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Algebraic Shifting and Spectral Sequences

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Abstract.

There is a canonical spectral sequence associated to any filtration of simplicial complexes. Algebraically shifting a finite filtration of simplicial complexes produces a new filtration of shifted complexes.

We prove that certain sums of the dimensions of the limit terms of the spectral sequence of a filtration weakly decrease by algebraically shifting the filtration. A key step is the combinatorial interpretation of the dimensions of the limit terms of the spectral sequence of a filtration consisting of near-cones.

§1. Introduction

The key step of Björner and Kalai's characterization [BK] of f-vectors and Betti numbers of simplicial complexes was that algebraically shifting a simplicial complex K produces a new complex $\Delta(K)$ whose homology Betti numbers are the same as those of K, *i.e.*,

$$\beta^q(K) = \beta^q(\Delta(K)). \tag{1}$$

But the Betti numbers of $\Delta(K)$ are much easier to compute, because $\Delta(K)$ is shifted and hence a near-cone.

Relative homology is a little less straightforward. First note that if $L \subseteq K$ are a pair of simplicial complexes, then $\Delta(L) \subseteq \Delta(K)$ [Ka2, Theorem 2.2]. The equality (1) above becomes merely an inequality for relative homology,

$$\beta^q(K, L) \leq \beta^q(\Delta(K), \Delta(L));$$

in other words, relative Betti numbers (weakly) increase in each dimension [Du2] (see also [Rö], where a more general result, on generic initial

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ideals and Gröbner bases, was subsequently proved). As with the Betti numbers of a single near-cone, the relative Betti numbers of a pair of near-cones are easy to compute.

We now examine what happens when a finite filtration

$$\mathcal{K}\colon \emptyset = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_m = K \tag{2}$$

of a simplicial complex K is algebraically shifted, *i.e.*, when each subcomplex in the filtration is algebraically shifted, giving a new filtration

$$\Delta(\mathcal{K})\colon \emptyset = \Delta(K_0) \subseteq \Delta(K_1) \subseteq \cdots \subseteq \Delta(K_m) = \Delta(K).$$

In particular, we will be concerned with a cohomology spectral sequence of this filtration whose limit terms $E^1_\infty, \ldots, E^m_\infty$ filter the cohomology $\widetilde{H}^*(K, \boldsymbol{k})$ of K over a field $\boldsymbol{k}$. That is, $\dim E^1_\infty + \cdots + \dim E^m_\infty = \beta^*(K) = \widetilde{H}^*(K, \boldsymbol{k})$; we can think of E^s_∞ as providing the contribution of $K_s \backslash K_{s-1}$ to the cohomology of K. Our main result (Theorem 6.1) is that the quantity $\dim E^1_\infty + \cdots + \dim E^p_\infty$ (weakly) decreases, and hence $\dim E^{p+1}_\infty + \cdots + \dim E^m_\infty$ (weakly) increases, by applying algebraic shifting. In some sense then, algebraic shifting moves more of the cohomology to later in the filtration of K. Relative homology is just the $n = 2, p = 1$ case, as $E^2_\infty = \widetilde{H}^*(K_2, K_1)$ for the filtration $\emptyset \subseteq K_1 \subseteq K_2$.

As with Betti numbers and relative Betti numbers, the quantity $\dim E^1_\infty + \cdots + \dim E^p_\infty$ is easy to compute for near-cones, and this is an important step of the proof.

Section 2 reviews the necessary background for simplicial complexes, including the exterior face ring, in which all our subsequent calculations take place. In Section 3, we first construct the spectral sequence corresponding to $\mathcal{K}$, and then use elementary manipulations to replace $\dim E^1_\infty + \cdots + \dim E^p_\infty$ by an expression not using spectral sequences. Then in Section 4 we interpret this expression combinatorially for near-cones; this combinatorial interpretation resembles and complements the combinatorial interpretations of the Betti numbers of a near-cone and the relative Betti numbers of a pair of near-cones. In Section 5, we briefly review algebraic shifting, and then modify arguments from [Du2] to prove the key inequality. Section 6 proves Theorem 6.1, which merely consists of tying together the results of the previous three sections.

