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A Duality of a Twisted Group Algebra of the Hyperoctahedral Group and the Queer Lie Superalgebra

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§1. Introduction

We establish a duality relation (Theorem 4.2) between one of the twisted group algebras of the hyperoctahedral group H_k (or the Weyl group of type B_k) and a Lie superalgebra $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ for any integers $k \geq 4$ and n_0, $n_1 \geq 1$. Here $\mathfrak{q}(n_0)$ and $\mathfrak{q}(n_1)$ denote the "queer" Lie superalgebras as called by some authors. The twisted group algebra $\mathcal{B}'_k$ in focus in this paper belongs to a different cocycle from the one $\mathcal{B}_k$ used by A. N. Sergeev in his work [8] on a duality with $\mathfrak{q}(n)$ and by the present author in a previous work [11]. This $\mathcal{B}'_k$ contains the twisted group algebra $\mathcal{A}_k$ of the symmetric group $\mathfrak{S}_k$ in a straightforward manner (cf. §1. 1. 1), and has a structure similar to the semidirect product of $\mathcal{A}_k$ and $\mathbb{C}[(\mathbb{Z}/2\mathbb{Z})^k]$. ($\mathcal{B}'_k$ and $\mathcal{B}_k$ were denoted by $\mathbb{C}^{[-1,+1,+1]}W_k$ and $\mathbb{C}^{[+1,+1,-1]}W_k$ respectively by J. R. Stembridge in [10].)

In §2, we construct the $\mathbb{Z}_2$-graded simple $\mathcal{B}'_k$-modules (where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$) using an analogue of the little group method. These simple $\mathcal{B}'_k$-modules are slightly different from the non-graded simple $\mathcal{B}'_k$-modules constructed by Stembridge in [10] because of the difference between $\mathbb{Z}_2$-graded and non-graded theories, but they can easily be translated into each other. We will use the algebra $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$, where $\mathcal{C}_k$ is the 2^k-dimensional Clifford algebra (cf. (3.2)) and $\dot{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product (cf. [1], [2], [11, §1]), as an intermediary for establishing our duality, as we explain below. The construction of the simple $\mathcal{B}'_k$-modules leads to a construction of the simple $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$-modules in §3.

In §4, we define a representation of $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$ in the k-fold tensor product $W = V^{\otimes k}$ of $V = \mathbb{C}^{n_0+n_1} \oplus \mathbb{C}^{n_0+n_1}$, the space of the natural representation of the Lie superalgebra $\mathfrak{q}(n_0 + n_1)$. This representation

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of $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$ depends on n_0 and n_1, not just $n_0 + n_1$. Note that $\mathcal{B}_k$ can be regarded as a subalgebra of $\mathcal{C}_k \otimes \mathcal{B}'_k$, since $\mathcal{B}_k$ is isomorphic to $\mathcal{C}_k \dot{\otimes} \mathcal{A}_k$ by our previous result (cf. (3.3) of [11]). Under this embedding, our representation of $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$ restricts to the representation of $\mathcal{B}_k$ in W defined by Sergeev (cf. Theorem A). We show that the centralizer of $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$ in $\mathrm{End}(W)$ is generated by the action of the Lie superalgebra $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ (Theorem 4.1). Moreover we show that $\mathcal{B}'_k$ and $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ act on a subspace W^ε of W "as mutual centralizers of each other" (Theorem 4.2). Note that $\mathcal{A}_k$ and $\mathfrak{q}(n)$ act on the same space W^ε "as mutual centralizers of each other" (cf. Theorem B).

In Appendix, we include short explanations of some known results, which we use in the previous sections.

In this paper, all vector spaces, and associative algebras, and representations are assumed to be finite dimensional over $\mathbb{C}$ unless specified otherwise. The precise statements of the results sketched in the introduction use the formulation of $\mathbb{Z}_2$-graded representations of $\mathbb{Z}_2$-graded algebras (superalgebras) (cf. §1.1.3) as was used in [1] and [2].

