

Principal blocks with extra-special defect groups of order 27

Yoko Usami

§1. Introduction

Let G be a finite group and p be a prime number. Let b be a p -block of G , P be a defect group of b and $k(b)$ (respectively, $l(b)$) be the number of irreducible ordinary characters (respectively, irreducible Brauer characters) in b . Suppose that

- two blocks b and b' of finite groups G and G' respectively,*
(1) *have the common defect group P and their Brauer categories $Br_{b,p}(G)$ and $Br_{b',p}(G')$ are equivalent.*

(See [FH] for Brauer categories.) When we consider only principal p -blocks, their defect groups are Sylow p -subgroups and having the same Brauer category is equivalent to having the same p -local structure. See the definition in section 4 in [R] : Finite groups G and H have the same p -local structure if they have a common Sylow p -subgroup P such that whenever Q_1 and Q_2 are subgroups of P and $f : Q_1 \rightarrow Q_2$ is an isomorphism, then there is an element $g \in G$ such that $f(x) = x^g$ for all $x \in Q_1$ if and only if there is an element $h \in H$ such that $f(x) = x^h$ for all $x \in Q_1$.

Under condition (1) there is a question whether we have

- (2) $k(b) = k(b')$ and $l(b) = l(b')$

or not. We have a following conjecture.

Conjecture 1. *When b and b' are principal blocks satisfying condition (1), the equalities in (2) hold.*

When P is an abelian group, it is known that a block b of G and its Brauer correspondent $Br_P(b)$ in $N_G(P)$ have the same Brauer category

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(Proposition 4.21 in [AB]), and Broué conjectured that they are derived equivalent (respectively, isotypic). See Conjecture 6.1 and Question 6.2 in [Br2]. Note that each of these conjectures implies that we have

$$(3) \quad k(b) = k(Br_P(b)) \quad \text{and} \quad l(b) = l(Br_P(b))$$

for any block b with abelian defect group P . As is stated in [Br2] Broué's conjectures above do not necessarily hold when P is not an abelian group. The principal 2-block b of any one of Suzuki groups $Sz(q)$ and its Brauer correspondent have the same Brauer category (actually, fusion of P is controlled by its normalizer, since Sylow 2-subgroups are T.I. sets), but they are not derived equivalent nor isotypic ; nevertheless (3) holds for them (cf. Consequences 5 and 7 in [A]). Here we have to add one more remark. M. Kiyota pointed out that a semidirect product of an elementary abelian 3-group $Z_3 \times Z_3$ of order 9 by a quaternion group of order 8 whose unique involution acts on $Z_3 \times Z_3$ trivially, has only two 3-blocks (i.e. the principal block b_0 and the other block b_1) and their Brauer categories are equivalent to each other but we have $l(b_0) \neq l(b_1)$.

In this paper we fix P as an extra-special group of order 27 and of exponent 3, and consider principal 3-blocks b having P as a defect group and check Conjecture 1. Note that in this case having the same Brauer category implies having the same inertial quotient $E (= N_G(P)/PC_G(P)$ here) and the same fusion of P . At any rate, using the classification of finite simple groups, we determine $k(b), l(b)$ and $k_0(b)$ completely and proves that Conjecture 1 is true for such blocks, and consequently we prove that Dade's conjecture of ordinary form holds for b . (Here $k_0(b)$ is the number of irreducible ordinary characters in b of height zero.)

When the author visited l'Université Paris 7, Lluís Puig suggested an idea of using his construction of characters as functions on local pointed elements which can be found in Corollary 4.4, Theorem 5.2 and Theorem 5.6 in [P]. The author uses his idea to prove Theorem 1 below.

In the following we denote a cyclic group of order m by Z_m , a quaternion group of order 8 by Q_8 , a dihedral group of order 8 by D_8 and a semidihedral group of order 16 by SD_{16} respectively.