§2. Simplicial complexes

For any subset S of a simplicial complex K, let S_q denote the set of q-dimensional faces of S. In particular, K_q is the set of q-dimensional faces of K itself; context should distinguish between K_q, for the q-dimensional faces of K, and K_s, for a member of the filtration (2).

Let $\boldsymbol{k}$ be a field, fixed throughout the paper. The qth *Betti number* of a simplicial complex K is $\beta^q = \beta^q(K) = \dim_{\boldsymbol{k}} \widetilde{H}^q(K)$, where $\widetilde{H}^q(K)$ is the qth reduced cohomology group of K (with respect to $\boldsymbol{k}$). Recall that over a field $\boldsymbol{k}$, $\dim_{\boldsymbol{k}} \widetilde{H}^q(K;\boldsymbol{k}) = \dim_{\boldsymbol{k}} \widetilde{H}_q(K;\boldsymbol{k})$, so that Betti numbers measure reduced homology as well as reduced cohomology.

Definition. Let K be a $(d-1)$-dimensional simplicial complex on vertex set $[n] := \{1,\dots,n\}$. Let $V = \{e_1,\dots,e_n\}$, and let $\Lambda(\boldsymbol{k}V)$ denote the exterior algebra of the vector space $\boldsymbol{k}V$; it has a $\boldsymbol{k}$-vector space basis consisting of all the monomials $e_S := e_{i_0} \wedge \cdots \wedge e_{i_q}$, where $S = \{i_0 < \cdots < i_q\} \subseteq [n]$ (and $e_\emptyset = 1$). Note that $\Lambda(\boldsymbol{k}V) = \oplus_{q=-1}^{n-1} \Lambda^{q+1}(\boldsymbol{k}V)$ is a graded $\boldsymbol{k}$-algebra, and that $\Lambda^{q+1}(\boldsymbol{k}V)$ has basis $\{e_S\colon |S| = q+1\}$. Let $(I_K)_q$ be the subspace of $\Lambda^{q+1}(\boldsymbol{k}V)$ generated by the basis $\{e_S\colon |S| = q+1,\ S \notin K\}$. Then $I_K := \oplus_{q=-1}^{d-1}(I_K)_q$ is the homogeneous graded ideal of $\Lambda(\boldsymbol{k}V)$ generated by $\{e_S\colon S \notin K\}$. Let $\Lambda^q[K] := \Lambda^{q+1}(\boldsymbol{k}V)/(I_K)_q$. Then the graded quotient algebra $\Lambda[K] := \oplus_{q=-1}^{d-1}\Lambda^q[K] = \Lambda(\boldsymbol{k}V)/I_K$ is called the *exterior face ring* of K (over $\boldsymbol{k}$).

The exterior face ring is the exterior algebra analogue to the Stanley-Reisner face ring of a simplicial complex [St]. For $x \in \boldsymbol{k}V$, let $\widetilde{x}$ denote the image of x in $\Lambda[K]$. For $S \subseteq K$, let

$$\widetilde{S} = \operatorname{span}\{\widetilde{e}_F\colon F \in S\}.$$

As with $I = I_K$ above, I_q will denote the q-dimensional part of *any* homogeneous graded subspace I contained in $\Lambda[K]$.

It is not hard to verify (or see equation (3) below) that the usual coboundary operator $\delta\colon \Lambda^q[K] \to \Lambda^{q+1}[K]$ used to compute cohomology may be given by $\delta\colon \widetilde{x} \mapsto \widetilde{f} \wedge \widetilde{x}$, where $f = e_1 + \cdots + e_n$. However, it will be necessary (see Section 5) to use a more "generic" coboundary operator, which will not change cohomology. Let $\hat{\boldsymbol{k}} = \boldsymbol{k}(\alpha_{11}, \alpha_{12}, \dots, \alpha_{nn})$ be the field extension over $\boldsymbol{k}$ by n^2 transcendentals, $\{\alpha_{ij}\}_{1\le i,j\le n}$, algebraically independent over $\boldsymbol{k}$. We will consider $\Lambda[K]$ as being over $\hat{\boldsymbol{k}}$ instead of $\boldsymbol{k}$ from now on. We are, in effect, simply adjoining these α_{ij}'s to our field of coefficients.