1.1. Preliminaries

1.1.1. A twisted group algebra $\mathcal{B}'_k$. For any $k \geq 1$, let $\mathcal{B}'_k$ denote the associative algebra generated by τ' and the elements γ_i, $1 \leq i \leq k-1$, with relations

(1.1)
$$\begin{aligned}
&\tau'^2 = 1, \gamma_i^2 = -1 \quad (-1 \leq i \leq k-1), \quad (\gamma_i\gamma_{i+1})^3 = -1 \quad (1 \leq i \leq k-2),\\
&(\gamma_i\gamma_j)^2 = -1 \quad (|i-j| \geq 2), \quad (\tau'\gamma_i)^2 = 1 \quad (2 \leq i \leq k-1),\\
&(\tau'\gamma_1)^4 = 1.
\end{aligned}$$

If $k \geq 4$, then $\mathcal{B}'_k$ is isomorphic to a twisted group algebra of the hyperoctahedral group H_k with a non-trivial 2-cocycle (cf. [10, Prop. 1.1]). We regard $\mathcal{B}'_k$ as a $\mathbb{Z}_2$-graded algebra by giving the generator τ' (resp. the generator γ_i, $1 \leq i \leq k-1$) degree 0 (resp. degree 1). Note that this grading of $\mathcal{B}'_k$ is different from that of $\mathcal{B}_k$ in (3.1) or in [11].

Let $\mathcal{A}_k$ denote the $\mathbb{Z}_2$-graded subalgebra of $\mathcal{B}'_k$ generated by γ_i, $1 \leq i \leq k-1$. If $k \geq 4$, then $\mathcal{A}_k$ is isomorphic to a twisted group algebra of the symmetric group $\mathfrak{S}_k$ with a non-trivial 2-cocycle, with the $\mathbb{Z}_2$-grading as in [2] and [11].

1.1.2. Partitions and symmetric functions. Let P_k denote the set of all partitions of k, and put $P = \coprod_{k\geq 0} P_k$. For $\lambda \in P$, we write $l(\lambda)$ for the length of λ, namely the number of non-zero parts of λ.

Also we write $|\lambda| = k$ if $\lambda \in P_k$. Let DP_k and OP_k denote the distinct partitions (or strict partitions, namely partitions whose parts are distinct) and the odd partitions (namely partitions whose parts are all odd) of k respectively. Let DP_k^+ and DP_k^- be the sets of all $\lambda \in DP_k$ such that $(-1)^{k-l(\lambda)} = +1$ and -1 respectively. Note that $(-1)^{k-l(\lambda)}$ equals the signature of permutations with cycle type λ. We also put $DP = \coprod_{k\geq 0} DP_k$ and $OP = \coprod_{k\geq 0} OP_k$. Let $(DP^2)_k$ (resp. $(OP^2)_k$) denote the set of all $(\lambda, \mu) \in DP^2$ (resp. OP^2) such that $|\lambda| + |\mu| = k$. Let $(DP^2)_k^+$ and $(DP^2)_k^-$ be the sets of all $(\lambda, \mu) \in (DP^2)_k$ such that $(-1)^{k-l(\lambda)-l(\mu)} = +1$ and -1 respectively.

Let Λ_x denote the ring of the symmetric functions in the variables $x = \{x_1, x_2, \dots\}$ with coefficients in $\mathbb{C}$; namely our Λ_x is the scalar extension of the Λ_x in [6], which is $\mathbb{Z}$-algebra, to $\mathbb{C}$.

Let Ω_x denote the subring of Λ_x generated by the power sums of odd degrees, namely the $p_r(x)$, $r = 1, 3, 5, \dots$. Then $\{p_\mu(x) \mid \mu \in OP\}$ is a basis of Ω_x, where $p_\mu = \prod_{i\geq 1} p_{\mu_i}$. For $\lambda \in DP$, let $Q_\lambda(x) \in \Lambda_x$ denote Schur's Q-function indexed by λ (cf. [7], [9, §6]). Then $\{Q_\lambda(x) \mid \lambda \in DP\}$ is also a basis of Ω_x.