Theorem 1. *Let b be the principal 3-block of a finite group G with an extra-special defect group P of order 27 and of exponent 3. Let E be the inertial quotient of b (i.e. $E = N_G(P)/PC_G(P)$) and let u be a non-trivial element in $Z(P)$. Then we have the following.*

- (1) *When $N_G(P) \subseteq C_G(Z(P))$, fusion of P in G is controlled by $N_G(P)$ and one of the following holds :*

- (i) If $E = 1$, then b is 3-nilpotent, $k(b) = 11$, $k_0(b) = 9$ and $l(b) = 1$.
- (ii) If $E \cong Z_2$, then $k(b) = 10$, $k_0(b) = 6$ and $l(b) = 2$. (In this case E acts on $P/Z(P)$ fixed-point-freely.)
- (iii) If $E \cong Z_4$, then $k(b) = 14$, $k_0(b) = 6$ and $l(b) = 4$.
- (iv) If $E \cong Q_8$, then $k(b) = 16$, $k_0(b) = 6$ and $l(b) = 5$.
- (2) When $N_G(P) \not\subseteq C_G(Z(P))$, E is isomorphic with either $Z_2, Z_2 \times Z_2, Z_8, D_8$ or SD_{16} and we have an estimate of $k(b)$ as below according to E and the number of conjugacy classes of elements of order 3. When $E \cong Z_2$, E does not act on $P/Z(P)$ fixed-point-freely. In each case $k(b) - l(b)$ takes a constant value. When $E \cong Z_8$, each case is further divided into two subcases according to fusion of a basic set of $C_G(u)$ in the extended centralizer $C_G^*(u)$ ($= \{g \in G \mid u^g = u \text{ or } u^{-1}\}$). The subcase where each element of a basic set of $C_G(u)$ is fixed by $C_G^*(u)$ corresponds to subcase 1. Otherwise it is subcase 2.
- (i) Suppose that $E \cong Z_2$.
- (i)-(1) If fusion of P is controlled by $N_G(P)$, then $P - \{1\}$ consists of 6 classes and $k(b) - l(b) = 8$ and $k(b) = 10$.
- (i)-(2) Otherwise, $P - \{1\}$ consists of 5 classes and $k(b) - l(b) = 7$ and $9 \leq k(b) \leq 11$.
- (ii) Suppose that $E \cong Z_2 \times Z_2$. Then one of the following holds.
- (ii)-(1) If fusion of P is controlled by $N_G(P)$, then $P - \{1\}$ consists of 4 classes and $k(b) - l(b) = 7$ and $k(b) = 11$.
- (ii)-(2) $P - \{1\}$ consists of 3 classes, $k(b) - l(b) = 5$ and $8 \leq k(b) \leq 11$.
- (ii)-(3) $P - \{1\}$ consists of 3 classes, $k(b) - l(b) = 6$ and $10 \leq k(b) \leq 12$.
- (ii)-(4) $P - \{1\}$ consists of 2 classes, $k(b) - l(b) = 4$ and $7 \leq k(b) \leq 12$.
- (ii)-(5) $P - \{1\}$ consists of 2 classes, $k(b) - l(b) = 3$ and $6 \leq k(b) \leq 11$.
- (ii)-(6) $P - \{1\}$ consists of 1 class, $k(b) - l(b) = 2$ and $5 \leq k(b) \leq 18$.
- (iii) Suppose that $E \cong Z_8$.
- (iii)-(1) If fusion of P is controlled by $N_G(P)$, then $P - \{1\}$ consists of 2 classes and $k(b) - l(b) = 5$. In subcase 1, $8 \leq k(b) \leq 14$. In subcase 2, $8 \leq k(b) \leq 12$.
- (iii)-(2) Otherwise, $P - \{1\}$ consists of 1 class and $k(b) - l(b) = 4$. In subcase 1, $8 \leq k(b) \leq 18$. In subcase 2, $7 \leq k(b) \leq 15$.

