We consider the time changed process $\check{\mathbf{M}} = (\check{X}_t, P_x)_{x \in \tilde{F}}$ defined by

$$\check{X}_t = X_{\tau_t} \qquad \tau_t = \inf\{s > 0 : A_s > t\}.$$

$\check{\mathbf{M}}$ is a μ-symmetric transient right process, whose Dirichlet form $(\check{\mathcal{E}}, \check{\mathcal{F}})$ on $L^2(F;\mu)$ and the extended Dirichlet space $\check{\mathcal{F}}_e$ can be described as follows (cf. [13, §6.2]) :

$$\check{\mathcal{F}}_e = \{\varphi = u|_F \, \mu - a.e. : u \in \mathcal{F}_e\} \qquad \check{\mathcal{F}} = \check{\mathcal{F}}_e \cap L^2(F;\mu) \tag{13}$$

$$\check{\mathcal{E}}(\varphi, \varphi) = \mathcal{E}\left(H_{\tilde{F}} u, H_{\tilde{F}} u\right) \quad \varphi = u|_F \in \check{\mathcal{F}}_e, \tag{14}$$

where

$$H_{\tilde{F}} u(x) = E_x\left(u\left(X_{\sigma_{\tilde{F}}}\right)\right) \quad x \in X,$$

E_x denoting the expectation with respect to P_x and $\sigma_{\tilde{F}}$ being the hitting time of the set $\tilde{F}$ by the sample path X_t. Two elements of $\check{\mathcal{F}}_e$ are regarded identical if they coincides μ-a.e. Since $\tilde{F}$ is a quasi-support of μ, the definition (14) of $\check{\mathcal{E}}$ makes sense.

We can restate (14) as follows (the Dirichlet principle):

$$\check{\mathcal{E}}(\varphi, \varphi) = \inf\{\mathcal{E}(u, u) : u \in \mathcal{F}_e,\ u = \varphi \ \mu\text{-a.e. on } F\}, \quad \varphi \in \check{\mathcal{F}}_e. \tag{15}$$

The first half of the next theorem is immediate from (9) and (15).

Theorem 2. *Suppose a measure μ satisfies $\Theta_\kappa(\mu) \in (0, \infty)$ for some $\kappa \in (0,1)$.*
Then the following holds for $S = S_{2/\kappa}(\mu)(\in (\Theta_\kappa(\mu), (4/\kappa)^\kappa \Theta_\kappa(\mu)))$.
(i)

$$||\varphi||^2_{L^{2/\kappa}(F;\mu)} \leq S \ \check{\mathcal{E}}(\varphi, \varphi) \qquad \forall \varphi \in \check{\mathcal{F}}_e. \tag{16}$$

(ii) *The transition function $\check{p}_t$ of the time changed process $\check{\mathbf{M}}$ on F satisfies the ultracontractivity* (2) *for $\mu \times \mu$–a.e. $(x, y) \in F \times F$, where H is some positive constant with*

$$H \leq \frac{1}{1-\kappa} \cdot S. \tag{17}$$

We know that (1) and (16) are equivalent by Voropoulos [24]. But we are more concerned with dependence of the isoperimetric constant Θ_κ and the heat constant H.

Simple mutual dependence of H and the constant N appearing in the Nash type inequality has been well studied in [8]. The Sobolev inequality (16) can be readily converted by a Hölder inequality into the Nash type inequality with $N = S$ and we can get the bound (17) easily. On the other hand, we know that the Sobolev inequality can be derived from the Nash type inequality under a certain control of S by N in view of [2, Cor, 4.4, Cor. 7.3], and we can get the following converse to Theorem 2.

Theorem 3. *Suppose that μ is a smooth Radon measure with support F and that the transition function $\check{p}_t$ of the time changed process $\check{\mathbf{M}}$ on F with respect to the PCAF with Revuz measure μ satisfies the ultracontractivity* (2) *for some $\kappa \in (0,1)$, $H > 0$. Then*
(i) *The Sobolev inequality* (16) *holds for some positive constant S with*

$$S \leq 48\, e^2\, \frac{1}{\kappa} \left(\frac{2-\kappa}{1-\kappa} \right)^{\frac{2-\kappa}{1-\kappa}} \cdot H. \tag{18}$$

(ii) *μ admits an isoperimetric constant $\Theta_\kappa(\mu)$ with a bound*

$$(4/\kappa)^{-\kappa}\, S \leq \Theta_\kappa(\mu) \leq S \tag{19}$$

by the constant S of (i).