1.1.3. Semisimple superalgebras. This theory of semisimple superalgebras was developed by T. Józefiak in [1], which we mostly follow. A $\mathbb{Z}_2$-graded algebra A, which is called a **superalgebra** in this paper, is called **simple** if it does not have non-trivial $\mathbb{Z}_2$-graded two-sided ideals. If A is a simple superalgebra, then it is either isomorphic to $M(m,n)$ (denoted by $M(m|n)$ in [2]) for some m and n, or isomorphic to $Q(n)$ for some n (see [2], [11, §1] for the definitions of simple superalgebras $M(m,n)$, $Q(n)$).

Let V be an A-**module**, namely a $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ together with a representation ρ: $A \to \mathrm{End}(V)$ satisfying $\rho(A_\alpha)V_\beta \subset V_{\alpha+\beta}$ (α, $\beta \in \mathbb{Z}_2$). We simply write $\rho(a)v = av$ for $a \in A$ and $v \in V$. By an A-submodule of V we mean a $\mathbb{Z}_2$-graded $\rho(A)$-stable subspace of V. We say that V is **simple** if it does not have non-trivial A-submodules.

Let V and W be two A-modules. Let $\mathrm{Hom}_A^\alpha(V,W)$ ($\alpha \in \mathbb{Z}_2$) denote the subspace of $\mathrm{Hom}^\alpha(V,W) = \{f \in \mathrm{Hom}(V,W)\,;\, f(V_\beta) \subset W_{\alpha+\beta}\}$ consisting of all elements $f \in \mathrm{Hom}^\alpha(V,W)$ such that $f(av) = (-1)^{\alpha\cdot\beta} af(v)$ for $a \in A_\beta$ ($\beta \in \mathbb{Z}_2$), $v \in V$. Put $\mathrm{Hom}_A^{\cdot}(V,W) = \mathrm{Hom}_A^0(V,W) \oplus \mathrm{Hom}_A^1(V,W)$ and put $\mathrm{End}_A^{\cdot}(V) = \mathrm{Hom}_A^{\cdot}(V,V)$. We call $\mathrm{End}_A^{\cdot}(V)$ the **supercentralizer** of A in $\mathrm{End}(V)$. Two A-modules V and W are called **isomorphic** if there exists an invertible linear map $f \in \mathrm{Hom}_A^{\cdot}(V,W)$. If this is the case, we write $V \cong_A W$ (or simply write $V \cong W$). If V and W are simple A-modules, then $V \cong W$ if and only if there exists an invertible element in $\mathrm{Hom}_A^0(V,W)$ or $\mathrm{Hom}_A^1(V,W)$. Note that, in [11] we distinguished between V and the shift of V which is defined to be the same vector space as V with the switched grading. In this paper, however, we identify V and the shift of V.

If V is a simple A-module, then $\mathrm{End}^{\cdot}_A(V)$ is isomorphic to either $M(1,0) \cong \mathbb{C}$ or $Q(1) \cong \mathcal{C}_1$ (cf. [1, Prop. 2.17], [2, Prop. 2.5, Cor. 2.6]). In the former (resp. latter) case, we say that V is of **type** M (resp. of **type** Q). This gives the following theorem (see [1], [2], [11, §1] for the definition of the "supertensor product" of the superalgebras or modules).

Theorem 1.1. *Let $C = A \dot{\otimes} B$ be the supertensor product of superalgebras A and B and let $V = U \otimes W$ be the supertensor product of a simple A-module U and a simple B-module W.*

(a) *If U, W are of type M, then V is a simple C-module of type M.*
(b) *If one of U and W is of type M and the other is of type Q, then V is a simple C-module of type Q.*
(c) *If U and W are of type Q, then V is a sum of two copies of a simple C-module X of type M: $V = X \oplus X$.*

Moreover, the above construction gives all simple $A \dot{\otimes} B$-modules.

