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Adele Geometry of Numbers

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Dedicated to Professors Ichiro Satake and Takashi Ono

This is a historical and expository account on adele geometry. The word *adele geometry* appeared in the lectures ([W4], 1959–1960) by A. Weil which were stimulated by works of Siegel–Tamagawa on quadratic forms. Our main concern here is to exhibit the strings of thoughts in the development of this topic originated from Minkowski's *geometry of numbers*. The subject was originally related to integral geometry and some diophantine problems, and we discuss such aspects in adele geometry on homogeneous spaces.

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§1. From geometry of numbers to adele geometry

1.1. *Minkowski's geometry of numbers.*

Following the footsteps of Gauss, Dirichlet, Eisenstein and Hermite, Minkowski developed the theory of quadratic forms and created the *geometry of numbers.* Suppose that f is a positive definite quadratic form on $\mathbb{R}^n$ whose determinant is assumed to be 1 for simplicity. Hermite showed, as a consequence of his reduction theory, the inequality

$$m(f) := \min_{x \in \mathbb{Z}^n \setminus \{0\}} f(x) \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}}.$$

The function $m(f)$ of f is bounded and $\mu_n := \max_f m(f)$ is called the *Hermite constant.* Minkowski improved this upper bound by a simple geometric idea which is the basis of his geometry of numbers. Consider the ellipsoid $B = \{x \in \mathbb{R}^n \colon f(x) < c\}$ for some $c > 0$. It is a symmetric convex body. Then, Minkowski's *first convex body theorem* ([M, VIII], 1891) says that

if the Euclidean volume $\mathrm{vol}(B) > 2^n$*, then* B *contains an integral point other than the origin.*

Since $\mathrm{vol}(B) = c^{n/2} V_n$, where V_n stands for the volume of the n-dimensional unit ball, we have

$$m(f) \leq \frac{4}{V_n^{2/n}}.$$

Minkowski then introduced the notion of *successive minima* for any convex body to deepen the first theorem. Using the above notations, the i-th successive minima of f is defined by

$$\lambda_i := \inf\{\lambda > 0 \colon \lambda B \text{ contains } i \text{ linearly independent integral points}\}$$

for $1 \leq i \leq n$ ($\lambda_1^2 = m(f)$). Minkowski's *second fundamental theorem* ([M, GZ], 1896) is then stated as

$$\frac{2^n}{n!} \leq \lambda_1 \cdots \lambda_n \mathrm{vol}(B) \leq 2^n.$$

It should be noted that Minkowski's convex body theorem can be seen as an analogue of Riemann-Roch theorem for an algebraic curve and has basic applications in algebraic number theory such as the finiteness of an ideal class group and so on.

Related to the lower bound for the Hermite constant, Minkowski asserted, in his letter to Hermite ([M, XI], 1893), the following:

Suppose B is an n-dimensional star body whose volume is less than $\zeta(n)$, where $\zeta(s)$ is the Riemann zeta function. Then, there is a lattice L of determinant 1 *such that $B \cap L = \{0\}$.*

This yields immediately the following inequality

$$\mu_n \geq \left(\frac{2\zeta(n)}{V_n}\right)^{\frac{2}{n}} .$$

Minkowski's assertion was proved by Hlawka about fifty years later.

In his reduction theory of quadratic forms ([M, XXI], 1905), Minkowski computed the volume of the set of all positive definite quadratic forms with determinant < 1. Namely, he gave a formula for the volume of $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ in terms of the special values of the Riemann zeta function.

Finally, we should mention Minkowski's two contributions to the theory of quadratic forms, preceding the geometry of numbers. One is the local-global principle on the equivalence of quadratic forms ([M, VII], 1890), which was completed by Hasse ([Ha], 1924) over any number fields based on the notion of a p-adic number field introduced by Hensel (1909). The other is the mass (weight) formula for a genus of a positive definite quadratic form ([M, IV], 1885), which was vastly developed by Siegel. This is our next focus.

1.2. *Siegel's main theorem and mean value theorem.*

Siegel's main theorem on quadratic forms is a precise quantitative version of the Hasse principle. Let S be a positive definite symmetric integral $n \times n$-matrix, $n \geq 3$. Consider the diophantine equation

$${}^tXSX = S\,,$$

which defines a $\frac{1}{2}n(n-1)$-dimensional affine variety over $\mathbb{Z}$. For a prime p, set

$$\alpha_p(S) := \lim_{e \longrightarrow \infty} \frac{[\{X \in M_n(\mathbb{Z}/p^e\mathbb{Z})\colon {}^tXSX \equiv S \mod p^e\}]}{2p^{\frac{1}{2}n(n-1)e}}\,,$$

where $[*]$ stands for the cardinality of a finite set $*$. For an infinite prime ∞, $\alpha_\infty(S)$ is defined as follows. Let U be a neighhborhood of S

in the space of positive definite symmetric matrices of size n and $\mathrm{vol}(U)$ denotes its Euclidean volume. Then, set

$$\alpha_\infty(S) := \lim_{U \to S} \frac{\mathrm{vol}(\{X \in M_n(\mathbb{R}) \colon {}^tXSX \in U\})}{\mathrm{vol}(U)}.$$

Here, the product $\prod_p \alpha_p(S)$ and $\alpha_\infty(S)$ are essentially the singular series and integral respectively in the work of Hardy–Littlewood on the Waring problem ([H-L], 1920). Compared with Hardy–Littlewood's asymptotic formula, Siegel's formula gives an accurate balance between the size of integral solutions and these local densities. For a complete set $S_1, \cdots, S_h$ of representatives of proper classes in the genus of S, let $E(S_i)$ denote the order of the unit group $SO(S_i)(\mathbb{Z})$. Then, the mass formula of Siegel ([S1], 1935) is stated as

$$\sum_{i=1}^{h} \frac{1}{E(S_i)} = 2 \left(\prod_{\text{all } v} \alpha_v(S) \right)^{-1}.$$

In fact, Siegel showed more general formulas including the case that S is indefinite where $1/E(S_i)$ is replaced by the volume of $SO(S_i)(\mathbb{R}) / SO(S_i)(\mathbb{Z})$ ([S2], 1936, 37, [S3], 1944).

After Hlawka proved Minkowski's assertion ([Hl], 1944), Siegel refined Hlawka's inequality to the equality ([S4], 1945). Let dg be an invariant volume element on $SL_n(\mathbb{R})$ normalized by $\mathrm{vol}_{dg}(SL_n(\mathbb{R}) / SL_n(\mathbb{Z})) = 1$ and dx be the Euclidean measure on $\mathbb{R}^n$. Denote the set of all primitive integral vectors by P. Then, Siegel's mean value theorem asserts: *for a compactly-supported continuous function φ on $\mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} \varphi(x)dx = \zeta(n) \int_{SL_n(\mathbb{R})/SL_n(\mathbb{Z})} \sum_{z \in P} \varphi(gz)dg.$$

This theorem gives the most satisfactory explanation to both Minkowski-Hlawka's theorem and Minkowski's computation of the volume of the space of positive definite quadratic forms with determinant ≤ 1.

Here, we wish to explain how the mean value theorem can be seen as a formula in integral geometry. Set $\mathcal{L} = \{g(P) \colon g \in SL_n(\mathbb{R})\}$ that is identified with the space of all lattices of determinant 1, namely $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$. Let Z be the space $\{(L, x) \in \mathcal{L} \times \mathbb{R}^n \colon x \in L\}$ describing the incidence relation and look at the diagram:

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr, "p"] & \\ SL_n(\mathbb{R})/SL_n(\mathbb{Z}) = \mathcal{L} & & \mathbb{R}^n \xrightarrow{\varphi} \mathbb{R} \end{tikzcd}$$

Then, compute the integral

$$I=\int_Z (p^*\varphi)(L,x)d(L,x)=\int_Z \varphi(x)d(g,x)$$

in two ways, noting that the volume $\mathrm{vol}(\{g\colon x\in g(P)\})$ is independent of x. Then we get the mean value theorem. In particular, we get Minkowski–Hlawka's theorem if we compute the volume $p^{-1}(B)$ in Z in two ways. This is the same idea as in Crofton's formula and others in integral geometry (cf. [San]).

