Advanced Studies in Pure Mathematics 30, 2001
Class Field Theory – Its Centenary and Prospect
pp. 307–312

Hodge Cycles and Unramified Class Fields

Hiromichi Yanai

In the theory of complex multiplication, we obtain ramified class fields by the torsion points of a CM abelian variety and unramified class fields as certain fields of moduli (cf. [S-T]). Several authors studied the class fields obtained by complex multiplication ([K], [R], [S], [Mau], [O]).

When the abelian variety has "many" Hodge cycles (hence the Hodge group is "small"), it is known that the ramified class fields are "small" (cf. [R]). On the contrary, we know no clear relation between Hodge cycles and unramified class fields. In certain cases, however, the existence of exceptional Hodge cycles helps us to say something about the unramified class fields.

In the previous paper [DCM], we have shown a method of constructing CM abelian varieties with exceptional Hodge cycles and we have given several examples in which we can determine the degrees of the unramified class fields obtained as fields of moduli.

In the present note, we shall generalize the result of [DCM] and explain how the exceptional Hodge cycles influence the unramified class fields.

§1. Exceptional Hodge Cycles

Let A be a CM abelian variety of type (K, S) defined over a subfield of $\mathbf{C}$ with $\dim A = d$, where K is a CM-field of degree $2d$ and S is a CM-type of K (cf. [S-T]). Let $\mathbf{Hg} = \mathbf{Hg}(A)$ be the Hodge (or the special Mumford-Tate) group of A, that is a sub algebraic torus of $GL(H^1(A, \mathbf{Q}))$. It is known that $\dim \mathbf{Hg} \leq d$. When A is simple and $\dim \mathbf{Hg} < d$, there exist certain Hodge cycles on a product A^k of several copies of A that are not generated by the divisor classes; we call such Hodge cycles *exceptional* (cf. [DCM]).

Received August 31, 1998.
Revised April 12, 1999.

In this section, we give a method of constructing CM abelian varieties with exceptional Hodge cycles. This generalizes [DCM Theorem 2.1].

Consider a chain of three distinct CM-fields

$$K \supset K_1 \supset K_2,$$

which we regard as subfields of $\mathbf{C}$. Let $2d,\ 2d_1,\ 2d_2$ be the degrees of $K,\ K_1,\ K_2$ over $\mathbf{Q}$, respectively. Let $\Gamma,\ \Gamma_1,\ \Gamma_2$ be the sets of the embeddings of these fields into $\mathbf{C}$ and let $\pi_1 : \Gamma \to \Gamma_1, \pi_{12} : \Gamma_1 \to \Gamma_2$ be the canonical surjections.

Take a CM-type S of K satisfying the following condition:

$$(*) \begin{cases} \text{For each } \mu \in \Gamma_2, \text{there exists a nonnegative integer } a_\mu \text{ such that} \\ \text{for each } \tau \in \Gamma_1 \text{ with } \pi_{12}(\tau) = \mu,\ \#\{\sigma \in S \mid \pi_1(\sigma) = \tau\} = a_\mu. \end{cases}$$

Under this condition, we have $a_\mu + a_{\mu\rho} = [K : K_1] = \dfrac{d}{d_1}$, where ρ denotes the complex conjugation.

Our theorem is

Theorem 1.1. *Take a CM-type (K, S) satisfying the condition $(*)$. Let A be a CM abelian variety of type (K, S) and let $\mathbf{Hg}$ be its Hodge group. Then we have*

$$\dim \mathbf{Hg} \leq d - (d_1 - d_2).$$

Remark 1.2. Our theorem gives an upper bound for the dimension of the Hodge group under the condition $(*)$. We should note that L. Mai [Mai] has discussed certain lower bounds.

If the abelian variety A considered in Theorem 1.1 is simple, we have exceptional Hodge cycles on A itself (see Section 2). This does not necessarily hold in general degenerate (i.e., $\dim \mathbf{Hg} < d$) cases (cf. [W]).

Proof of Theorem. Since our argument is similar to that of [DCM], we give only an outline.

For a CM-field F, $T_F = Res_{F/\mathbf{Q}}(\mathbf{G_m})$ denotes the algebraic torus corresponding to the multiplicative group $F^\times$ and T_F^+ denotes the kernel of the norm map $N_{F/F^+} : T_F \to T_{F^+}$, where F^+ is the maximal real subfield of F.

Since K acts on the cohomology group $H^1(A, \mathbf{Q})$, we can regard T_K as a subgroup of $GL(H^1(A, \mathbf{Q}))$. It is known that $\mathbf{Hg} \subseteq T_K^+ \subset T_K$ (cf. [DCM]).