Tierry Coulhon has called author's attention to the relevance of the capacitary isoperimetric inequality (1) to the Faber-Krahn inequality.

For an open set $G \subset X$, we put

$$\mathcal{F}_G = \{u \in \mathcal{F} : u = 0 \quad \text{q.e. on } X \setminus G\}.$$

Due to the spectral synthesis theory for the Dirichlet space, $\mathcal{E}$ with domain $\mathcal{F}_G$ can be considered as a regular Dirichlet form on $L^2(G;m)$ which is called the part of $(\mathcal{E}, \mathcal{F})$ on G ([13, §4.4]). For a measure μ on X, we let

$$\lambda_1(\mu; G) = \inf_{u \in \mathcal{F}_G} \frac{\mathcal{E}(u,u)}{||u||^2_{L^2(\mu)}} \left(= \inf_{u \in \mathcal{F} \cap C_0(G)} \frac{\mathcal{E}(u,u)}{||u||^2_{L^2(\mu)}} \right),$$

which may be regarded, on account of the Dirichlet principle (15), as the first eigenvalue for the part of the trace Dirichlet space $(\check{\mathcal{F}}, \check{\mathcal{E}})$ on the relatively open sunset $F \cap G$ of F. Since $\lambda_1(\mu; G)$ is the reciprocal of the

Poincaré constant $S_2(\mu; G)$ defined by (11) for the part form $(\mathcal{E}, \mathcal{F}_G)$, we get from (12)

$$\frac{1}{\lambda_1(\mu; G)} \leq 4 \sup_{K \subset G} \frac{\mu(K)}{\mathrm{Cap}(K; G)}, \tag{20}$$

where $\mathrm{Cap}(K; G)$ is defined by (8) with X being replaced by G.

Let us assume that $\Theta_\kappa(\mu)$ is finite for some $\kappa \in (0, 1)$. Since $\mathrm{Cap}(K; G) \geq \mathrm{Cap}(K)$ for $K \subset G$, we have

$$\frac{1}{\mathrm{Cap}(K; G)} \leq \frac{\Theta_\kappa(\mu)}{\mu(K)^\kappa}. \tag{21}$$

(20) and (21) lead us to

$$\frac{1}{\lambda_1(\mu; G)} \leq 4 \sup_{K \subset G} \Theta_\kappa(\mu) \cdot \mu(K)^{1-\kappa} = 4\Theta_\kappa(\mu) \cdot \mu(G)^{1-\kappa}$$

and

$$\lambda_1(\mu; G) \geq \frac{1}{4\Theta_\kappa(\mu)} \cdot \frac{1}{\mu(G)^{1-\kappa}} \tag{22}$$

for any open set $G \subset X$ of finite μ-measure.

(22) is called the *Faber-Krahn inequality* and the above procedure of getting (22) from (1) using the capacitary strong type inequality has been indicated by Grigor'yan [18]. Very intimate relationship among the Faber-Krahn inequality, ultracontractivity (2) and the Nash type inequality has been studied in [19]. However, in order to recover the capacitary isoperimetric inequality (1) from the ultracontractivity (2), one may need to path through Nash type inequality to Sobolev's one as being done in this section.

§3. Application to time changes of symmetric stable processes on d-sets

In this section, we consider the symmetric 2α-stable process $\mathbf{M} = (X_t, P_x)$ on $\mathbb{R}^n$ for $0 < \alpha \leq 1, 2\alpha < n$. The transition function of $\mathbf{M}$ is a convolution semigroup $\{\nu_t, t > 0\}$ of symmetric probability measures on $\mathbb{R}^n$ with

$$\hat{\nu}_t(x) \left(= \int_{\mathbb{R}^n} e^{i(x,y)} \nu_t(dy) \right) = e^{-tc|x|^{2\alpha}},$$

c being a fixed positive constant. For simplicity, we take $c = 1$. In case that $\alpha = 1$, $\mathbf{M}$ is the n-dimensional Brownian motion with variance of