1.3. *Ono's G-idele and Tamagawa's interpretation of Siegel's formula.*

Inspired by Chevalley's work on class field theory and linear algebraic groups, T. Ono introduced the *adele group* (or *adelization*) of a linear algebraic group G ([O1], 1957). It is called G-idele. Let G be a linear algebraic group in GL_n defined over a number field k. The adele group $G(\mathbb{A})$ is a locally compact group obtained as a restricted product of the local groups $G(k_v)$ and the group of global rational points $G(k)$ is embedded in $G(\mathbb{A})$ as a discrete subgroup. As applications of the adelizations, Ono generalized the finiteness of an ideal class group and Dirichlet's unit theorem to those for any solvable algebraic group ([O2], 1959).

On the other hand, at the Tokyo-Nikko conference in 1955, M. Kuga already had formulated Siegel's mean value theorem in the adelic manner and posed a problem to interpret the mass formula in that context ([Ku1]).

It was T. Tamagawa who discovered new interpretation and proof of Siegel's mass formula by introducing the *Tamagawa measure* on the adele group in late fifties (cf. [T]). Notation being as in 1.2, let $G=SO(S)$ be the algebraic group defined by

$${}^tXSX=S \ \text{ and } \ \det(X)=1\,.$$

An invariant gauge form ω on G defined over $\mathbb{Q}$ defines a local measure ω_p on $G(\mathbb{Q}_p)$ for each p and ω_∞ on $G(\mathbb{R})$. We then have the volume-theoretic interpretation of $\alpha_p(S)$ and $\alpha_\infty(S)$ as follows:

$$\alpha_p(S)=\int_{G(\mathbb{Z}_p)} d\omega_p\,,\quad (p=\text{prime});\qquad \alpha_\infty(S)=\int_{G(\mathbb{R})} d\omega_\infty\,.$$

For the open subgroup $G(\mathbb{A}^\infty)=G(\mathbb{R})\times\prod_p G(\mathbb{Z}_p)$ of the adele group $G(\mathbb{A})$, there is a bijection between the set of classes in the genus of S and the double coset space $G(\mathbb{A}^\infty)\backslash G(\mathbb{A})/G(\mathbb{Q})$, under which S_i is supposed to correspond to $G(\mathbb{A}^\infty)g_iG(\mathbb{Q})$. Then, we have the decomposition

$$\begin{aligned} G(\mathbb{A})/G(\mathbb{Q}) &= \bigsqcup_{i=1}^{h}(G(\mathbb{A}^\infty)g_iG(\mathbb{Q}))/G(\mathbb{Q}) \\ &\simeq \bigsqcup_{i=1}^{h} g_iG(\mathbb{A}^\infty)g_i^{-1}/(g_iG(\mathbb{A}^\infty)g_i^{-1}\cap G(\mathbb{Q}))\,. \end{aligned}$$

The product $\omega_{\mathbb{A}} = \prod_v \omega_v$ defines an invariant measure, called the *Tamagawa measure*, on $G(\mathbb{A})$ and $\mathrm{vol}_{\omega_{\mathbb{A}}}(g_iG(\mathbb{A}^\infty)g_i^{-1}) = \prod_v \alpha_v(S)$. Since $g_iG(\mathbb{A}^\infty)g_i^{-1}\cap G(\mathbb{Q})$ is isomorphic to the unit group $SO(S_i)(\mathbb{Z})$, we have

$$\int_{G(\mathbb{A})/G(\mathbb{Q})} d\omega_{\mathbb{A}} = \prod_v \alpha_v(S)\left(\sum_{i=1}^{h}\frac{1}{E(S_i)}\right).$$

Hence, Siegel's formula is equivalent to

$$\text{(T)} \qquad \tau(G) := \mathrm{vol}_{\omega_{\mathbb{A}}}(G(\mathbb{A})/G(\mathbb{Q})) = 2\,.$$

In fact, the adelic formulation works over any number field and (T) also yields Siegel's formulas in the indefinite case. Weil ([W5], 1962) showed that $\tau(G) = 2$ essentially contains Siegel's main theorems in general.

1.4. *Weil's integration theory and adele geometry.*

Weil's contribution to this subject is based on his theory of integrations on topological transformation spaces ([W1], 1940). As an application of this general theory, Weil viewed Siegel's mean value theorem as a special case of the following Fubini-type theorem ([W2], 1946). Let X be a topological space on which a locally compact unimodular group G acts transitively. Let L be a discrete subspace on which a discrete subgroup Γ of G acts. Let H be the stabilzer of $x \in L$ and set $\gamma = H\cap\Gamma$. Suppose we have a compactly supported continuous function φ on X. Here is a typical situation:

$$\begin{array}{ccccc} G & \curvearrowright & X & \xrightarrow{\varphi} & \mathbb{C} \\ \cup & & \cup & & \\ \Gamma & \curvearrowright & L & & \end{array}$$

Then, Weil showed the equality

$$\int_X \varphi(x)dx = \frac{1}{\mathrm{vol}(H/\gamma)}\int_{G/\Gamma}\sum_{z\in\Gamma/\gamma}\varphi(gz)dg\,,$$

where dx and dg are suitable matching measures.

Stimulated by Tamagawa's discovery (T), Weil introduced the *Tamagawa number* of a linear algebraic group over a global field and posed the *Weil conjecture* in [W3], 1959 (See 2.1 below for the precise forms). In 1959-60, Weil gave lectures on Tamagawa numbers ([W4]) which have played an important role in the development of the arithmetic of algebraic groups. In these lectures, first of all, Weil introduced the basic notions in adele geometry such as an intrinsic definition of the *adelization* of any algebraic variety defined over a global field k, convergence factors for a global measure on the adele group and so on. As for the Tamagawa numbers, he computed systematically those for classical groups applying the Poisson summation formula to the above equality in the adelic setting. Pushing this way, Weil later generalized the relation obtained by Siegel ([S5], 1951, 52) between theta functions and Eisenstein series to the case of classical groups in the adelic and representation-theoretic language ([W6], [W7], 1964, 65).

In the following, we shall see that the topics dicussed above have been developed in the context of adele geometry in the latter half of this century.

§2. Tamagawa numbers and the mean value theorem in adele geometry

2.1. *Tamagawa numbers.*

From now on, k denotes an algebraic number field and $\mathbb{A}$ the adele ring of k. Let $\mathcal{V}_\infty$ and $\mathcal{V}_f$ be the set of infinite and finite places of k, respectively. For $v \in \mathcal{V} = \mathcal{V}_\infty \cup \mathcal{V}_f$, k_v stands for the completion of k at v. As usual, the multiplicative valuation $|\cdot|_v$ of k_v is normalized so that $|a|_v = \mu_v(aC)/\mu_v(C)$ for $a \in k_v$, where μ_v is a Haar measure of k_v and C is an arbitrary compact subset of k_v with nonzero measure. Then $|\cdot|_{\mathbb{A}} = \prod_{v\in\mathcal{V}} |\cdot|_v$ is the idele norm of $\mathbb{A}^\times$. The precise definition of Tamagawa number is given as follows. Let G be a connected linear algebraic group defined over k. We denote by $\mathbf{X}^*(G)$ and by $\mathbf{X}^*_k(G)$ the free $\mathbb{Z}$-modules consisting of all rational characters and all k-rational characters of G, respectively. The absolute Galois group Γ_k of k acts on $\mathbf{X}^*(G)$. The representation of Γ_k in the space $\mathbf{X}^*(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is denoted by σ_G and the corresponding Artin L-function is denoted by $L(s, \sigma_G) = \prod_{v\in\mathcal{V}_f} L_v(s, \sigma_G)$. We set $\sigma_k(G) = \lim_{s\to 1}(s-1)^n L(s, \sigma_G)$, where $n = \operatorname{rank}\mathbf{X}^*_k(G)$. For a left invariant gauge form ω^G on G defined over k, we associate a left invariant Haar measure ω^G_v on $G(k_v)$. Then, the Tamagawa measure on $G(\mathbb{A})$ is well defined by $\omega^G_{\mathbb{A}} = |d_k|^{-\dim G/2}\omega^G_\infty \omega^G_f$, where $\omega_\infty^G = \prod_{v\in\mathcal{V}_\infty} \omega_v^G$, $\omega_f^G = \sigma_k(G)^{-1}\prod_{v\in\mathcal{V}_f} L_v(1,\sigma_G)\omega_v^G$ and $|d_k|$ is the absolute value of the discriminant of k. For $\chi \in \mathbf{X}_k^*(G)$, let $|\chi|_{\mathbb{A}}$ be the continuous homomorphism $G(\mathbb{A}) \to \mathbb{R}_+^\times$ defined by $|\chi|_{\mathbb{A}}(g) = |\chi(g)|_{\mathbb{A}}$. We write $G(\mathbb{A})^1$ for the intersection of kernels of all such $|\chi|_{\mathbb{A}}$'s. If $\chi_1, \cdots, \chi_n$ is a $\mathbb{Z}$-basis of $\mathbf{X}_k^*(G)$, then the mapping