For now, we will only need the first n transcendentals, $\alpha_{11},\dots,\alpha_{1n}$. Let $f_1 = \alpha_{11}e_1 + \cdots + \alpha_{1n}e_n$. Then define the *weighted coboundary* *operator* $\delta\colon \Lambda[K] \to \Lambda[K]$ by $\delta\colon \widetilde{x} \mapsto \widetilde{f}_1 \wedge \widetilde{x}$, so

$$(3) \qquad \delta(\widetilde{e}_S) = \widetilde{f}_1 \wedge \widetilde{e}_S = \sum_{j=1}^{n} \alpha_{1j}\widetilde{e}_j \wedge \widetilde{e}_S = \sum_{\substack{j \notin S \\ S\cup\{j\}\in K}} \pm\alpha_{1j}\widetilde{e}_{S\cup\{j\}}$$

(hence the name weighted coboundary operator). Betti numbers may be computed using this δ, *i.e.*, $\beta^q(K) = \dim_{\boldsymbol{k}}(\ker\delta)_q/(\operatorname{im}\delta)_q$ [BK, pp. 289–290].

§3. Spectral sequences

The filtration (2) in Section 1 naturally gives rise to a filtration of ideals in $\Lambda[K]$, as follows. For $0 \le s \le m$, define

$$Q^s = K \backslash K_s$$

so the ideals $\widetilde{Q}^s$ form a filtration

$$(4) \qquad \Lambda[K] = \widetilde{Q}^0 \supseteq \cdots \supseteq \widetilde{Q}^m = \widetilde{0} = I_K.$$

By *e.g.* [Sp, p. 493], there is a convergent spectral sequence E_r corresponding to this filtration. By this we mean that there is a sequence of pairs $\{(E_r, d_r)\}_{r\ge1}$, where: E_r is a bigraded vector space over a field $\boldsymbol{k}$; d_r is a differential on E_r of bidegree $(r, 1-r)$ (so $d_r\colon E_r^{s,t} \to E_r^{s+r,t-r+1}$); $H(E_r) := (\ker d_r/\operatorname{im} d_r) \cong E_{r+1}$; $E_1^{s,t} \cong \widetilde{H}^{s+t}(K_s \backslash K_{s-1})$; and E_∞ is associated to a filtration on $\widetilde{H}^*(K)$, in that $E_\infty^{s,t} \cong \ker(H^{s+t}(K) \to H^{s+t}(K_{s+1}))/\ker(H^{s+t}(K) \to H^{s+t}(K_s))$. For every $E_r^{s,t}$ expression in this section, the "total degree" $s+t$ is fixed, at say q, so we will suppress the "complementary degree" t, and write E_r^s to mean $E_r^{s,q-s}$ (s is called the "filtered degree"). Similarly, every subspace of $\Lambda[K]$ is understood to be just the q-dimensional component, and so we will write I to mean I_q. For further details on spectral sequences of filtrations, see, *e.g.*, [Sp, Section 9.1].

It is straightforward to verify that

$$E_r^s = \frac{Z_r^{s-1}}{Z_{r-1}^s + \delta Z_{r-1}^{s-r}} = \frac{Z_r^{s-1} + \widetilde{Q}^s}{\delta Z_{r-1}^{s-r} + \widetilde{Q}^s}$$

and $d_r = \delta$ form a spectral sequence corresponding to the filtration (4) as described above, where

$$Z_r^s = \{c \in \widetilde{Q}^s \colon \delta c \in \widetilde{Q}^{s+r}\}.$$

(The verification is analagous to that for the homology spectral sequence of a filtration [Sp, pp. 469–470].) Then, letting $r \to \infty$,

$$
\begin{aligned}
E_\infty^s &= \frac{Z_\infty^{s-1}}{Z_\infty^s + (\operatorname{im}\delta \cap \widetilde{Q}^{s-1})} \\
&= \frac{Z_\infty^{s-1} + \widetilde{Q}^s}{(\operatorname{im}\delta \cap \widetilde{Q}^{s-1}) + \widetilde{Q}^s},
\end{aligned} \tag{5}
$$

where

$$Z_\infty^s = \{c \in \widetilde{Q}^s \colon \delta c = \widetilde{0}\} = \widetilde{Q}^s \cap \ker\delta.$$