μ_t being equal to $2t$. $\mathbf{M}$ is transient. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of $\mathbf{M}$ on $L^2(\mathbb{R}^n)$ is given by

$$(23)\qquad \begin{cases} \mathcal{E}(u,u) &= \displaystyle\int_{\mathbb{R}^n} \hat{u}(x)\bar{\hat{v}}(x)|x|^{2\alpha}dx \\ \mathcal{F} &= \{u \in L^2(\mathbb{R}^n) : \displaystyle\int_{\mathbb{R}^n} |\hat{u}(x)|^2|x|^{2\alpha}dx < \infty\}. \end{cases}$$

The extended Dirichlet space $(\mathcal{F}_e, \mathcal{E})$ of $\mathbf{M}$ can then be identified with the Riesz potential space $\dot{L}^{\alpha,2}(\mathbb{R}^n) = \{I_\alpha * f : f \in L^2(\mathbb{R}^n)\}$, where the Riesz potential of a measure ν on $\mathbb{R}^n$ is defined by

$$I_\alpha * \nu(x) = \gamma_\alpha \int_{\mathbb{R}^n} |x-y|^{-(n-\alpha)}\nu(dy), \quad \gamma_\alpha = \frac{\Gamma((n-\alpha)/2))}{\pi^{n/2}2^\alpha\Gamma(\alpha/2)}.$$

The capacity defined by (8) for the present Dirichlet form coincides with the Riesz capacity defined for any compact set $K \subset \mathbb{R}^n$ by

$$(24)\quad \dot{C}_{\alpha,2}(K) = \inf\{||f||^2_{L^2(\mathbb{R}^n)} : f \in L^2_+(\mathbb{R}^n),\ I_\alpha * f(x) \geq 1\ \forall x \in K\}.$$

We call a closed subset F of $\mathbb{R}^n$ a (semi global) *d-set* for $0 < d \leq n$ if there exists a positive measure μ supported by F satisfying, for some constants $0 < c_1 \leq c_2$,

$$c_1 r^d \leq \mu(B(x,r)) \quad \forall x \in F, \forall r \in (0,1)$$

$$\mu(B(x,r)) \leq c_2 r^d \quad \forall x \in F, \forall r \in (0,\infty),$$

where $B(x,r)$ denotes the n-dimensional ball with center x and radius r. Such a measure is called a *d-measure.* It is known that the restriction of the d-dimensional Hausdorff measure to a d-set F is a d-measure (cf.[20]).

For a d-measure μ, we will be concerned with its *d-bound* defined by (3). We consider a d-measure μ on a d set F with

$$n - 2\alpha < d \leq n.$$

Otherwise, $\dot{C}_{\alpha,2}(F) = 0$ and μ can not satisfy the isoperimetric inequality with respect to the present Dirichlet form. Since

$$\dot{C}_{\alpha,2}(B(x,r)) = \dot{c}_{\alpha,2}\ r^{n-2\alpha}, \qquad \dot{c}_{\alpha,2} = \dot{C}_{\alpha,2}(B(0,1)),$$

we can immediately obtain a lower bound of the isoperimetric constant for μ by its d-bound:

$$(25)\qquad \dot{c}_{\alpha,2}^{-1}\ v_d(\mu)^{\frac{n-2\alpha}{d}} \leq \Theta_{\frac{n-2\alpha}{d}}(\mu).$$

We can also obtain an inequality in the opposite direction:

Theorem 4. *For any Radon measure μ with finite d-bound, it holds that*

$$\Theta_{\frac{n-2\alpha}{d}}(\mu) \ \leq \ c(n,\alpha,d) \ v_d(\mu)^{\frac{n-2\alpha}{d}} \tag{26}$$

for

$$c(n,\alpha,d) = \frac{4d^2 \ \gamma_\alpha^2 \ v_n \ (n-\alpha)^2}{(n-2\alpha)^2\{d-(n-2\alpha)\}^2}, \tag{27}$$

where v_n is the volume of the n dimensional unit ball.

By setting $\kappa = \frac{n-2\alpha}{d}$ in Corollary 1 and using (25) and (26), we get the bound of the Sobolev constant $S = S_{\frac{2d}{n-2\alpha}}(\mu)$ for μ in terms of its d-bound $v_d(\mu)$:

$$\dot{c}_{\alpha,2}^{-1} \ v_d(\mu)^{\frac{n-2\alpha}{d}} \ \leq \ S \ \leq \ (4d/(n-2\alpha))^{\frac{n-2\alpha}{d}} \ c(n,\alpha,d) \ v_d(\mu)^{\frac{n-2\alpha}{d}} \tag{28}$$

for the constant $c(n,\alpha,d)$ of (27).