Using the above U, W, V and X, define an $A \dot{\otimes} B$-module $U \dot{\circ} W$ by

$$U \dot{\circ} W = \begin{cases} V & \text{if } U \text{ or } W \text{ is of type } M, \\ X & \text{if } U \text{ and } W \text{ are of type } Q. \end{cases} \tag{1.2}$$

Let $\mathrm{Irr}\, A$ denote the set of all isomorphism classes of simple A-modules for any superalgebra A.

Corollary 1.2. *We have a bijection*

$$\dot{\circ}\colon \mathrm{Irr}\, A \times \mathrm{Irr}\, B \ni (U,W) \overset{\sim}{\mapsto} U \dot{\circ} W \in \mathrm{Irr}\, A \dot{\otimes} B.$$

§2. Simple modules for $\mathcal{B}'_k$

The simple $\mathcal{A}_k$-modules are parametrized by DP_k (cf. [2], [7], [9]). For $\lambda \in DP_k$, let V_λ denote a simple $\mathcal{A}_k$-module indexed by λ. Then V_λ is of type M (resp. of type Q) if $\lambda \in DP_k^+$ (resp. $\lambda \in DP_k^-$). We construct a $\mathcal{B}'_k$-module $V_{\lambda,\mu}$ for $(\lambda,\mu) \in (DP^2)_k$ as follows. Define a surjective homomorphism of superalgebras $\pi_k\colon \mathcal{B}'_k \to \mathcal{A}_k$ (resp. $\pi'_k\colon \mathcal{B}'_k \to \mathcal{A}_k$) by $\pi_k(\tau') = 1$, $\pi_k|_{\mathcal{A}_k} = \mathrm{id}_{\mathcal{A}_k}$ (resp.$\pi'_k(\tau') = -1$, $\pi'_k|_{\mathcal{A}_k} = \mathrm{id}_{\mathcal{A}_k}$). The simple $\mathcal{A}_{k'}$ (resp. $\mathcal{A}_{k-k'}$)-module V_λ (resp. V_μ) can be lifted to a $\mathcal{B}'_{k'}$ (resp.

$\mathcal{B}'_{k-k'}$)-module via $\pi_{k'}$ (resp. $\pi'_{k-k'}$), where $k' = |\lambda|$. This (simple) $\mathcal{B}'_{k'}$ (resp. $\mathcal{B}'_{k-k'}$)-module is denoted by $V_{\lambda,\phi}$ (resp. $V_{\phi,\mu}$). Let $V_{\lambda,\mu}$ denote the $\mathcal{B}'_k$-module induced from the $\mathcal{B}'_{k'} \dot{\otimes} \mathcal{B}'_{k-k'}$-module $V_{\lambda,\phi} \dot{\circ} V_{\phi,\mu}$, namely

$$V_{\lambda,\mu} = \mathcal{B}'_k \otimes_{\mathcal{B}'_{k'} \dot{\otimes} \mathcal{B}'_{k-k'}} (V_{\lambda,\phi} \dot{\circ} V_{\phi,\mu})$$

(see the definition of $\dot{\circ}$ in (1.2)), where $\mathcal{B}'_{k'} \dot{\otimes} \mathcal{B}'_{k-k'}$ is embedded into $\mathcal{B}'_k$ via

$$\tau' \dot{\otimes} 1 \mapsto \tau', \quad \gamma_i \dot{\otimes} 1 \mapsto \gamma_i \quad (1 \le i \le k'-1),$$
$$1 \dot{\otimes} \tau' \mapsto \tau'_{k'+1}, \quad 1 \dot{\otimes} \gamma_j \mapsto \gamma_{k'+j} \quad (1 \le j \le k-k'-1)$$

where $\tau'_i = \gamma_{i-1}\gamma_{i-2}\cdots\gamma_1\tau'\gamma_1\cdots\gamma_{i-2}\gamma_{i-1}$, $1 \le i \le k$.