- (iv) Suppose that $E \cong D_8$. Then one of the following holds.
- (iv)-(1) If fusion of P is controlled by $N_G(P)$, then $P-\{1\}$ consists of 3 classes, $k(b)-l(b)=8$ and $k(b)=13$.
 - (iv)-(2) $P-\{1\}$ consists of 2 classes, $k(b)-l(b)=6$ and $9 \leq k(b) \leq 13$.
 - (iv)-(3) $P-\{1\}$ consists of 1 class, $k(b)-l(b)=4$ and $7 \leq k(b) \leq 15$.
- (v) Suppose that $E \cong SD_{16}$.
- (v)-(1) If fusion of P is controlled by $N_G(P)$, then $P-\{1\}$ consists of 2 classes, $k(b)-l(b)=7$ and $10 \leq k(b) \leq 15$.
 - (v)-(2) Otherwise, $P-\{1\}$ consists of 1 class, $k(b)-l(b)=5$ and $7 \leq k(b) \leq 14$.

Using the classification of finite simple groups we obtain the following theorem. As is well known, we can assume that $O_{p'}(G)=1$ when we treat the principal p -block of G .

Theorem 2. (Using the classification of finite simple groups.) Let G be a finite group with $O_3(G)=1$ having an extra-special Sylow 3-subgroup P of order 27 and of exponent 3. Let M be a minimal normal subgroup of G . Then one of the following holds :

- (i) $M \cong Z_3$ and $Z(P)$ is a normal subgroup of G and fusion of P in G is controlled by $N_G(P)$. As for the principal 3-block b , $k(b)$ and $l(b)$ are uniquely determined according to its inertial quotient.
- (ii) $M \cong Z_3 \times Z_3$ and G/M is embedded in $GL(2, 3)$. In particular, G is 3-solvable.
- (iii) $M \cong PSL(3, q)$ where $q \equiv 4, 7 \pmod{9}$. Furthermore we have

$$PGL(3, q) \subseteq G \subseteq Aut(PSL(3, q))$$

- (iv) $M \cong PSU(3, q^2)$ where $q \equiv 2, 5 \pmod{9}$. Furthermore we have

$$PGU(3, q^2) \subseteq G \subseteq Aut(PSU(3, q^2)).$$

- (v) $M \cong M_{24}$, Ru or J_4 . Furthermore $G=M$.
- (vi) $M \cong PSL(3, 3)$, $PSU(3, 3^2)$, ${}^2F_4(2)'$, M_{12} , J_2 or He . Furthermore $G=M$ or $Aut(M)$.
- (vii) $M \cong G_2(q)$ where $q \equiv 2, 4, 5, 7 \pmod{9}$. Furthermore $M \subseteq G \subseteq Aut(M)$.
- (viii) $M \cong {}^2F_4(q)$ where $2^{2m+1} = q \equiv 2, 5 \pmod{9}$. Furthermore $M \subseteq G \subseteq Aut(M)$.

The number $k(b)$ in case of $N_G(P) \subseteq C_G(Z(P))$ (see Theorem 1 (2)) is uniquely determined by E as follows: If $E \cong Z_2$ (respectively $Z_2 \times Z_2$, Z_8 , D_8 and SD_{16}), then $k(b) = 10$ (respectively, 11, 13, 13 and 14). When $N_G(P) \not\subseteq C_G(Z(P))$, we have always $k_0(b) = 9$. Furthermore, Dade's conjecture of ordinary form holds for b in any case. The above groups in (ii) through (viii) fall into the cases described in Theorem 1 (2) as follows. The numbers in the statements below correspond to those in Theorem 1 (2). The semidirect product of $Z_3 \times Z_3$ by $SL(2, 3)$, some groups in (iii) above and $PGU(3, q^2) \cdot (\text{odd order})$ with $q \equiv 2, 5 \pmod{9}$ satisfy (i)-(2). The semidirect product of $Z_3 \times Z_3$ by $GL(2, 3)$, all the remaining groups in (iii) above and $PGU(3, q^2) \cdot (\text{even order})$ with $q \equiv 2, 5 \pmod{9}$ satisfy (ii)-(2). $PSL(3, 3)$ and M_{12} satisfy (ii)-(5). $PSU(3, 3^2)$ and J_2 satisfy (iii)-(1). M_{24} , $Aut(M_{12})$, $Aut(PSL(3, 3))$, He and $Aut(He)$ satisfy (iv)-(2). ${}^2F'_4(2)$ satisfies (iv)-(3). $Aut(PSU(3, 3^2))$, $Aut(J_2)$ and all the groups in (vii) above satisfy (v)-(1). Ru , J_4 and all the groups in (viii) above satisfy (v)-(2).