By setting $\kappa = \frac{n-2\alpha}{d}$ in Theorem 1 and Theorem 2, we have

Theorem 5. *Suppose μ is a d-measure on $\mathbb{R}^n$ with $n-2\alpha < d \leq n$. Then we have the following for S satisfying the bounds* (28)*:*
(i)

$$||u||^2_{L^{\frac{2d}{n-2\alpha}}(\mathbb{R}^n;\mu)} \leq S \ \mathcal{E}(u,u) \qquad \forall u \in \dot{L}^{\alpha,2}(\mathbb{R}^n). \tag{29}$$

(ii) *Let $\check{\mathbf{M}}$ be the time changed process on the support F of μ of $\mathbf{M}$ by the PCAF with Revuz measure μ. Then its transition function $\check{p}_t$ satisfies*

$$\check{p}_t(x,y) \leq \left(\frac{H}{t}\right)^{\frac{d}{d-(n-2\alpha)}}, \qquad t > 0, \tag{30}$$

for $\mu \times \mu$-a.e. $(x,y) \in F \times F$, where H is some positive constant with

$$H \leq \frac{d}{d-(n-2\alpha)} \ S. \tag{31}$$

Actually inequality (29) together with the bounds

$$c_3 \ v_d(\mu)^{\frac{n-2\alpha}{d}} \leq S \leq c_4 \ v_d(\mu)^{\frac{n-2\alpha}{d}}$$

holding for some positive constants c_3, c_4 independent of μ goes back to the work of Adams ([23, 1.4.1]). Here we have made these contants c_3 and c_4 more explicit in (28).

We can also derive from Theorem 3 the following converse to Theorem 4.

Theorem 6. *Suppose that μ is a smooth Radon measure on $\mathbb{R}^n$ with support F and that the transition function $\tilde{p}_t$ of the time changed process $\check{\mathbf{M}}$ on F with respect to the PCAF with Revuz measure μ satisfies the bound* (30) *for some $d \in (n-2\alpha, n]$ and $H > 0$. Then*
(i) *The inequality* (29) *holds for some positive constant S with*

$$S \le \frac{48de^2}{n-2\alpha}\left(\frac{2d-(n-2\alpha)}{d-(n-2\alpha)}\right)^{\frac{2d-(n-2\alpha)}{d-(n-2\alpha)}} \cdot H. \tag{32}$$

(ii) *μ is a d-measure whose d-bound $v_d(\mu)$ satisfies*

$$\frac{n-2\alpha}{4d}\left(\frac{S}{c(n,\alpha,d)}\right)^{\frac{d}{n-2\alpha}} \le v_d(\mu) \le (\dot{c}_{\alpha,2}\ S)^{\frac{d}{n-2\alpha}} \tag{33}$$

for the constant S of (i) *and for $c(n,\alpha,d)$ of* (27).

Let $\mu, F, \check{\mathbf{M}}$ be as in Theorem 5 and $(\check{\mathcal{E}}, \check{\mathcal{F}})$ be the Dirichlet form of $\check{\mathbf{M}}$ on $L^2(F;\mu)$ the trace Dirichlet form of (23) on the d-set F. Put

$$\delta = \alpha - \frac{n-d}{2} \in (0,1] \tag{34}$$

and consider the Besov space $B_\delta^{2,2}(F)$ over F defined by
(35)

$$\begin{cases} (\varphi,\psi)_{B_\delta^{2,2}(F)} &= \displaystyle\int_{F\times F\backslash d} \frac{(\varphi(x)-\varphi(y))(\psi(x)-\psi(y))}{|x-y|^{d+2\delta}}\mu(dx)\mu(dy) \\ B_\delta^{2,2}(F) &= \{\varphi \in L^2(F;\mu) : (\varphi,\varphi)_{B_\delta^{2,2}(F)} < \infty\}. \end{cases}$$

$B_\delta^{2,2}(F)$ is a Dirichlet form on $L^2(F;\mu)$ equipped with the norm

$$||\varphi; B_\delta^{2,2}(F)||^2 = (\varphi,\varphi)_{L^2(F;\mu)} + (\varphi,\varphi)_{B_\delta^{2,2}(F)}.$$