$$g \mapsto (|\chi_1(g)|_{\mathbb{A}}, \cdots, |\chi_n(g)|_{\mathbb{A}})$$

yields an isomorphism from the quotient group $G(\mathbb{A})/G(\mathbb{A})^1$ to $(\mathbb{R}_+^\times)^n$. We put the Lebesgue measure dx on $\mathbb{R}$ and the invariant measure dx/x on $\mathbb{R}_+^\times$. Then there exists uniquely a Haar measure $\omega_{G(\mathbb{A})^1}$ of $G(\mathbb{A})^1$ such that the Haar measure on $G(\mathbb{A})/G(\mathbb{A})^1$ matching with $\omega_{\mathbb{A}}^G$ and $\omega_{G(\mathbb{A})^1}$ is equal to the pull-back of the measure $\prod_{i=1}^n dx_i/x_i$ on $(\mathbb{R}_+^\times)^n$ by the above isomorphism. The measure $\omega_{G(\mathbb{A})^1}$ is independent of the choice of a $\mathbb{Z}$-basis of $\mathbf{X}_k^*(G)$. Since $G(k)$ is a discrete subgroup of $G(\mathbb{A})^1$, we put the counting measure $\omega_{G(k)}$ on $G(k)$. Then the Tamagawa number $\tau(G)$ is defined to be the volume of the quotient space $G(\mathbb{A})^1/G(k)$ with respect to $\omega_{G(\mathbb{A})^1}/\omega_{G(k)}$.

The basic problem on the measure-finiteness and compactness of $G(\mathbb{A})^1/G(k)$, posed by Weil ([W3], 1959), was settled by Mostow–Tamagawa ([M-T]) and Borel–Harish-Chandra ([B-HC], [B1]) in 1962, and by Harder ([H1], 1969) for the positive characteristic case. The following conjecture was also stated by Weil in [W3].

If G is a connected, simplyconnected semisimple group over a global field, the Tamagawa number $\tau(G) = 1$.

Inspired by the Weil conjecture on Tamagawa number, it had been a fundamental problem to get the 'arithmetic index theorem' for the Tamagawa number after the model of the Gauss-Bonnet theorem for Riemannian manifolds and the Cauchy integral formula for complex manifolds. For the works of Ono, Demazure, Mars, Langlands and Lai on this problem, we refer to Ono's appendix to [W4]. Finally, the Weil conjecture was settled by Kottwitz ([Ko], 1988) assuming the Hasse principle of H^1 of the group of type E_8, which was proved by Chernousov ([Ch], 1989). Combined with Ono's relative theory ([O4]), the final formula for the Tamagawa number of a unimodular connected linear algebraic group G over k is given as follows. Let $\pi_1(G)$ be Borovoi's algebraic fundamental group ([Bo]) and $(\pi_1(G)_{\Gamma_k})_{tors}$ denote the torsion part of the coinvariant quotient of $\pi_1(G)$ under the absolute Galois group Γ_k of k. As an abstract group, $\pi_1(G)$ is canonically isomorphic to the topological fundamental group of the complex Lie group $G(\mathbb{C})$. The correspondence

$G \mapsto \pi_1(G)$ yields an exact functor from the category of connected affine k-groups to the category of Γ_k-module generated finitely over $\mathbb{Z}$. Set

$$\mathrm{Ker}^1(k, G) := \mathrm{Ker}(H^1(k, G) \longrightarrow \prod_v H^1(k_v, G)).$$

Theorem 2.1.1. *The Tamagawa number of G is given by*

$$\tau(G) = \frac{[(\pi_1(G)_{\Gamma_k})_{tors}]}{[\mathrm{Ker}^1(k, G)]}.$$

Hence, the Tamagawa number $\tau(G)$ is determined by the Galois module $\pi_1(G)$, for $\mathrm{Ker}^1(k, G)$ is described by $\pi_1(G)$ ([Bo]).

Remarks 2.1.2. 1) The Weil conjecture in positive characteristic case is still open, except the case of Chevalley groups proved by Harder ([H2], 1974). We are informed that E. Kushnirsky recently proved the Weil conjecture for quasi-split groups in positive characteristic case in a part of his thesis ([Kus]).

2) The proof of the Weil conjecture for number field case involves several steps of quite different features: tori ([O3]), quasi-split groups ([Lan], [Lai]), comparison between quasi-split groups and inner forms ([K]) and Hasse principle for H^1 of simply-connected groups. It would be desirable to get a unified proof appealing to the simplyconnectedness of the group.

2.2. *Mean value theorem in adele geometry.*

Ono ([O5]) introduced the notion of Tamagawa number for a homogeneous space in connection with the mean value theorem in the following adelic setting. Let G be a unimodular connected linear algebraic group and H be a unimodular connected subgroup of G and set $X = G/H$. Let $\omega_{\mathbb{A}}^X$ be the canonical measure on the adele space $X(\mathbb{A})$ so that the matching $\omega_{\mathbb{A}}^G = \omega_{\mathbb{A}}^X \omega_{\mathbb{A}}^H$ holds. Assume further that G and H have no non-trivial k-rational characters. Then X is quasi-affine. Using Kottwitz's theorem, we have the following *uniformity.*

Theorem 2.2.1 ([Mo-Wa1], [Mo]). *Notations and assumptions being as above, there is a constant $\tau(G, X)$ so that the following equality holds for any compactly supported continuous function φ on $G(\mathbb{A})X(k)$:*

$$\int_{G(\mathbb{A})X(k)} \varphi(x) d\omega_{\mathbb{A}}^X(x) = \frac{\tau(G, X)}{\tau(G)} \int_{G(\mathbb{A})/G(k)} \sum_{z \in X(k)} \varphi(gz) d\omega_{\mathbb{A}}^G(g).$$

Here, the constant $\tau(G,X)$ is called the *Tamagawa number of a homogeneous space* (G,X) and given by the following formula.

Theorem 2.2.2 ([ibid]).

$$\tau(G,X) = \frac{[\pi_1(G)_{\Gamma_k}]}{[\pi_1(H)_{\Gamma_k}][\mathrm{Cok}(\mathrm{Ker}^1(k,H) \longrightarrow \mathrm{Ker}^1(k,G))]}.$$

In particular, if $\pi_1(X(\mathbb{C})) = \pi_2(X(\mathbb{C})) = 1$, *then the mean value property* $\tau(G,X) = 1$ *holds.*

Remarks 2.2.3. 1) F. Sato ([Sat]) developed a theory in another direction which gives a natural interpretation of Siegel's mass in the case of indefinite forms and extends Siegel's main theorem to that for a homogeneous space.

2) As we explained in 1.2, the mean value theorem can be seen as a formula in the integral geometry. The familiar formulas such as Poincaré's one in integral geometry deal with the integrals of intersection numbers of subvarieties in an ambient space (cf. [San]). It would be nice if there exist such kind of formulas in adele geometry.

§3. Geometry of numbers over adele spaces

3.1. *Adelic fundamental theorems of Minkowski.*

A generalization of Minkowski's fundamental theorems in the geometry of numbers to those over any number field was first considered by Weyl ([We]) and later by Rogers and Swinnerton-Dyer ([R-S]). Mahler considered the geometry of numbers over rational function fields ([Mah]). The difficulty to work over a number field k arises from a fact that a lattice may not be free in general unless the class number of k is 1. This difficulty is automatically resolved by working over the adeles.