Lemma 3.1. *For the spectral sequence defined above,*

$$\dim E_\infty^{1,q-1} + \cdots + \dim E_\infty^{p,q-p} = \dim \frac{(\ker\delta + \widetilde{Q}^p)_q}{(\operatorname{im}\delta + \widetilde{Q}^p)_q}.$$

Proof. Recall that the total degree $s+t$ of every $E_r^{s,t}$ is fixed at q, as is the dimension of every subspace of $\Lambda[K]$, and so we suppress the q's in the proof.

By equation (5),

$$
\begin{aligned}
E_\infty^s &= \frac{(\ker\delta \cap \widetilde{Q}^{s-1}) + \widetilde{Q}^s}{(\operatorname{im}\delta \cap \widetilde{Q}^{s-1}) + \widetilde{Q}^s} \\
&= \frac{(\ker\delta + \widetilde{Q}^s) \cap \widetilde{Q}^{s-1}}{(\operatorname{im}\delta + \widetilde{Q}^s) \cap \widetilde{Q}^{s-1}}.
\end{aligned} \tag{6}
$$

The result now follows by an easy induction on p. For $p = 1$, by equation (6),

$$E_\infty^1 = \frac{(\ker\delta + \widetilde{Q}^1) \cap \widetilde{Q}^0}{(\operatorname{im}\delta + \widetilde{Q}^1) \cap \widetilde{Q}^0} = \frac{\ker\delta + \widetilde{Q}^1}{\operatorname{im}\delta + \widetilde{Q}^1}.$$

If $p > 1$, then

$$\begin{aligned}
\dim E^1_\infty + \cdots + \dim E^p_\infty &=^{(1)} (\dim E^1_\infty + \cdots + \dim E^{p-1}_\infty) + \dim E^p_\infty \\
&=^{(2)} \dim \frac{\ker \delta + \widetilde{Q}^{p-1}}{\operatorname{im} \delta + \widetilde{Q}^{p-1}} \\
&\qquad + \dim \frac{(\ker \delta + \widetilde{Q}^p) \cap \widetilde{Q}^{p-1}}{(\operatorname{im} \delta + \widetilde{Q}^p) \cap \widetilde{Q}^{p-1}} \\
&=^{(3)} \dim \frac{(\ker \delta + \widetilde{Q}^p) + \widetilde{Q}^{p-1}}{(\operatorname{im} \delta + \widetilde{Q}^p) + \widetilde{Q}^{p-1}} \\
&\qquad + \dim \frac{(\ker \delta + \widetilde{Q}^p) \cap \widetilde{Q}^{p-1}}{(\operatorname{im} \delta + \widetilde{Q}^p) \cap \widetilde{Q}^{p-1}} \\
&=^{(4)} \dim \frac{\ker \delta + \widetilde{Q}^p}{\operatorname{im} \delta + \widetilde{Q}^p}.
\end{aligned}$$

Equality $=^{(2)}$ above is by induction and equation (6), equality $=^{(3)}$ follows from $\widetilde{Q}^p \subseteq \widetilde{Q}^{p-1}$, and equality $=^{(4)}$ is a routine exercise in linear algebra (or see [Du2, Lemma 5.1]). Q.E.D.

§4. Near-cones

Let v be a vertex of a simplicial complex K. Let

$$\operatorname{del}_K v = \operatorname{del} v := \{F \in K\colon v \cup F \notin K\}$$

be the *deletion* of v (in K), let

$$\operatorname{lk}_K v = \operatorname{lk} v := \{F \in K\colon v \notin F,\ v \,\dot\cup\, F \in K\}$$

be the *link* of v (in K), and let the *star* of v (in K) be

$$v * \operatorname{lk} v = \{F \in K\colon v \cup F \in K\}$$

the cone over $\operatorname{lk} v$. Then K may be partitioned

$$K = (v * \operatorname{lk} v) \,\dot\cup\, \operatorname{del} v.$$

The link and star of v are subcomplexes of K.