By virtue of a Jonsson-Wallin trace theorem [20, chap. V], this space is related to the Bessel potential space $L_{\alpha,2}(\mathbb{R}^n)$ as

$$B_\delta^{2,2}(F) = L_{\alpha,2}(\mathbb{R}^n)|_F \tag{36}$$

both the restriction and extension operators involved being continuous. Since the present Dirichlet space (23) equipped with $\mathcal{E}_1$-norm is known to be equivalent to the Bessel potential space, we are led from the Dirichlet principle (15) and (36) to the following continuous embedding:

$$(37) \quad B_\delta^{2,2}(F) \subset \check{\mathcal{F}}_e, \qquad \check{\mathcal{E}}(\varphi,\varphi) \le C\ ||\varphi; B_\delta^{2,2}(F)||^2,\ \forall\varphi \in B_\delta^{2,2}(F),$$

for some positive constant C.

Nevertheless 0-oder forms $\check{\mathcal{E}}$ and $(\cdot,\cdot)_{B_\delta^{2,2}(F)}$ are not necessarily equivalent. For instance, let $\mathbf{M}$ be the standard Brownian motion on $\mathbb{R}^n$ with $n \ge 3$, F be the unit sphere Σ contered at the origin and μ be the surface measure σ on Σ. Then we have the following expression of the trace Dirichlet form $\check{\mathcal{E}}(f,f)$ for $f \in \check{\mathcal{F}}$ ([17]):

(38)

$$\check{\mathcal{E}}(f,f) = \frac{1}{\Omega}\int_{\Sigma\times\Sigma\setminus d}(f(\xi)-f(\eta))^2\frac{1}{|\xi-\eta|^n}\sigma(d\xi)\sigma(d\eta)+v_0\int_\Sigma f(\xi)^2\sigma(d\xi),$$

where Ω is the area of Σ and $v_0 = \frac{n-2}{2}$. The first term on the right hand side correponds to the form (35) for $d = n-1, \delta = 1/2$. But the additional second term appears due to the transience of the Brownian motion.

§4. Spectral synthesis for contractive p-norms and Besov spaces

Let X be a locally compact separable metric space and m a positive Radon measure on X with $\text{supp}[m] = X$. Let $N(x,dy)$ be a positive kernel on $(X,\mathcal{B}(X))$ such that $N(x,\{x\}) = 0,\ x \in X$, and $N(x,dy)m(dx)$ is a symmetric measure over $X\times X - d$, where $d = \{(x,x) : x \in X\}$. For a fixed $1 \le p < \infty$, we introduce the pseudo-norm $|||\cdot|||_p$ and the function space $\mathcal{F}_p$ by (5). Denoting the norm of the space $L^p(X;m)$ by $||\cdot||_p$, we equip $\mathcal{F}_p$ with the norm

$$(39) \qquad |||u|||_{p,1} = |||u|||_p + ||u||_p \qquad u \in \mathcal{F}_p \cap L^p(X;m).$$

We assume the regularity of this space in the sense that $\mathcal{F}_p \cap C_0(X)$ is dense in $\mathcal{F}_p$ with norm (39) and in $C_0(X)$ with uniform norm.

Denote by $\mathcal{O}$ the family of all open sets in X. We define the *p-capacity* of $A \in \mathcal{O}$ by

(40)

$$\text{Cap}_p(A) = \inf\{|||u|||_p^p + ||u||_p^p : u \in \mathcal{F}_p,\ u \ge 1\ m\text{-a.e. on } A\} \quad A \in \mathcal{O},$$

and extend it to any set $B \subset X$ by

$$\text{Cap}_p(B) = \inf\{\text{Cap}_p(A) : A \in \mathcal{O},\ B \subset A\}.$$

'q.e.' will mean *'except for a set of zero p-capacity'*. Cap_p-quasicontinuous function will be called simply *quasicontinuous*. In what follows, we also assume that $1 < p < \infty$.

Although the space $(\mathcal{F}_p, |||\cdot|||_{p,1}$ is slightly more complicated than the ordinary L^p space, we can well adopt the uniform convexity argument to ensure the unique existence of the equilibrium potential for any $A \in \mathcal{O}$ with finite p-capacity. Thus Cap_p on open sets can be seen to enjoy the continuity along the increasing limit as in [12]. It is also strongly subadditive as in [21]. Hence Cap_p is a Choquet capacity, each element $u \in \mathcal{F}_p$ has a quasicontinuous version $\tilde{u}$, each set of finite p-capacity has a unique equilibrium potential just as in the case of the Dirichlet space. We also have the following nice property:

(41)

u is quasi-continuous and $u = 0$ m-a.e. on $G(\in \mathcal{O}) \Longrightarrow u = 0$ q.e. on G.