Theorem 2.1. (cf. [10, Th. 7.1]) *$\{V_{\lambda,\mu} \mid (\lambda,\mu) \in (DP^2)_k\}$ is a complete set of the isomorphism classes of simple $\mathcal{B}'_k$-modules. $V_{\lambda,\mu}$ is of type M (resp. of type Q) if $(\lambda,\mu) \in (DP^2)_k^+$ (resp. $(\lambda,\mu) \in (DP^2)_k^-$)*

The proof is analogous to the little group method, and is omitted. It can also be shown that this parametrization coincides with that by Stembridge in [10, Th. 7.1] modulo the usual difference between $\mathbb{Z}_2$-graded and non-graded modules.

If $(\lambda,\mu) \in (DP^2)_k^-$, then fix a non-zero homogeneous element $x_{\lambda,\mu}$ of $\mathrm{End}_{\mathcal{B}'_k}(V_{\lambda,\mu}) \cong Q(1)$ of degree 1.

§3. The algebras $\mathcal{B}_k$ and $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$

For any $k \ge 1$, let $\mathcal{B}_k$ denote the associative algebra generated by τ and the elements σ_i, $1 \le i \le k-1$, with relations

(3.1)
$$\tau^2 = \sigma_i^2 = 1 \quad (1 \le i \le k-1), \quad (\sigma_i\sigma_{i+1})^3 = 1 \quad (1 \le i \le k-2),$$
$$(\sigma_i\sigma_j)^2 = 1 \quad (|i-j| \ge 2), \quad (\tau\sigma_i)^2 = 1 \quad (2 \le i \le k-1),$$
$$(\tau\sigma_1)^4 = -1.$$

We regard $\mathcal{B}_k$ as a superalgebra by giving the generator τ' (resp. the generator σ_i, $1 \le i \le k-1$) degree 1 (resp. degree 0). The subgroup of $(\mathcal{B}_k)^\times$ generated by σ_i, $1 \le i \le k-1$, is isomorphic to the symmetric group of degree k and it is denoted by $\mathfrak{S}_k$.

Let $\mathcal{C}_k$ denote the 2^k-dimensional Clifford algebra, namely $\mathcal{C}_k$ is generated by $\xi_1, \ldots, \xi_k$ with relations

$$(3.2) \qquad \xi_i^2 = 1, \quad \xi_i\xi_j = -\xi_j\xi_i \quad (i \ne j)\ .$$

We regard $\mathcal{C}_k$ as a superalgebra by giving the generator ξ_i, $1 \le i \le k$, degree 1. $\mathcal{C}_k$ is a simple superalgebra. Let X_k be a unique simple $\mathcal{C}_k$-module. If k is even (resp. odd), then X_k is of type M (resp. of type Q). If k is odd, then fix a non-zero element z_k of $\mathrm{End}^1_{\mathcal{C}_k}(X_k)$.

Define a linear map $\vartheta\colon \mathcal{B}_k \to \mathcal{C}_k \dot{\otimes} \mathcal{A}_k$ by

$$
\begin{aligned}
&\vartheta(\tau_i) \mapsto \xi_i \otimes 1 \qquad (1 \le i \le k),\\
&\vartheta(\sigma_j) \mapsto \frac{1}{\sqrt{2}}(\xi_j - \xi_{j+1}) \otimes \gamma_j \;\; (1 \le j \le k-1)
\end{aligned}
\tag{3.3}
$$

where $\tau_i = \sigma_{i-1} \cdots \sigma_1 \tau \sigma_1 \ldots \sigma_{i-1}$. Then ϑ is an isomorphism of algebras (cf. [11, Th. 3.2]). For $\lambda \in DP_k$, define a $\mathcal{B}_k$-module W_λ by $W_\lambda = X_k \dot{\circ} V_\lambda$. By Corollary 1.2, $\{W_\lambda \mid \lambda \in DP_k\}$ is a complete set of isomorphism classes of simple $\mathcal{B}_k$-modules.