We will say K is a *near-cone* with *apex* v if every face F in $\operatorname{del}_K v$ has its entire boundary $\{F \backslash w\colon w \in F\}$ contained in $v * \operatorname{lk}_K v$. In this case, every face of $\operatorname{del}_K v$ is a facet (*i.e.*, is maximal in K), since $v * \operatorname{lk} v$ is a subcomplex. If we contract the subcomplex $v * \mathrm{lk}\, v$ to v, what remains is a sphere for every face in $\mathrm{del}\, v$; therefore

$$\beta^q(K) = \#\{F \in \mathrm{del}_K\, v\colon \dim F = q\} \tag{7}$$

when K is a near-cone with apex v [BK, Theorem 4.3].

Lemma 4.1. *If K is a near-cone with apex v, then $\delta(\mathrm{lk}\, v) = \{\delta\widetilde{e}_F\colon F \in \mathrm{lk}\, v\}$ is a basis for* $\mathrm{im}\,\delta$.

Proof. The members of $\delta(\mathrm{lk}\, v)$ are linearly independent because if $F \in \mathrm{lk}\, v$ then $\delta\widetilde{e}_F$ has nontrivial support on $\widetilde{e}_{v\dot\cup F}$, but if $G \in \mathrm{lk}\, v$ and $G \neq F$, then $\delta\widetilde{e}_G$ has no support on $\widetilde{e}_{v\dot\cup F}$. Thus for each member of $\delta(\mathrm{lk}\, v)$ there is a face on which it alone has nontrivial support; linear independence follows immediately.

On the other hand, we will show that if $G \notin \mathrm{lk}\, v$, then $\delta\widetilde{e}_G$ is in the span of $\delta(\mathrm{lk}\, v)$. If $G \in \mathrm{del}\, v$, then $\delta\widetilde{e}_G = 0$, since G is a facet. The only possibility remaining is that $G = v \dot\cup F$ for $F \in \mathrm{lk}\, v$. In that case

$$\delta\widetilde{e}_F = \pm\alpha_{1v}\widetilde{e}_G + \sum_{\substack{w\neq v\\ F\dot\cup w\in K}} \pm\alpha_{1w}\widetilde{e}_{F\dot\cup w}$$

so

$$0 = \delta^2\widetilde{e}_F = \pm\alpha_{1v}\delta\widetilde{e}_G + \sum_{\substack{w\neq v\\ F\dot\cup w\in K}} \pm\alpha_{1w}\delta\widetilde{e}_{F\dot\cup w},$$

and so

$$\delta\widetilde{e}_G = \sum_{\substack{w\neq v\\ F\dot\cup w\in K}} \pm\left(\frac{\alpha_{1w}}{\alpha_{1v}}\right)\delta(F\dot\cup w).$$

Now, if $v \dot\cup (F \dot\cup w) \notin K$, then $F \dot\cup w \in \mathrm{del}\, v$, so $F \dot\cup w$ is a facet and so $\delta\widetilde{e}_{F\dot\cup w} = 0$. But if $v\dot\cup(F\dot\cup w) \in K$, then $F\dot\cup w \in \mathrm{lk}\, v$, so $\delta\widetilde{e}_{F\dot\cup w} \in \delta(\mathrm{lk}\, v)$. Thus $\delta\widetilde{e}_G$ is in the span of $\delta(\mathrm{lk}\, v)$. Q.E.D.

Lemma 4.2. *If K is a near-cone with apex v, then*

$$\ker\delta = \widetilde{D} + \mathrm{im}\,\delta,$$

where $D = \mathrm{del}_K\, v$.

Proof. Since every face in $D = \mathrm{del}_K\, v$ is a facet, $\widetilde{D} \subseteq \ker\delta$, so $\widetilde{D} + \mathrm{im}\,\delta \subseteq \ker\delta$.