Now, let us explain adelic fundamental theorems in the geometry of numbers following Macfeat ([Mac]), Bombieri–Vaaler ([Bom-V]) and Thunder ([Th2]). In the following, $\mathcal{O}$ (resp. $\mathcal{O}_v$) denotes the ring of integers of k (resp. k_v for $v \in \mathcal{V}_f$) and r_1 (resp. r_2) the number of real (resp. complex) places of k. Let V be an n-dimensional affine space over k. Fix a basis $e_1, \cdots, e_n$ of $V(k)$ and identify $GL(V(k))$ with $GL_n(k)$. Let $L = \mathcal{O}e_1 + \cdots + \mathcal{O}e_n$ be a free $\mathcal{O}$-module and set $L_v = L \otimes_{\mathcal{O}} \mathcal{O}_v$ for $v \in \mathcal{V}_f$. For finite adele $g_f = (g_v)_{v \in \mathcal{V}_f} \in GL_n(\mathbb{A}_f)$, define an $\mathcal{O}$-lattice $g_f L$ by

$$g_f L = \bigcap_{v \in \mathcal{V}_f} (V(k) \cap g_v L_v).$$

Then, the set of $\mathcal{O}$-lattices in $V(k)$ is identified with

$$GL_n(\mathbb{A}_f)/\prod_{v\in\mathcal{V}_f} GL_n(\mathcal{O}_v)\,.$$

Choose a nonempty open bounded convex symmetric subset C_v of $V(k_v)$ for $v\in\mathcal{V}_\infty$ and set $C_\infty=\prod_{v\in\mathcal{V}_\infty} C_v$. Take $g_f=(g_v)_{v\in\mathcal{V}_f}\in GL_n(\mathbb{A}_f)$ and set

$$\mathcal{S}(C_\infty,g_f):=C_\infty\times\prod_{v\in\mathcal{V}_f} g_vL_v\,.$$

Then, $\mathcal{S}(C_\infty,g_f)$ is an open relatively-compact subset of $V(\mathbb{A})$ and its volume is

$$\omega_{\mathbb{A}}^V(\mathcal{S}(C_\infty,g_f))=|d_k|^{-n/2}|\det g_f|_{\mathbb{A}}\omega_\infty^V(C_\infty)\,.$$

An *adelic first fundamental theorem* is stated as follows.

Theorem 3.1.1 ([Mac, Theorem 2], [Th2, Theorem 3]). *Notation being as above, suppose $2^{n[k:\mathbb{Q}]}<\omega_{\mathbb{A}}^V(\mathcal{S}(C_\infty,g_f))$. Then, $\mathcal{S}(C_\infty,g_f)\cap V(k)\neq\{0\}$.*

Next, for $1\le i\le n$, define *i-th successive minimum* for $\mathcal{S}(C_\infty,g_f)$ by

$$\begin{aligned}
&\lambda_i(\mathcal{S}(C_\infty,g_f))\\
&\quad=\inf\{\lambda>0\colon \mathcal{S}(\lambda C_\infty,g_f)\cap V(k)\text{ contains}\\
&\qquad\qquad\qquad k\text{-linearly independent } i \text{ vectors.}\}\\
&\quad=\inf\{\lambda>0\colon \lambda C_\infty\cap g_fL\text{ contains } k\text{-linearly independent } i\text{ vectors.}\}
\end{aligned}$$

where $\lambda C_\infty=\prod_{v\in\mathcal{V}_\infty}\lambda C_v$. Setting $\lambda_i=\lambda_i(\mathcal{S}(C_\infty,g_f))$, $1\le i\le n$ for simplicity, an *adelic second fundamental theorem* is stated as follows.

Theorem 3.1.2 ([Mac, Theorem 5,6], [Bom-V, Theorem 3,6]). *One has the inequality*

$$(\lambda_1\cdots\lambda_n)^{[k:\mathbb{Q}]}\omega_{\mathbb{A}}^V(\mathcal{S}(C_\infty,g_f))\le 2^{n[k:\mathbb{Q}]}\,.$$

Further, assume that for each complex place v,

$$zC_v=C_v \textit{ for } z\in k_v \textit{ with } |z|_v=1.$$

Then, one has

$$\frac{2^{n[k:\mathbb{Q}]}\pi^{nr_2}}{(n!)^{r_1}(2n!)^{r_2}|d_k|^{n/2}}\le(\lambda_1\cdots\lambda_n)^{[k:\mathbb{Q}]}\omega_{\mathbb{A}}^V(\mathcal{S}(C_\infty,g_f))\,.$$

Theorems 3.1.1 and 3.1.2 are proved by reworking with the proofs of the classical case in the adelic setting. They can be applied to deal with problems in diophantine approximation. Let us present typical ones.

Let V_m be the m-th exterior product of V for $1 \leq m \leq n-1$. If we put $e_I = e_{i_1} \wedge \cdots \wedge e_{i_m}$, $(i_1 < \cdots < i_m)$, for every subset $I = \{i_1, \cdots, i_m\} \subset \{1, 2, \cdots, n\}$, then $\{e_I\}_{|I|=m}$ is a base of $V_m(k)$. For each $v \in \mathcal{V}$, the local height H_v on $V_m(k_v)$ is defined as follows:

$$H_v\Big(\sum_{|I|=m} a_I e_I\Big) = \begin{cases} \left(\displaystyle\sum_{|I|=m} |a_I|_v^2\right)^{1/(2[k:\mathbb{Q}])} & (v \text{ is real}), \\ \left(\displaystyle\sum_{|I|=m} |a_I|_v\right)^{1/[k:\mathbb{Q}]} & (v \text{ is complex}), \\ \left(\displaystyle\sup_{|I|=m} |a_I|_v\right)^{1/[k:\mathbb{Q}]} & (v \text{ is finite}). \end{cases}$$

If $x_1, \cdots, x_m \in V(k_v)$, we write $H_v(x_1, \cdots, x_m)$ for $H_v(x_1 \wedge \cdots \wedge x_m)$. The global height H on $V_m(k)$ is defined by

$$H(x_1, \cdots, x_m) = \prod_{v \in \mathcal{V}} H_v(x_1, \cdots, x_m), \qquad (x_1, \cdots, x_m \in V(k)).$$

Let W be an m-dimentional k-subspace of $V(k)$ and $w_1, \cdots, w_m$ a basis of W. Since we have

$$H(\gamma w_1, \cdots, \gamma w_m) = |\det \gamma|_{\mathbb{A}} H(w_1, \cdots, w_m) = H(w_1, \cdots, w_m)$$

for any $\gamma \in GL(W)$, the height $H(w_1, \cdots, w_m)$ is independent of the choice of a basis of W, and hence $H(W) = H(w_1, \cdots, w_m)$ is well defined. In other words, H is regarded as a height on the Grassmanian variety of m-dimentional subspaces of $V(k)$. We set

$$B_{\mathbb{A}}^n = \prod_{v \in \mathcal{V}} \{x_v \in V(k_v) \colon H_v(x_v) \leq 1\}.$$

The volume of $B_{\mathbb{A}}^n$ is given by

$$V_k(n) := \omega_{\mathbb{A}}^V(B_{\mathbb{A}}^n) = |d_k|^{-n/2} \left(\frac{\pi^{n/2}}{\Gamma(1+n/2)}\right)^{r_1} \left(\frac{(2\pi)^n}{\Gamma(1+n)}\right)^{r_2}.$$