§2. Remarks on Theorem 1

(1) After the author obtained Theorem 1, Masao Kiyota told the author that several years ago he already determined $k(b)$, $k_0(b)$ and $l(b)$ for principal blocks b when $N_G(P) \subseteq C_G(Z(P))$ by Brauer and Olsson's method using the orthogonality relation between columns of generalized decomposition matrix.

(2) Outline of the proof is as follows. First, list up all possible Broué's (or Alperin's) conjugation families for b -subpairs (with an aid of 3-strongly embedded subgroups) in order to determine fusion of b -subpairs in G ([Br1, CP]). This work means that we list up all possible Brauer categories as in [CP]. Note that when b is a principal p -block, b -subpairs are equivalent to p -subgroups. Second, collect information about blocks b_Q such that

$$(1, b) \not\subseteq (Q, b_Q) \subseteq (P, e),$$

where (P, e) is a fixed maximal b -subpair. Third, construct a Z -basis of generalized characters in b which vanish on 3-regular elements. Here we apply L.Puig's Theorem 5.6 in [P], where he gave some equivalent conditions of a function on local pointed elements to be a generalized character. Fourthly, determine the decomposition of each character in the above Z -basis into irreducible characters in order to know $k(b)$. It is known that any irreducible character in b appears in some generalized character in this Z -basis. In order to determine these decompositions the

author used a computer and also checked the elementary divisors of Cartan matrices by a computer. Unfortunately, when $N_G(P) \not\subseteq C_G(Z(P))$, we can not determine $k(b)$ uniquely. There are huge number of possible decompositions. But, as for $k(b)$, it seems that we can get almost the same estimate of $k(b)$ as this by hand.

(3) When E is of order 2, either G has a normal subgroup of index 3, or G is a 3-solvable group of 3-length 1 by S.D. Smith and A.P. Tyrer's theorem in [ST].

§3. Remarks on Theorem 2

(1) Using the strong assumption that $Z(P) \triangleleft G$, $k(b)$ in (i) is determined. Here we already use the classification of finite simple groups to determine the number of irreducible ordinary characters in the principal 3-block with an elementary abelian defect group of order 9 and with the cyclic inertial quotient of order 8.

(2) If G is a 3-solvable group with $O_{3'}(G) = 1$ and has an extraspecial Sylow 3-subgroup of order 27 and of exponent 3, then G is completely determined, that is, *either* the semidirect product of P and a group E isomorphic with 1 , Z_2 , $Z_2 \times Z_2$, Z_4 , Q_8 , D_8 , or SD_{16} or the semidirect product of $Z_3 \times Z_3$ by $SL(2, 3)$ or $GL(2, 3)$ (with faithful actions). (cf. Proposition 53.4 in [Ka] or [Ko]).

(3) It is not easy to choose the irreducible characters in b among all irreducible characters in G when G belongs to one of infinite series in (iii), (iv), (vii) and (viii). Fortunately, any nonprincipal 3-block of a simple group in these infinite series has some proper subgroup of P as a defect group. So using the estimate of $k(b)$ in Theorem 1 and the known facts on the number of irreducible ordinary characters in other 3-blocks and some more information about b itself, we determine $k(b)$ effectively in these cases. The author thanks Ken-ichi Shinoda and Meinolf Geck for information about ${}^2F_4(q)$.