By equation (7),

$$\dim(\ker\delta) - \dim(\mathrm{im}\,\delta) = \beta^*(K) = |\mathrm{del}\, v| = \dim\widetilde{D}.$$

Thus

$$\begin{aligned}\dim(\widetilde{D} + \operatorname{im}\delta) &= \dim\widetilde{D} + \dim(\operatorname{im}\delta) - \dim(\widetilde{D} \cap \operatorname{im}\delta)\\ &= \dim(\ker\delta) - \dim(\widetilde{D} \cap \operatorname{im}\delta).\end{aligned}$$

So now it only remains to show that

$$\widetilde{D} \cap \operatorname{im}\delta = 0. \tag{8}$$

To this end, recall from the proof of Lemma 4.1 that each $\delta\widetilde{e}_F$ in $\delta(\operatorname{lk} v)$ is the unique element of $\delta(\operatorname{lk} v)$ with nonzero support on $\widetilde{e}_{v\dot{\cup}F}$, but now note further that $v \mathbin{\dot{\cup}} F \notin \operatorname{del} v$. Thus any nonzero element of $\operatorname{im}\delta = \delta(\widetilde{\operatorname{lk} v})$ has nontrivial support outside $\operatorname{del} v$, which establishes equation (8), and hence the lemma. Q.E.D.

Lemma 4.3. *If $K = L \mathbin{\dot{\cup}} Q$ is a partition of the faces of a near-cone K into two disjoint subsets, then*

$$\dim\frac{(\ker\delta + \widetilde{Q})_q}{(\operatorname{im}\delta + \widetilde{Q})_q} = \#\{F \in L_q \colon v \notin F,\ v \mathbin{\dot{\cup}} F \notin K\}.$$

Proof. Again let $D = \operatorname{del}_K v$. Then

$$\begin{aligned}\frac{\ker\delta + \widetilde{Q}}{\operatorname{im}\delta + \widetilde{Q}} &= \frac{\widetilde{D} + \operatorname{im}\delta + \widetilde{Q}}{\operatorname{im}\delta + \widetilde{Q}} && \text{by Lemma 4.2}\\ &\cong \frac{\widetilde{D}}{\widetilde{D} \cap (\operatorname{im}\delta + \widetilde{Q})}\\ &= \frac{\widetilde{D}}{\widetilde{D} \cap \widetilde{Q}} && \text{by equation (8)}\\ &= \frac{\widetilde{D}}{\widetilde{D \cap Q}}.\end{aligned}$$

Thus,

$$\begin{aligned}\dim\frac{(\ker\delta + \widetilde{Q})_q}{(\operatorname{im}\delta + \widetilde{Q})_q} &= |D_q| - |(D \cap Q)_q|\\ &= |\operatorname{del}_K v \cap L_q|\\ &= \#\{F \in L_q \colon v \notin F,\ v \mathbin{\dot{\cup}} F \notin K\}.\end{aligned}$$

Q.E.D.

§5. Algebraic shifting

Algebraic shifting transforms a simplicial complex into a shifted simplicial complex with many of the same algebraic properties of the original complex. Algebraic shifting was introduced by Kalai in [Ka1]; our exposition is summarized from [BK] and included for completeness.

Definition. If $R = \{r_0 < \cdots < r_q\}$ and $S = \{s_0 < \cdots < s_q\}$ are $(q+1)$-subsets of $[n] = \{1, \ldots, n\}$, then:

- $R \leq_P S$ under the standard *partial order* if $r_i \leq s_i$ for all i; and
- $R <_L S$ under the *lexicographic order* if there is a j such that $r_j < s_j$ and $r_i = s_i$ for $i < j$.

Lexicographic order is a total order which refines the partial order.

Definition. A collection $\mathcal{C}$ of $(q+1)$-subsets of $[n]$ is *shifted* if $R \leq_P S$ and $S \in \mathcal{C}$ together imply that $R \in \mathcal{C}$. A simplicial complex Δ is *shifted* if the set of q-dimensional faces of Δ is shifted for every q.

It is not hard to see that shifted simplicial complexes are near-cones with apex 1.