Theorem 3.1.3 ([Bom-V, Theorem 8], [Th2, Corollary 2 to Theorem 3]). *Let W be an m-dimentional subspace of $V(k)$. Then there is a basis $w_1, \cdots, w_m$ of W such that*

$$H(w_1)\cdots H(w_m) \leq \frac{2^m}{V_k(m)^{1/[k:\mathbb{Q}]}} H(W).$$

This kind of result is called Siegel's lemma, which usually asserts the existence of integral solutions with small height of a system of linear equations. Bombieri and Vaaler used the adelic second fundamental theorem and the cube slicing theorem to prove this. Incidentally, since the cube slicing theorem is also interesting from a viewpoint of adele geomety, we explain it here. We set for each infinite place $v \in \mathcal{V}_\infty$

$$C_v^n = \left\{ \sum_{i=1}^n a_i e_i \in V(k_v) \colon \sup_i |a_i|_v < \begin{cases} 2^{-1} & \text{(if } v \text{ is real)} \\ (2\pi)^{-1} & \text{(if } v \text{ is complex)} \end{cases} \right\}$$

and

$$C_{\mathbb{A}}^n = \mathcal{S}(\prod_{v\in\mathcal{V}_\infty} C_v^n, 1) = \prod_{v\in\mathcal{V}_\infty} C_v^n \times \prod_{v\in\mathcal{V}_f} L_v .$$

The volume of $C_{\mathbb{A}}^n$ is $\omega_{\mathbb{A}}^V(C_{\mathbb{A}}^n) = |d_k|^{-n/2}$. The cube slicing theorem is stated as follows:

Theorem 3.1.4 ([Bom-V, Theorem 7]). *If V_1 be an affine subspace of V defined over k, then*

$$H(V_1(k))^{-[k:\mathbb{Q}]} |d_k|^{-\dim V_1/2} \leq \omega_{\mathbb{A}}^{V_1}(C_{\mathbb{A}}^n \cap V_1(\mathbb{A})).$$

Another application of the adelic fundamental theorems is the adelization of Jarnik's theorem. O'Leary and Vaaler [O'L-V] introduced an inhomogeneous minimum of $\mathcal{S}(C_\infty, g_f)$ by

$$\mu(\mathcal{S}(C_\infty, g_f)) = \inf\{\mu > 0 \colon V(\mathbb{A}) = \bigcup_{x\in V(k)} (\mathcal{S}(\mu C_\infty, g_f) + x)\}$$

and they proved

Theorem 3.1.5 ([O'L-V, Theorem 5]). *Let λ_i be the successive minima of $\mathcal{S}(C_\infty, g_f)$. Then one has*

$$\frac{\lambda_n}{2} \leq \mu(\mathcal{S}(C_\infty, g_f)) \leq \mu(B_{\mathbb{A}}^1)(\lambda_1 + \lambda_2 + \cdots + \lambda_n).$$

In general, the exact value of $\nu(k) := \mu(B^1_{\mathbb{A}})$ is not known. We have $\nu(\mathbb{Q}) = 1/2$, $\nu(\mathbb{Q}(\sqrt{-1})) = 1/\sqrt{2}$ and $\nu(\mathbb{Q}(\sqrt{-3})) = 1/\sqrt{3}$ for example. O'Leary and Vaaler showed the following estimate ([O'L-V, Theorem 6]); If $k \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$, then

$$\nu(k) \leq \frac{1}{2}\left(\frac{2}{\pi}\right)^{r_2} |d_k|^{1/2}.$$

Theorem 3.1.5 was used to generalize Schinzel's theorem, which gives a refeinment of Theorem 3.1.3. Let W be an m-dimentional subspace of $V(k)$. A basis $w_1, \cdots, w_m$ of W is called primitive if the equality

$$H_v(a_1x_1 + \cdots + a_mx_m) = \sup(H_v(a_1x_1), \cdots, H_v(a_mx_m))$$

holds for all $a_1, \cdots, a_m \in k_v$ and all finite $v \in \mathcal{V}_f$.

Theorem 3.1.6 ([Bu-V, Theorem 2]). *There is a primitive basis $w_1, \cdots, w_m$ of W such that*

$$H(w_1)\cdots H(w_m) \leq (m-1)!\left(1+\left(\frac{2\nu(k)}{V_k(1)^{1/[k:\mathbb{Q}]}}\right)^2\right)^{\frac{m-1}{2}} \frac{2^m}{V_k(m)^{1/[k:\mathbb{Q}]}} H(W).$$

Besides these, the adelic second fundamental theorem is useful for seeking a nontrivial integral solution with small height of a homogeneous quadratic equation. To explain this, let φ be a quadratic form on $V(k)$ and Φ the symmetric matrix corresponding to φ with respect to $e_1, \cdots, e_n$. The height $H(\Phi)$ of Φ is defined in a similar fashion as the global height of $V(k)$. The next theorem due to Vaaler is the adelization of Schlickewei's result.

Theorem 3.1.7 ([V, I, Theorem 1]). *Let W be a subspace of $V(k)$. Assume that φ is not identically zero and not anisotropic on W. Then there exists a maximally totally isotorpic subspace X of W such that*

$$H(X) \leq \frac{H(W)\{2^{2\dim X+3}H(\Phi)\}^{(\dim W - \dim X)/2}}{V_k(\dim W - \dim X)^{1/[k:\mathbb{Q}]}}.$$

As a corollary, it follows that there is a nonzero integral solution $a_1e_1 + \cdots + a_ne_n \in L \cap W$ of a homogeneous quadratic equation $\varphi(x) = 0$

such that

$$\prod_{v\in\mathcal{V}}\left(\sup_{1\le i\le n}|a_i|_v\right)^{1/[k:\mathbb{Q}]}$$
$$\le \frac{2}{V_k(1)^{1/[k:\mathbb{Q}]}}\left(\frac{H(W)\{2^{2\dim X+3}H(\Phi)\}^{(\dim W-\dim X)/2}}{V_k(\dim W-\dim X)^{1/[k:\mathbb{Q}]}}\right)^{\frac{1}{\dim X}}.$$

Some generalization and refinement of Theorem 3.1.7 were given at [V, II] and [Wa3].

3.2. *Adelic Minkowski–Hlawka theorem.*

As in the classical case, the mean value theorem 2.2.1 yields an adelic Minkowski–Hlawka type result. Suppose the situation is as in 2.2 and let C be a compact subset of $X(\mathbb{A})$. Taking the characteristic function of C as φ in 2.2.1, we get the following

Theorem 3.2.1. *Suppose $\omega_{\mathbb{A}}^X(C) < \tau(G,X)/\tau(G)$. Then there is $g\in G(\mathbb{A})$ such that $C\cap gX(k)=\emptyset$.*

As a special case, we let $X = V\backslash\{0\}$, V being as in 3.1, $G = SL_n, n\ge 2$ and $H=\{g\in SL_n\colon ge_1=e_1\}$. Then, $\tau(G)=\tau(G,X)=1$ by 2.1.1 and 2.2.2. Following Thunder [Th2], let C be a relatively compact star domain in $V(\mathbb{A})$. Namely, $C=\prod_v C_v$ is a relatively compact open subset of $V(\mathbb{A})$ containing 0 and satisfying the property

$$aC\subset C,\ \text{for } a\in\{(a_v)\in\mathbb{A}^\times\colon |a_v|_v\le 1\}.$$

Then, $FC=\{ax\colon a\in F, x\in C\}$ is also a relatively compact star domain and we can show (cf. [Th2, Lemma2])

$$\frac{\omega_{\mathbb{A}}^V(FC\cap X(\mathbb{A}))}{\omega_{\mathbb{A}}^V(C)}=\frac{h_kR_kn^{r_1+r_2-1}}{w_k\zeta_k(n)},$$

where h_k, R_k, w_k and ζ_k denote the class number, the regulator, the number of root of unity and the zeta function of k, respectively. From this and 3.2.1, we have

Theorem 3.2.2. *Suppose that C is a relatively compact star domain in an affine adele space $V(\mathbb{A})$ such that*

$$\omega_{\mathbb{A}}^V(C)<\frac{w_k\zeta_k(n)}{h_kR_kn^{r_1+r_2-1}}.$$

Then, there is $g \in SL_n(\mathbb{A})$ *such that* $C \cap gV(k) = \{0\}$.

3.3. *Generalized Hermite constants.*

Watanabe [Wa2] generalized the notion of *Hermite constant* to that for any strongly rational representation of a reductive algebraic group using the Borel reduction theory.