Let $\hat{\mathcal{B}}_k$ denote the supertensor product (cf. [1], [2], [11, §1]) of the algebras $\mathcal{C}_k$ and $\mathcal{B}'_k$, namely $\hat{\mathcal{B}}_k = \mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$. Since $\mathcal{B}_k \cong \mathcal{C}_k \dot{\otimes} \mathcal{A}_k$, $\mathcal{B}_k$ can be regarded as a subalgebra of $\hat{\mathcal{B}}_k$. For $(\lambda, \mu) \in (DP^2)_k$, put $W_{\lambda,\mu} = X_k \dot{\circ} V_{\lambda,\mu}$. By Theorem 1.1 and (1.2), $W_{\lambda,\mu}$ is of type M (resp. of type Q) if $l(\lambda) + l(\mu)$ is even (resp. odd). By Corollary 1.2, $\{W_{\lambda,\mu} \mid (\lambda,\mu) \in (DP^2)_k\}$ is a complete set of isomorphism classes of simple $\hat{\mathcal{B}}_k$-modules.

§4. A duality of $\mathcal{B}'_k$ and $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$

Let $\mathfrak{q}(n)$ denote the Lie subsuperalgebra of $\mathfrak{gl}(n,n)$ (denoted by $l(n,n)$ in [5]) consisting of the matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$. The Jacobi product $[\ ,\ ]\colon \mathfrak{q}(n) \times \mathfrak{q}(n) \to \mathfrak{q}(n)$ is defined by $[X,Y] = XY - (-1)^{\overline{X}\cdot\overline{Y}}YX$ for all homogeneous elements $X, Y \in \mathfrak{q}(n)$, where the symbol $^{-}$ expresses the degree of a homogeneous element. This Lie superalgebra is called the queer Lie superalgebra. Let $\mathcal{U}_n = \mathcal{U}(\mathfrak{q}(n))$ denote the universal enveloping algebra of $\mathfrak{q}(n)$, which can be regarded as a superalgebra. Let W denote the k-fold supertensor product of the $2n$-dimensional natural representation $V = V_0 \oplus V_1$, $\mathbf{dim}\, V = (n,n)$, namely $W = V^{\otimes k}$, where $\mathbf{dim}\, V$ denotes the pair $(\dim V_0, \dim V_1)$. We define a representation $\Theta\colon \mathcal{U}_n \to \mathrm{End}(W)$ by

$$\Theta(X)(v_1 \otimes \cdots \otimes v_k) = \sum_{j=1}^{k} (-1)^{\overline{X}\cdot(\overline{v_1} + \cdots + \overline{v_{j-1}})} v_1 \otimes \cdots \otimes \underbrace{Xv_j}_{j} \otimes \cdots \otimes v_k$$

for all homogeneous elements $X \in \mathfrak{q}(n)$ and $v_i \in V$ $(1 \le i \le k)$. Note that $\mathcal{U}_n$ is an infinite dimensional superalgebra. However, for a fixed number k, $\mathcal{U}_n$ acts on W through its finite dimensional image in $\mathrm{End}(W)$. Therefore we can use the results in §1.1.3 on finite dimensional superalgebras and their finite dimensional modules.

Let n_0 and n_1 be two positive integers such that $n_0 + n_1 = n$. The Lie superalgebra $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ can be embedded into $\mathfrak{q}(n)$ via

(4.1)

$$\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1) \ni \left(\begin{pmatrix} A & B \\ B & A \end{pmatrix}, \begin{pmatrix} C & D \\ D & C \end{pmatrix} \right) \mapsto \begin{pmatrix} A & 0 & B & 0 \\ 0 & C & 0 & D \\ B & 0 & A & 0 \\ 0 & D & 0 & C \end{pmatrix} \in \mathfrak{q}(n).$$

The universal enveloping algebra of $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ is isomorphic to $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$ which can be embedded into $\mathcal{U}_n$ as a subalgebra generated by the elements of $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$.