For $G \in \mathcal{O}$, we let

$$\mathcal{F}_{p,0}^G = \overline{\mathcal{F}_p \cap C_0(G)}^{|||\cdot|||_{p,1}}, \tag{42}$$

where $C_0(G)$ denotes the family of continuous functions on X whose support is compact and contained in G. We say that *the spectral synthesis is possible for* $G \in \mathcal{O}$ if

$$\mathcal{F}_{p,0}^G = \{u \in \mathcal{F}_p : \tilde{u} = 0 \text{ } q.e. \text{ on } X \setminus G\}. \tag{43}$$

Following the method of [1, §9.2] for the space $W^{1,p}(\mathbb{R}^n)$ and using the contraction property of the space $\mathcal{F}_p$ together with the above mentioned properties of Cap_p, we can prove the next theorem.

Theorem 7. (i) *The spectral synthesis is possible for $G \in \mathcal{O}$ if $X \setminus G$ is compact.*

(ii) *The spectral synthesis is possible for any $G \in \mathcal{O}$ under the next assumption:*

(A) *There exist non-negative functions $w_n \in C_0(X)$ increasing to* 1 *such that*

$$\sup_n \sup_{x\in X} W_n(x) < \infty, \quad \lim_{n\to\infty} \sup_{x\in K} W_n(x) = 0 \text{ for any compact } K \subset X,$$

where

$$W_n(x) = \int_X |w_n(x) - w_n(y)|^p N(x, dy) \qquad x \in X. \tag{44}$$

As a consequence of Theorem 7 (i), the next useful identity holds for any compact set $K \subset X$:

$$(45)\quad \mathrm{Cap}_p(K) = \inf\{|||u|||_p^p + ||u||_p^p : u \in \mathcal{F}_p \cap C_0(X),\ u \geq 1 \text{ on } K\}.$$

We now let

$$0 < d \leq n, \quad < \alpha < 1, \quad 1 < p < \infty,$$

and consider the contractive Besov space $B_\alpha^{p,p}(F)$ on a d-set $F \subset \mathbb{R}^n$ defined by (6). This is a special example of the space $\mathcal{F}_p$ with contractive p-norm $|||\cdot|||_{p,1}$. The associated p-capacity of a set $A \subset F$ is denoted by $\mathrm{Cap}_{\alpha,p}(A; F)$. It can be shown that condition **A** is satisfied by this space. By Theorem 7, the spectral synthesis is therefore possible for any relatively open set $H \subset F$ with respect to $B_\alpha^{p,p}(F)$, which immediately implies the equivalence

$$(46)\qquad B_{\alpha,0}^{p,p}(H) = B_\alpha^{p,p}(F) \Longleftrightarrow \mathrm{Cap}_{\alpha,p}(F \setminus H; F) = 0,$$

where $B_{\alpha,0}^{p,p}(H)$ denotes the closure of $B_\alpha^{p,p}(F) \cap C_0(H)$ in the space $B_\alpha^{p,p}(F)$.

On the other hand, the next implications have been proved in [14] by making use of the property (45) of $\mathrm{Cap}_{\alpha,p}(\cdot; F)$, a Jonsson-Wallin trace theorem ([20]) and the metric properties of the Bessel capacity on $\mathbb{R}^n$ ([1]):

$$(47)\qquad \mathrm{Cap}_{\alpha,p}(\Lambda; F) = 0 \Longrightarrow \mathcal{H}_{dim}(\Lambda) \leq d - \alpha p,$$

$$(48)\qquad H_{d-\alpha p}(\Lambda) < \infty \Longrightarrow \mathrm{Cap}_{\alpha,p}(\Lambda; F) = 0.$$

Here $\mathcal{H}_{dim}$ and H_γ denote the Hausdorff dimension and γ-dimensional Hausdorff measure respectively.

(46),(47) and (48) lead us to the next desired theorem.

Theorem 8. *Assume that H is a relatively open subset of F and that $F \setminus H$ has a locally finite positive $\tilde{d}$-dimensional Hausdorff measure with $\tilde{d} < d$. Then $B_{\alpha,0}^{p,p}(H) = B_\alpha^{p,p}(F)$ if and only if $\alpha \leq \dfrac{d - \tilde{d}}{p}$.*

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