Let G be a connected reductive group over k and $\rho\colon G \longrightarrow GL(V)$ be a strongly k-rational, absolutely irreducible representation of G on an affine k-space V. Let D be the highest weight space whose stabilizer is the parabolic subgroup P. Then, $X = G/P$ is a smooth projective variety isomorphic to the closed image of the cone $\rho(G)D$ in the projective space $\mathbb{P}(V)$ attached to V. We fix a suitable maximal compact subgroup K of the adele group $G(\mathbb{A})$. We also fix a norm $||\cdot||_v$ on $V(k_v)$ compatible with $|\cdot|_v$ for each place v so that $||\cdot||_v$ is the maximum norm for almost all v when a base of $V(k)$ is once fixed, and set $||x||_{\mathbb{A}} = \prod_v ||x_v||_v$ for $x \in GL(V(\mathbb{A}))V(k)$. We suppose that $||\cdot||_{\mathbb{A}}$ is K-invariant and normalized by $||x_0||_{\mathbb{A}} = 1$ for $x_0 \in D(k)\backslash\{0\}$. Let $G(\mathbb{A})^1$ be the subgroup of $G(\mathbb{A})$ defined in 2.1. The situation is as follows:

$$\begin{array}{ccccc} G(\mathbb{A})^1 & \curvearrowright & G(\mathbb{A})^1/P(\mathbb{A})^1 & \xrightarrow{||\rho(\cdot)x_0||_{\mathbb{A}}} & \mathbb{R}^n \\ \cup & & \cup & & \\ G(k) & \curvearrowright & X(k) & & \ni x_0 \end{array}$$

By the Borel reduction theory ([B2], [G]), for each $g \in G(\mathbb{A})^1$, the twisted height function $H_g(x) = ||\rho(g\gamma)x_0||_{\mathbb{A}}^{1/[k:\mathbb{Q}]}$, $x = \rho(\gamma)x_0$, attains the mimimum at a point in the intersection of $G(k)$ and a Siegel set in $G(\mathbb{A})$, and $g \mapsto \min_{x \in X(k)} H_g(x)$ is a bounded continuous function on $K\backslash G(\mathbb{A})^1/G(k)$.

Definition 3.3.1 ([Wa2, Proposition 2]). *The maximum*

$$\mu(\rho, ||\cdot||_{\mathbb{A}}) := \max_{g \in G(\mathbb{A})^1} \min_{x \in X(k)} H_g(x)^2$$

exists for a connected reductive group over k. *We call it the generalized Hermite constant associated to* $(\rho, ||\cdot||_{\mathbb{A}})$.

When $G = GL_n$ and ρ is an exterior power representation, this notion is due to Rankin ([Ra], $k = \mathbb{Q}$) and Thunder ([Th3], for any k). It was also studied by Baeza and Icaza [Ba-Ic], [Ic] when $G = GL_n$ and ρ is the natural representation.

Now, by Weil's integration theory in 1.4, for a suitable function φ on $\mathbb{R}^n$, we have

$$\int_{G(\mathbb{A})^1/P(\mathbb{A})^1} \varphi(||\rho(g)x_0||_{\mathbb{A}})d(\omega_{G(\mathbb{A})^1}/\omega_{P(\mathbb{A})^1})(g)$$
$$= \frac{1}{\tau(P)} \int_{G(\mathbb{A})^1/G(k)} \sum_{x\in X(k)} \varphi(||\rho(g)x||_{\mathbb{A}})d(\omega_{G(\mathbb{A})^1}/\omega_{G(k)})(g)\,.$$

The basic idea to give a lower bound for $\mu(\rho, ||\cdot||_{\mathbb{A}})$ is same as in the classical case: Take a characteristic function of the interval $[0, T]$ for $T > 0$ as φ and set $B(T) = \{g \in G(\mathbb{A})^1/P(\mathbb{A})^1\colon ||\rho(g)x_0||_{\mathbb{A}} \leq T\}$. If $T < M := \sup\{T\colon \mathrm{vol}(B(T)) < \tau(G)/\tau(P)\}$, then there is $g \in G(\mathbb{A})^1$ so that $H_g(x) > T^{1/[k:\mathbb{Q}]}$ for all $x \in X(k)$. It means $\mu(\rho, ||\cdot||_{\mathbb{A}}) \geq M^{2/[k:\mathbb{Q}]}$. The computation of the volume of $B(T)$ was worked out in [W2] for the case that ρ is maximal, namely the restriction of the heighest weight of ρ to a maximal k-split torus is a positive integer multiple of a fundamental k-weight. In that case, P is a standard maximal parabolic subgroup and the integral over $G(\mathbb{A})^1/P(\mathbb{A})^1$ boils down to easier one to handle. The main theorem in [Wa2] is stated in the following vague form.

Theorem 3.3.2. *Assume that ρ is maximal. Then, the volume* $\mathrm{vol}(B(T))$ *is of the form $c_1T^{c_2}, c_2 \in \mathbb{Q}$, and hence*

$$\mu(\rho, ||\cdot||_{\mathbb{A}}) \geq \left(\frac{\tau(G)}{c_1\tau(P)}\right)^{\frac{2}{c_2[k:\mathbb{Q}]}}.$$

The constants c_1 can be computed using the argument in Langlands' computation [Lan] and given more explicitly for k-split G. But, it would take long here to give all definitions of quantities involved. So, we will just give simple examples.

Example 3.3.3 ([Th3], [Wa2]). $G = GL_n (n \geq 2)$, $\rho = m$-th exterior power representation, $0 < m < n$. Then X is a Grassmannian of m-planes in an affine n-space. The norm is Euclidean norm at infinite places and the maximum norm of coordinates at finite places. In this cases, we set $\gamma_{n,m}(k) = \mu(\rho, ||\cdot||_{\mathbb{A}})$. Then, we have

$$\gamma_{n,m}(k) \geq \left(\frac{n|d_k|^{m(n-m)/2}}{\mathrm{Res}_{s=1}\zeta_k(s)} \frac{\prod_{i=n-m+1}^{n} Z_k(i)}{\prod_{i=2}^{m} Z_k(i)}\right)^{\frac{2}{n[k:\mathbb{Q}]}},$$

where $Z_k = (\pi^{-s/2}\Gamma(s/2))^{r_1}((2\pi)^{1-s}\Gamma(s))^{r_2}\zeta_k(s)$. Thunder gave the following upper bound:

$$\gamma_{n,m}(k) \leq 2^{2m}\left(\frac{1}{V_k(n)}\right)^{\frac{2m}{n[k:\mathbb{Q}]}}.$$

Example 3.3.4 ([Wa2], [Wa3]). Let φ be a nondegenerate quadratic form on an n-dimentional vector space $V(k)$ and Φ the symmetric matrix corresponding to φ with respect to a basis $e_1, \cdots, e_n$. We consider the special orthogonal group $G = SO_\varphi$ of φ and a natural representation $\rho\colon G \to GL(V)$. Let q be the Witt index of φ. It is assumed to be $n \geq 3$ and $q \geq 1$. Then $\mu(\rho, ||\cdot||_{\mathbb{A}})$ is interpreted as

$$\mu(\rho, ||\cdot||_{\mathbb{A}}) = \max_{g\in G(\mathbb{A})} \min_{\substack{x\in V(k)\setminus\{0\}\\ \varphi(x)=0}} ||gx||_{\mathbb{A}}^{2/[k:\mathbb{Q}]}.$$

It was proved in [Wa3] that

$$\mu(\rho, ||\cdot||_{\mathbb{A}}) \leq (2\gamma_{n-1,1}(k)H(\Phi))^{n-1},$$

where $H(\Phi)$ is a height of Φ. If $n = 2q$ or $2q+1$, then G is split over k and we have

$$\mu(\rho, ||\cdot||_{\mathbb{A}}) \geq \begin{cases} \left\{\dfrac{|D_k|^{n-1/2}(2n-1)}{\mathrm{Res}_{s=1}\zeta_k(s)} Z_k(2n)\right\}^{2/((2n-1)[k:\mathbb{Q}])} & (n = 2q+1), \\ \left\{\dfrac{|D_k|^{n-1}(2n-2)}{\mathrm{Res}_{s=1}\zeta_k(s)} \dfrac{Z_k(2(n-1))Z_k(n)}{Z_k(n-1)}\right\}^{1/((n-1)[k:\mathbb{Q}])} & \\ & (n = 2q). \end{cases}$$

Similar estimates are proved for the fundamental k-representations of G.