Recall (see Section 2) that $\{\alpha_{ij}\}_{1 \leq i,j \leq n}$ are n^2 transcendentals adjoined to our field of coefficients.

Definition (Kalai). For $1 \leq i \leq n$, let

$$f_i = \sum_{j=1}^{n} \alpha_{ij} e_j,$$

so $\{f_1, \ldots, f_n\}$ forms a "generic" basis of $\hat{\boldsymbol{k}}V$. (Note this is consistent with our definition of f_1 in Section 2.) Define $f_S := f_{i_0} \wedge \cdots \wedge f_{i_q}$ for $S = \{i_0 < \cdots < i_q\}$ (and set $f_\emptyset = 1$). Let

$$\Delta(K, \boldsymbol{k}) := \{S \subseteq [n] \colon \widetilde{f_S} \notin \operatorname{span}\{\widetilde{f_R} \colon R <_L S\}\}$$

be the *algebraically shifted complex* obtained from K; we will write $\Delta(K)$ instead of $\Delta(K, \boldsymbol{k})$ when the field is understood to be $\boldsymbol{k}$. In other words, the $(q+1)$-subsets of $\Delta(K)$ can be chosen by listing all the $(q+1)$-subsets of $[n]$ in lexicographic order and omitting those that are in the span of earlier subsets on the list, modulo I_K and with respect to the f-basis.

The algebraically shifted complex $\Delta(K)$ is (as its name suggests) shifted, and is independent of the numbering of the vertices of K [BK, Theorem 3.1].

Recall from Section 1 that if $L \subseteq K$ is a pair of simplicial complexes, then $\Delta(L) \subseteq \Delta(K)$. Thus for $Q = K \backslash L$, we may define $\Delta(Q) = \Delta(K) \backslash \Delta(L)$.

Lemma 5.1. *Let $L \subseteq K$ be a pair of simplicial complexes and $Q = K \backslash L$. Then*

$$\dim \frac{(\ker \delta + \widetilde{Q})_q}{(\operatorname{im} \delta + \widetilde{Q})_q} \geq \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \mathbin{\dot\cup} F \notin \Delta(K)\}.$$

Proof. This is implicit in the proof of [Du2, Theorem 5.2]. As it is not stated there explicitly, we reproduce here some of the details. From [Du2, Lemma 4.4]

$$\dim(\operatorname{im} \delta \cap \widetilde{Q})_{q+1} \leq \#\{F \in \Delta(K)_q \colon 1 \notin F,\ 1 \mathbin{\dot\cup} F \in \Delta(Q)\},$$

and from [Du2, Lemma 4.5]

$$\dim(\delta \widetilde{Q})_{q+1} \geq \#\{F \in \Delta(Q)_q \colon 1 \notin F,\ 1 \mathbin{\dot\cup} F \in \Delta(Q)\}.$$

Then, since $L = K \backslash Q$,

$$\dim \frac{(\operatorname{im} \delta \cap \widetilde{Q})_{q+1}}{(\delta \widetilde{Q})_{q+1}} \leq \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \mathbin{\dot\cup} F \in \Delta(Q)\}.$$

By equations (1) and (7), respectively,

$$\beta^q(L) = \beta^q(\Delta(L)) = \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \mathbin{\dot\cup} F \notin \Delta(L)\}.$$

But, with the notation $\delta^{-1}\widetilde{Q} := \{\widetilde{x} \in \Lambda[K] \colon \delta\widetilde{x} \in \widetilde{Q}\}$, we also have

$$\begin{aligned}
\beta^q(L) &= \dim \frac{(\delta^{-1}\widetilde{Q})_q}{(\operatorname{im} \delta + \widetilde{Q})_q} && \text{by [Du1, Lemma 3.3]} \\
&= \dim \frac{(\delta^{-1}\widetilde{Q})_q}{(\ker \delta + \widetilde{Q})_q} + \dim \frac{(\ker \delta + \widetilde{Q})_q}{(\operatorname{im} \delta + \widetilde{Q})_q} \\
&= \dim \frac{(\operatorname{im} \delta \cap \widetilde{Q})_{q+1}}{(\delta\widetilde{Q})_{q+1}} + \dim \frac{(\ker \delta + \widetilde{Q})_q}{(\operatorname{im} \delta + \widetilde{Q})_q} && \text{by [Du1, Lemma 3.6],}
\end{aligned}$$