(4) In order to prove Dade's conjecture in this case, we consider the set of G -conjugacy classes of radical 3-chains as the disjoint union of two subsets, one of which consists of classes of chains whose final subgroups are defect groups of the principal blocks of the normalizers of the chains and the other consists of the rest. There is a bijection from the former subset to the latter given by the Brauer correspondence between the corresponding principal blocks, sending a class of chains of length m into that of length $m-1$. Then by cancellation we get the conclusion (cf. 2.3 in [U1]).

§4. Perfect isometries and Morita equivalences

Having the same p -local structure does not always guarantee a derived category equivalence between the principal p -blocks (see counter examples in §1). But the author thinks that we can still expect something. Recall Broué's theorem :

Theorem 3 (Broué, Theorem3.1 [Br2]). *If two blocks are derived category equivalent, then there is a perfect isometry between these blocks.*

In view of this theorem, we can expect a derived equivalence between blocks if there exists a perfect isometry between them, although it is not proved that they are equivalent. In any case, it is meaningful to check whether a perfect isometry exists, as the first step towards checking the existence of a derived equivalence. The author and her student M.Nakabayashi did it in the following cases. (cf. Theorem 2 and [N]).

Proposition 4. *The groups in (i) (respectively (ii), (iii), (iv), (v), (vi) and (vii)) below have the same 3-local structure and there is a perfect isometry between the principal 3-blocks of any two of them.*

- (i) $PSU(3, 3^2)$, J_2 .
- (ii) $PSL(3, 3)$, M_{12} .
- (iii) M_{24} , He , $Aut(He)$.
- (iv) $Aut(M_{12})$, $Aut(PSL(3, 3))$.
- (v) Ru , J_4 .
- (vi) *the semidirect product of $Z_3 \times Z_3$ by $SL(2, 3)$, $PGU(3, q^2)$ with $q \equiv 2, 5 \pmod{9}$, $PGL(3, q)$ with $q \equiv 4, 7 \pmod{9}$.*
- (vii) $G_2(q)$ with q a power of 2 and $q \equiv 2, 4, 5, 7 \pmod{9}$.

Proposition 5. *The groups in (i)' (respectively (ii)', (iii)', (iv)', (v)' and (vi)') have the same 3-local structure, but there is no perfect isometry between their principal 3-blocks which sends the trivial character to the trivial character. Here P is the extra-special group of order 27 and of exponent 3.*

- (i)' *the semidirect product of P by Z_8 , $PSU(3, 3^2)$,*
- (ii)' M_{24} , $Aut(M_{12})$
- (iii)' Ru , ${}^2F_4(2)$
- (iv)' $G_2(2)$, $Aut(J_2)$
- (v)' $Aut(J_2)$, *the semidirect product of P by SD_{16} with the faithful action*
- (vi)' $G_2(4)$, *the semidirect product of P by SD_{16} with the faithful action.*

On the other hand, there are Koshitani and Kunugi's results on the principal 3-blocks of $PSU(3, q^2)$ and $PSL(3, q)$ with elementary abelian

defect groups of order 9 ([KK], [Ku]). Based on them we have got the following theorem.

Theorem 6 (N. Kunugi and Y. Usami [KU], [U2]). *The principal 3-blocks of all the groups in (i) (respectively (ii), (iii) and (iv)) below are Morita equivalent.*

- (i) $PGU(3, q^2)$ defined over the finite field $GF(q^2)$ satisfying $q \equiv 2, 5 \pmod{9}$.
- (ii) $PGL(3, q)$ satisfying $q \equiv 4, 7 \pmod{9}$.
- (iii) $SU(3, q^2)$ defined over the finite field $GF(q^2)$ satisfying $q \equiv 2, 5 \pmod{9}$.
- (iv) $SL(3, q)$ satisfying $q \equiv 4, 7 \pmod{9}$.

Moreover, let q be a power of 2 and satisfying $q \equiv 2$ or $5 \pmod{9}$. Then the author and M. Nakabayashi have almost finished proving that the principal 3-blocks of $G_2(q)$ and $G_2(2)$ are Morita equivalent to each other.

For the characters of groups in Theorem 2, see the following:

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*Department of Mathematics
Ochanomizu University
Tokyo 112-8610, Japan*