§4. Distribution of rational points in adele transformation spaces

4.1. *Hardy–Littlewood variety and Weyl–Kuga's criterion on uniform distribution.*

As for the asymptotic distribution of integral solutions under local-global principle, Hardy, Littlewood and Ramanujan invented the *circle method* in the course of their study of the Waring problem ([H-L]) which stimulated Siegel's work. The notion of a *Hardy–Littlewood variety* was introduce by Borovoi and Rudnick ([Bo-R]) to describe the asymptotic distribution of integral points of an affine variety in terms of local densities, namely the product of singular series and singular integral the circle method expects. Let X be an affine variety defined over k embedded into an affine space V over k. Assume that there is a gauge form ω^X on X. The associated Tamagawa measure is denoted by $\omega_{\mathbb{A}}^X$ (we attach a suitable convergence factor if necessary). Take a finite nonempty subset S of $\mathcal{V}$ such that $X(k_v)$ is non-compact for $v \in S$, and fix a norm $||\cdot||_v$ on $V(k_v)$ for $v \in S$. We set $k_S = \prod_{v\in S} k_v$ and $\mathbb{A}^S$ = the ring of S-adeles. Let $B_S = \prod_{v\in S\cap\mathcal{V}_\infty} B_v \times \prod_{S\cap\mathcal{V}_f} X(k_v)$, where B_v is a topological connected component of $X(k_v)$ for $v \in \mathcal{V}_\infty$, and set, for $T > 0$, $B_S^T = \{(x_v) \in B_S \colon ||x_v||_v \le T\}$. Choose an open relatively-compact subset B^S of $X(\mathbb{A}^S)$ and define the counting function by

$$N_S(T, X, B) = [X(k) \cap (B_S^T \times B^S)]\,.$$

Then, following Borovoi and Rudnick, we call X a *relatively S-Hardy-Littlewood variety* if there is a non-negative function δ on $X(\mathbb{A})$ satisfying the conditions

1) δ is locally constant, not identically zero,
2) for any B_S and B^S as above,

$$N_S(T, X, B) \sim \int_{B_S^T\times B^S} \delta(x) d\omega_{\mathbb{A}}^X(x) \ \text{ as } \ T \to \infty.$$

In addition to the above conditions, if we can take the constant 1 as δ, X is called *strongly S-Hardy-Littlewood.*

Borovoi and Rudnick investigated the Hardy–Littlewood property for the case $k = \mathbb{Q}$ and $S = \mathcal{V}_\infty$, and in particular when X is an affine homogeneous space of a simplyconnected, semisimple group, based on the work of Duke, Rudnick and Sarnak ([D-R-Sar]). Among other things, they showed a certain affine symmetric space is indeed relatively Hardy-Littlewood. A typical example is the quadric $\{x \colon f(x) = a\}$ of an indefinite integral quadratic form f in n variables, $n \ge 4$, $a \in \mathbb{Z}\backslash\{0\}$. In the above definition, the density function δ may not be unique. In the next paragraph, we shall see that δ is determined in a simple form for a wide class of affine homogeneous spaces and general S when we give a 'right' definition of an *S-Hardy-Littlewood homogeneous space.*

In the circle method, a key role is played by Weyl's inequality having its origin in the work of Weyl [We1] on the uniform distribution of sequences. In [Ku2], Kuga extended Weyl's criterion for a sequence of numbers to that for a family of subsets of rational points on a linear algebraic group, and later Murase ([Mu]) generalized Kuga's results in the adelic setting.

Remark 4.1.1. There have been some works to reconstruct adelically theHardy–Littlewood circle method for special cases ([Mar], [I], [Lac],..). As was discussed in [Pa], it would be interesting to develope a general framework for the adelic Hardy–Littlewood method.

4.2. *Hardy–Littlewood homogeneous spaces.*

An *S-Hardy-Littlewood homogeneous space* was introduced in [MoWa2] as a framework to describe the asymptotic distribution of S-integral points on an affine homogeneous space.

Let G be an affine connected unimodular group. Suppose G acts on an affine space V and let X be a closed G-orbit in V. Suppose that G, V and the action are defined over a number field k and that X has a k-rational point x_0. We assume the stabilizer H of x_0 in G to be connected and unimodular. We also fix invariant gauge forms ω^G, ω^H and a G-invariant gauge form ω^X on G, H and X, respectively, which match together, and the associated Tamagawa measures are denoted by $\omega_{\mathbb{A}}^G, \omega_{\mathbb{A}}^H$ and $\omega_{\mathbb{A}}^X$. Let S be a finite nonempty subset of $\mathcal{V}$ so that G is isotropic over k_v for $v \in S$. We fix a norm $||\cdot||_v$ on $V(k_v)$ for $v \in S$. Choose a $G(k_S)$-orbit B_S and set $B_S(T) = \{(x_v)\colon ||x_v||_v \leq T\}$ for $T > 0$. Choose an open, relatively compact, 'convex' subset B^S of $X(\mathbb{A}^S)$. Here, the 'convex' condition is added only when there is an infinite place $v \notin S$ and, roughly, it means $\mathcal{V}_\infty \backslash S$-component of B^S is the intersection of $X(k_{\mathcal{V}_\infty \backslash S})$ and a convex subset of $V(k_{\mathcal{V}_\infty \backslash S})$. Set $B(T) = B_S(T) \times B^S$.

Definition 4.2.1. *An affine homogeneous space $X = G/H$ is called an S-Hardy-Littlewood homogeneous space if there exists a function $\delta : X(\mathbb{A}) \to \mathbb{R}_{\geq 0}$, called density function, such that*

H1) *δ is locally constant, $G(k_S)$-invariant, and not identically zero,*

H2) *for any B_S and B^S as above and $g \in G(\mathbb{A})$, one has*

$$\#(gX(k) \cap B(T)) \sim \int_{B(T)} \delta(g^{-1}x) d\omega_{\mathbb{A}}^X(x), \quad as\ T \to \infty.$$

Further, if we can take $\delta \equiv 1$, X is called strongly S-Hardy-Littlewood.

We easily see that the density function is unique by H1) and H2). Moreover, using the mean value theorem in 2.2, we can determine the density function in the following simple form.

Theorem 4.2.2. *Let G, H and X be as above. Assume further that G has the strong approximation property with respect to S and that H has no non-trivial k-rational characters. Then, if X is S-Hardy-Littlewood, the density function δ is given by*

$$\delta(x) = \begin{cases} \tau(G, X)^{-1} & x \in G(\mathbb{A})X(k), \\ 0 & x \notin G(\mathbb{A})X(k). \end{cases}$$

Our method to supply examples of S-Hardy-Littlewood homogeneous spaces is an extension of those of [D-R-Sar] and [Mu], namely based on Weyl–Kuga's criterion and Howe–Moore type vanishing theorem of spherical functions at infinity. Let the notations and assumptions be as in 4.2.2 and G is supposed to be semisimple and simplyconnected. We may assume $B_S \times B^S$ is contained in a $G(\mathbb{A})$-orbit of $x \in X(k)$ and count $\#(gG(k)x \cap B(T))$, since the difference from $\#(gX(k) \cap B(T))$ can be computed by Galois cohomology. For this, let χ^T be the characteristic function of $B(T)$ and set

$$F^T(g) = V(T)^{-1} \sum_{\gamma \in G(k)/H(k)} \chi^T(g\gamma x), \quad (g \in G(\mathbb{A})),$$

where $V(T) = \omega_{\mathbb{A}}^X(B(T))$. Since $F^T(g) = V(T)^{-1}\#(gG(k)x \cap B(T))$, the following condition (A) yields the S-Hardy-Littlewood property for each orbit.

(A) $\lim_{T \to \infty} F^T(g) = \tau(H)$.

Next, consider the following condition.