We consider several algebraic tori and morphisms between them:

$$T_K \underset{N_{K/K_1}}{\longrightarrow} T_{K_1} \underset{j}{\supset} T_{K_2} \supset T_{K_2}^+,$$

where N_{K/K_1} is the norm map and j is the closed immersion.

Put

$$T = N_{K/K_1}^{-1}(j(T_{K_2}^+)) \cap T_K^+.$$

Then we can show

$$\mathbf{Hg} \subseteq T \quad \text{and} \quad \dim(T_K^+/T) = d_1 - d_2$$

(see the proof of [DCM Theorem 2.1]). Since $\dim T_K^+ = d$, these imply our Theorem. □

Example 1.3. Let K be the cyclotomic field of 37-th roots of unity. We naturally identify the Galois group $\Gamma = Gal(K/\mathbf{Q})$ with the group $(\mathbf{Z}/37\mathbf{Z})^\times$. Let K_1, K_2 be the subfields of K of degree 12 and 4 over $\mathbf{Q}$, and H_1, H_2 be the corresponding subgroups of Γ, respectively. Then $H_1 = \{1, 26, 10\}$ and $H_2 = H_1 \cup 16H_1 \cup 34H_1$. The coset decomposition of Γ is

$$\Gamma = H_2 \cup 2H_2 \cup 36H_2 \cup 35H_2.$$

Put $\mu = H_2$, $\mu' = 2H_2 \in \Gamma_2$, so that $\rho\mu = 36H_2$, $\rho\mu' = 35H_2$. Taking $a_\mu = 2$, $a_{\mu'} = 3$, $a_{\rho\mu} = 1$, $a_{\rho\mu'} = 0$, we choose two elements from each of H_1, $16H_1$, $34H_1$ and choose all elements of $2H_2$. From $36H_2$ and $35H_2$, we must choose the elements that are not complex conjugate to any elements selected before. In this way, we get a simple CM-type S satisfying the condition $(*)$; for example,

$$S = \{1, 2, 3, 7, 12, 14, 15, 16, 18, 20, 24, 26, 27, 28, 29, 31, 32, 33\}.$$

In this case, we have $\dim \mathbf{Hg} = 18 - (6 - 2) = 14$. Moreover, we can construct an exceptional Hodge cycle by the elements of (for example) $H_1 \cup 21H_1 = \{1, 10, 21, 25, 26, 28\}$ (cf. Remark 2.2 below).

§2. Unramified Class Fields

In this section, we explain how the exceptional Hodge cycles obtained in Theorem 1.1 restrict the degree of the field of moduli.

From now on, we assume that the CM-field K is abelian over $\mathbf{Q}$. Let A be a simple, principal (in the sense of [S-T]) CM abelian variety of type (K, S) and let $\mathbf{Hg}$ be its Hodge group. Let $X_K = X(T_K)$ be the character group of T_K, that is naturally identified with the group ring $\mathbf{Z}[\Gamma]$; each element of X_K can act on the ideal classes of K.

Let $\mathbf{Hg}^{\perp} \subset X_K$ be the group of the annihilators of $\mathbf{Hg}$ in X_K. Take an embedding $\theta : K \to \mathrm{End}(A) \otimes \mathbf{Q}$ inducing the CM-type S and take a polarization $\mathcal{C}$ of A. Then the field of moduli M of the triple $(A, \theta, \mathcal{C})$ is an unramified class field over K. We denote the ideal class group of K by C_K and denote the subgroup of C_K corresponding to M by $C_K(S)$.

The next proposition is none other than [DCM Proposition 3.1].

Proposition 2.1. *For $\mathbf{A} \in C_K$ and $x \in \mathbf{Hg}^{\perp}$, we have $\mathbf{A}^x \in C_K(S)$.*

In the rest of this section, we assume that K, K_1, K_2 and S are as in Theorem 1.1. We can describe certain elements of $\mathbf{Hg}^{\perp}$ explicitly:

Proposition 2.2. *For $\mu \in \Gamma_2$ and $\tau, \tau' \in \Gamma_1$ satisfying $\tau \neq \tau'$ and $\tau_{12}(\tau) = \tau_{12}(\tau') = \mu$, put*

$$y = y(\mu, \tau, \tau') = \sum_{\substack{\sigma \in \Gamma \\ \pi_1(\sigma) = \tau}} \sigma + \sum_{\substack{\sigma' \in \Gamma \\ \pi_1(\sigma') = \tau' \rho}} \sigma'.$$

Then we have

(i) *$y \in \mathbf{Hg}^{\perp}$ and $y \notin (T_K^{+})^{\perp}$,*
(ii) *for each $\mathbf{A} \in C_K$, $\mathbf{A}^y \in C_1$,*
where C_1 is the image of the natural map $C_{K_1} \to C_K$.