and so

$$\begin{aligned}
\dim \frac{(\ker \delta + \widetilde{Q})_q}{(\operatorname{im} \delta + \widetilde{Q})_q} &= \beta^q(L) - \dim \frac{(\operatorname{im} \delta \cap \widetilde{Q})_{q+1}}{(\delta\widetilde{Q})_{q+1}} \\
&\geq \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \mathbin{\dot\cup} F \notin \Delta(L)\} \\
&\quad - \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \mathbin{\dot\cup} F \in \Delta(Q)\} \\
&= \#\{F \in \Delta(L)_q \colon 1 \notin F,\ 1 \mathbin{\dot\cup} F \notin \Delta(K)\}.
\end{aligned}$$

Q.E.D.

§6. Proof of Main Theorem

Given a filtration $\mathcal{K}$ of a simplicial complex K, let $E_r^{s,t}$ refer to the terms of the corresponding spectral sequence given in Section 3, and let

$$e^{s,t}(\mathcal{K}) = \dim E_\infty^{s,t}(\mathcal{K}).$$

Theorem 6.1. *For all p, q,*

$$e^{1,q-1}(\mathcal{K}) + \cdots + e^{p,q-p}(\mathcal{K}) \geq e^{1,q-1}(\Delta(\mathcal{K})) + \cdots + e^{p,q-p}(\Delta(\mathcal{K})).$$

Proof. For $0 \leq s \leq m$, let $\Sigma^s = \Delta(K) \backslash \Delta(K_s)$, so

$$\Lambda[\Delta(K)] = \widetilde{\Sigma}^0 \supseteq \widetilde{\Sigma}^1 \supseteq \cdots \supseteq \widetilde{\Sigma}^m = I_{\Delta(K)}$$

is the filtration of ideals of $\Lambda[\Delta(K)]$ corresponding to the filtration $\Delta(\mathcal{K})$. By Lemmas 3.1 and 5.1,

$$\begin{aligned} e^{1,q-1}(\mathcal{K}) + \cdots + e^{p,q-p}(\mathcal{K}) &= \dim \frac{(\ker \delta_K + \widetilde{Q}^p)_q}{(\operatorname{im} \delta_K + \widetilde{Q}^p)_q} \\ &\geq \#\{F \in \Delta(K_p)_q \colon 1 \notin F,\ 1 \dot\cup F \notin \Delta(K)\}. \end{aligned}$$

On the other hand, because $\Delta(K)$ is shifted and hence a near-cone, Lemmas 3.1 and 4.3 give

$$\begin{aligned} e^{1,q-1}(\Delta(\mathcal{K})) + \cdots + e^{p,q-p}(\Delta(\mathcal{K})) &= \dim \frac{(\ker \delta_{\Delta(K)} + \widetilde{\Sigma}^p)_q}{(\operatorname{im} \delta_{\Delta(K)} + \widetilde{\Sigma}^p)_q} \\ &= \#\{F \in \Delta(K_p)_q \colon 1 \notin F,\ 1 \dot\cup F \notin \Delta(K)\}. \end{aligned}$$

Q.E.D.

Note that $e^{1,q-1}(\mathcal{K}) + \cdots + e^{m,q-m}(\mathcal{K}) = \beta^q(K)$, which, by equation (1) is unchanged under algebraic shifting. Thus, Theorem 6.1 says that algebraic shifting puts less of the fixed sum of the $e^{s,q-s}$'s into the earlier part of the filtration, and hence puts more into the later part. In particular,

$$\begin{aligned} &e^{p+1,q-p-1}(\mathcal{K}) + \cdots + e^{m,q-m}(\mathcal{K}) \\ &\qquad \leq e^{p+1,q-p-1}(\Delta(\mathcal{K})) + \cdots + e^{m,q-m}(\Delta(\mathcal{K})). \end{aligned}$$

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