(B) For any compactly supported function ψ on $G(\mathbb{A})/G(k)$, one has

$$\lim_{T \to \infty} \int_{G(\mathbb{A})/G(k)} F^T(g)\psi(g)d\omega_{\mathbb{A}}^G(g) = \tau(H) \int_{G(\mathbb{A})/G(k)} \psi(g)d\omega_{\mathbb{A}}^G(g).$$

An extension of Weyl–Kuga–Murase's criterion is, roughly speaking, that the condition (A) holds if (B) holds for any F_ϵ^T, where F_ϵ^T is a certain modified function of F^T (For the precise, refer to [Mo-Wa2]). To certify (B), we consider the vanishing condition of certain spherical functions at infinity. Let K_v be a maximal compact subgroup of $G(k_v)$ for $v \in S$ so that the norm $|| \cdot ||_v$ is K_v-invariant. For the stabilizer H_x of x in G, let $\mathcal{H}(G(k_S)/H_x(k_S))$ be the space of smooth functions on $G(k_S)/H_x(k_S)$ which is $\prod_{v \in S \setminus \mathcal{V}_\infty} K_v$-finite and let π be the representation of $G(k_S)$ via left translation on this space. For $f \in \mathcal{H}(G(k_S)/H_x(k_S))$, let π_f be the space generated by $\pi(g)f$, $g \in G(k_S)$. Then, (B) follows from the following condition

(C) For $f \in \mathcal{H}(G(k_S)/H_x(k_S))$, if π_f is an infinite dimensional, unitarizable, irreducible representation, then $f(g) \to 0$ as $g \to \infty$.

Now, we assume that G is anisotropic over k and that G has no anisotropic simple factor over k_v for some $v \in S$. Then we have

Theorem 4.2.3. *Assumptions being as above, further assume that the condition* (C) *holds for all H_x, $x \in X(k)$. Then $X = G/H$ is S-Hardy-Littlewood with density function given in* Theorem 4.2.2.

For instance, using Rudnick–Schlichtkrull's vanishing theorem, we can show

Theorem 4.2.4. *Assume that G is semisimple, simplyconnected, anisotropic over k, and X is affine symmetric. Further assume that S contains an infinite place v such that G is k_v-almost simple, k_v-isotropic and $|| \cdot ||_v$ is invariant under a maximal compact subgroup of $G(k_v)$. Then, X is an S-Hardy-Littlewood homogeneous space.*

Remark 4.2.5. We expect that S-Hardy-Littlewood property still holds when G is k-isotropic. To show this, we need harder analysis involved with Eisenstein series.

Example 4.2.6. Let D be a skew field of index d and center k_1, with an involution τ such that $k_1^\tau = k$. Let V be an n-dimensional space over D, which is also regarded as an affine k-space, and let φ be a non-degenerate ε-hermitian form relative to τ on V. Here $\varepsilon = \pm 1$. It is assumed to be $n \geq 4$. Let G be the simplyconnected covering group of the special unitary group SU_φ, which is regarded as an algebraic group defined over k. For $a \in D^\times$, consider the affine variety $X = \{x \in V \colon \varphi(x) = a\}$. Assume that $X(k) \neq \emptyset$. By Witt's theorem, X is a homogeneous space of G. Let S be a finite set of places of k. If φ is k-anisotropic and S contains an infinite place v such that φ is isotropic over k_v, then X is strongly S-Hardy-Littlewood.

Example 4.2.7. Let φ be an integral k-anisotropic tenary quadratic form with discriminant d_φ. Consider the affine quadric $X = \{x = (x_1, x_2, x_3) : \varphi(x) = a\}$ for $a \in k^\times$. Assume $X(k) \neq \emptyset$ and fix $x_0 \in X(k)$. Let $G = Spin_\varphi$ be the spinor group of φ and H the stabilizer of x_0 in G. Then H is k-anisotropic torus and $X \cong G/H$. If S contains an infinite place v, or that S contains a finite prime v which does not divide $2d_\varphi$, where φ is isotropic over k_v in both cases, then X is relatively S-Hardy-Littlewood with density function

$$\delta(x) = \begin{cases} 2 & x \in G(\mathbb{A})X(k), \\ 0 & x \notin G(\mathbb{A})X(k). \end{cases}$$

To prove this, we use more general result ([Mo-Wa2, Theorem 7.2]) than Theorem 4.2.4.

4.3. *Rational points on flag varieties.*

Let G be a connected reductive group defined over a number field k and $\rho: G \to GL(V)$ a finite dimensional k-rational representation. Suppose a parabolic k-subgroup P stabilizes a line D through a k-rational point $x_0 \in V(k)$. Thus ρ yields an embedding of the flag variety $X = G/P$ into the projective space $\mathbb{P}(V)$. We fix a norm $||\cdot||_v$ on $V(k_v)$ for each place v of k and set $||x||_{\mathbb{A}} = \prod_v ||x_v||_v$ for $x \in GL(V(\mathbb{A}))V(k)$. Suppose $||\cdot||_{\mathbb{A}}$ is normalized by $||x_0||_{\mathbb{A}} = 1$ and is invariant under a maximal compact subgroup K of $G(\mathbb{A})$. Let χ be the k-rational character obtained from the action of P on D via ρ. We then define a twisted height H_g on $X(k)$ for $g \in G(\mathbb{A})$ by

$$H_g(x) = ||\rho(g\gamma)x_0||_{\mathbb{A}}^{1/[k:\mathbb{Q}]}, \quad (x = \rho(\gamma)x_0,\ \gamma \in G(k)).$$

We have the associated counting function

$$N(X,T,g) = \#\{x \in X(k) \colon H_g(x) \leq T^{1/[k:\mathbb{Q}]}\}.$$

When g is the identity, we have the usual height and counting function and we shall drop g. Note $H_g(x)^{[k:\mathbb{Q}]} = |\chi(p)|_{\mathbb{A}}$ if we write $g\gamma = kp$ with $k \in K$ and $p \in P(\mathbb{A})$.

A basic problem here is to describe the asymptotic behavior of $N(X,T,g)$ as $T \to \infty$ and some special cases were investigated. A theorem of Schanuel ([Sca]) describes $N(\mathbb{P}^{n-1},T)$ for a projective space, where ρ is the natural representation of GL_n. A generalization to a Grassmann variety $Gr(n,m)$, where ρ is the m-th exterior power representation of GL_n, was carried out by Schmidt ([Scm], $k = \mathbb{Q}$) and by Thunder ([Th1], for any k). Using the norm defined in 3.1, their results show

$$N(Gr(n,m),T,g) \sim cT^n \text{ as } T \to \infty$$

where c is the constant depending only on k and m. As for general $X = G/P$, G being semisimple, Franke–Manin–Tschinkel ([F-M-T]) and Peyre ([Pe]) obtained the asymptotic formula for $N(X,T)$ by using the special height via the anticanonical bundle corresponding to $\chi = -2\rho_P$, where ρ_P is one half sum of k-roots occuring in the unipotent radical of P. For instance, this height is the n-th power of the usual one for $\mathbb{P}^{n-1}$. Their method is to study the analytic property, the residue at $s = 1$, of the height zeta function

$$\sum_{x \in X(k)} H(x)^{-s},$$

which can be regarded as Eisenstein series owing to the ralation $H_g(x)^{[k:\mathbb{Q}]} = |\chi(p)|_{\mathbb{A}}$ mentioned as above. Actually, their result holds for the twisted height H_g and takes the following form

$$N(X,T,g) \sim cT(\log T)^{r-1} \text{ as } T \to \infty,$$

where r is the k-rank of the center of a Levi subgroup of P and c is the constant which is independent of g.

Finally, we remark that for the set

$$B(T) = \{gP(\mathbb{A})^1 \in G(\mathbb{A})^1/P(\mathbb{A})^1 \colon H_g(x_0) \leq T\},$$

we have the equality as in 3.3,

$$\mathrm{vol}(B(T)) = \frac{1}{\tau(P)} \int_{G(\mathbb{A})^1/G(k)} N(X,T,g) d(\omega_{G(\mathbb{A})^1}/\omega_{G(k)})(g)\,.$$

So, it would be interesting to know when there is a function $f(T)$ of T such that $N(X,T,g) \sim f(T)$ uniformly with respect to $g \in G(\mathbb{A})^1$ as $T \to \infty$, since in that case $N(X,T,g)$ is given asymptotically as $\tau(P)\tau(G)^{-1}\mathrm{vol}(B(T))$ by the above integral formula. For example, Schmidt–Thunder's asymptotic formula is consistent with Theorem 3.3.2.

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