Remark 2.3. The element y is corresponding to a (complex valued) exceptional Hodge cycle on A. See [H], [P].

Proof of Proposition 2.2. The following characterization of the elements of $\mathbf{Hg}^{\perp}$ can be found in the proof of [DCM Proposition 1.1].

For $x = \sum_{\gamma \in \Gamma} n_\gamma \gamma \in X_K$,

$$x \in \mathbf{Hg}^{\perp} \Leftrightarrow \text{for each } g \in \Gamma, \sum_{\gamma g \in S} n_\gamma = \sum_{\gamma' g \in S\rho} n_{\gamma'}.$$

We rewrite y so that $y = \sum_{\gamma \in \Gamma} m_\gamma \gamma$. Then, for each $g \in \Gamma$,

$$\sum_{\gamma g \in S} m_\gamma = a_{\mu g} + a_{\mu\rho g} = \frac{d}{d_1} = \left(\frac{d}{d_1} - a_{\mu g}\right) + \left(\frac{d}{d_1} - a_{\mu\rho g}\right) = \sum_{\gamma' g \in S\rho} m_{\gamma'}.$$

Hence $y \in \mathbf{Hg}^{\perp}$. The assumption $\tau \neq \tau'$ implies $y\rho \neq y$; this proves $y \notin (T_K^+)^{\perp}$. We have thus obtained (i).

We can easily see $\mathbf{A}^y = (N_{K/K_1}(\mathbf{A}))^{\tau+\tau'\rho} \in C_1$. This proves (ii). $\square$

Let D be the subgroup of C_K generated by the ideal classes of the form $\mathbf{A}^{y(\mu,\tau,\tau')}$, where $\mathbf{A} \in C_K$ and μ, τ, τ' are as in Proposition 2.2. Then, by Proposition 2.1 and 2.2, we have

$$D \subseteq C_1 \cap C_K(S) \subseteq C_K.$$

In some cases, it may happen that $D = C_1 \subseteq C_K(S)$. If this is the case then we can say that a sub CM-field K_1 with "large" class number causes a "small" unramified class field.

Finally, we recall the examples in [DCM].

In [DCM Example 3.2], K is the cyclotomic field of 31-st roots of unity; K_1 and K_2 are the sub fields of degree 6 and 2 over $\mathbf{Q}$, respectively. Here we have $h_K = h_K^- = h_{K_1} = h_{K_1}^- = 9$ and $\#C_K(S) = 3$.

In [DCM Example 3.4], K is the cyclotomic field of 61-st roots of unity; K_1 and K_2 are the sub fields of degree 20 and 4 over $\mathbf{Q}$, respectively. Here we have $h_K = h_K^- = 41 \cdot 1861$, $h_{K_1} = h_{K_1}^- = 41$ and $\#C_K(S) = 41$.

In both examples, we have $D = C_1 = C_K(S)$.

References

[H] F. Hazama, *Hodge cycles on Abelian varieties of CM-type*, Res. Act. Fac. Sci. Engrg. Tokyo Denki Univ., **5** (1983), 31–33.

[K] T. Kubota, *On the field extension by complex multiplication*, Trans. AMS, **118**, No.6 (1965), 113–122.

[Mai] L. Mai, *Lower bounds for the rank of a CM-type*, J. Number Theory, **32**, No.2 (1989), 192–202.

[Mau] E. Maus, *On the field of moduli of an abelian variety with complex multiplication*, Symp. Math., **XV** (1973), 551–555.

[O] A. I. Ovseevic, *Abelian extensions of fields of CM type*, Func. Anal. Appl., 8, No.1 (1974), 14–20.

[P] H. Pohlmann, *Algebraic cycles on abelian varieties of complex multiplication type*, Ann. of Math., (2) **88** (1968), 161–180.

[R] K. A. Ribet, *Division fields of abelian varieties with complex multiplication*, Mém. Soc. Math. France, No.2 (1980), 75–94.

[S] G. Shimura, *On the class fields obtained by complex multiplication of abelian varieties*, Osaka Math. J., **14** (1962), 33–44.

[S-T] G. Shimura and Y. Taniyama, *Complex Multiplication of Abelian Varieties and Its Application to Number Theory*, Publ. Math. Soc. Japan (1961).

[W] S. P. White, *Sporadic cycles on CM abelian varieties*, Compositio Math., **88** (1993), 123–142.

[DCM] H. Yanai, *On Degenerate CM-types*, J. Number Theory, **49**, No.3 (1994), 295–303.

Aichi Institute of Technology
Yakusa, Toyota 470-0392, Japan
E-mail address: `yanai@ge.aitech.ac.jp`