Now we define a representation $\Psi\colon \hat{\mathcal{B}}_k \to \mathrm{End}(W)$, which depends on n_0 and n_1, by

(4.2)

$$\Psi(\xi_i \otimes 1)(v_1 \otimes \cdots \otimes v_k) = (-1)^{\overline{v_1} + \cdots + \overline{v_{i-1}}} v_1 \otimes \cdots \otimes P v_i \otimes \cdots \otimes v_k \qquad (1 \le i \le k),$$

$$\Psi(1 \otimes \tau')(v_1 \otimes \cdots \otimes v_k) = (Qv_1) \otimes v_2 \otimes \cdots \otimes v_k,$$

$$\begin{aligned} &\Psi(1 \otimes \gamma_j)(v_1 \otimes \cdots \otimes v_k) \\ &= \frac{(-1)^{\overline{v_1} + \cdots + \overline{v_{j-1}}}}{\sqrt{2}} v_1 \otimes \cdots \otimes (Pv_j) \otimes v_{j+1} \otimes \cdots \otimes v_k \\ &\quad - \frac{(-1)^{\overline{v_1} + \cdots + \overline{v_{j-1}} + \overline{v_j}}}{\sqrt{2}} v_1 \otimes \cdots \otimes v_j \otimes (Pv_{j+1}) \otimes \cdots \otimes v_k \\ &\qquad (1 \le j \le k-1) \end{aligned}$$

for all homogeneous elements $v_j \in V$, $1 \le j \le k$, where

$$P = \begin{pmatrix} 0 & -\sqrt{-1} I_n \\ \sqrt{-1} I_n & 0 \end{pmatrix} \in M(n,n)_1,$$

$$Q = \begin{pmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & -I_{n_1} & 0 & 0 \\ 0 & 0 & I_{n_0} & 0 \\ 0 & 0 & 0 & -I_{n_1} \end{pmatrix} \in Q(n)_0.$$

Note that, by the isomorphism $\vartheta\colon \mathcal{B}_k \cong \mathcal{C}_k \dot{\otimes} \mathcal{A}_k \subset \hat{\mathcal{B}}_k$, W can be regarded as a $\mathcal{B}_k$-module and this $\mathcal{B}_k$-module was investigated by Sergeev in [8] (cf. Theorem A). Then, observing the actions of $\vartheta(\tau)$, $\vartheta(\sigma_i) \in$

$\mathcal{C}_k \dot{\otimes} \mathcal{A}_k$, $1 \leq i \leq k-1$, on W, we have

(4.3)
$$\Psi(\vartheta(\tau))(v_1 \otimes \cdots \otimes v_k) = (Pv_1) \otimes \cdots \otimes v_k,$$
$$\Psi(\vartheta(\sigma_i))(v_1 \otimes \cdots \otimes v_k) = (-1)^{\overline{v_i} \cdot \overline{v_{i+1}}} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k$$

for all homogeneous elements $v_j \in V$, $1 \leq j \leq k-1$.

Let W' be a $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-submodule of W. Since $(\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1))_0 \cong \mathfrak{gl}(n_0, \mathbb{C}) \oplus \mathfrak{gl}(n_1, \mathbb{C})$ as a Lie algebra, and V is a sum of two copies (V_0 and V_1) of the natural representation of $\mathfrak{gl}(n_0, \mathbb{C}) \oplus \mathfrak{gl}(n_1, \mathbb{C})$, $W'|_{(\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1))_0}$ is embedded into a sum of tensor powers of the natural representation, so that this representation of $\mathfrak{gl}(n_0, \mathbb{C}) \oplus \mathfrak{gl}(n_1, \mathbb{C})$ can be integrated to a polynomial representation $\theta_{W'}$ of $GL(n_0, \mathbb{C}) \times GL(n_1, \mathbb{C})$. Let $\mathrm{Ch}[W']$ denote the character of $\theta_{W'}$, namely

$$\begin{aligned}&\mathrm{Ch}[W'](x_1, x_2, \ldots, x_{n_0}, y_1, y_2, \cdots, y_{n_1}) \\ &= \mathrm{tr}\, \theta_{W'}(\mathrm{diag}(x_1, x_2, \ldots, x_{n_0}), \mathrm{diag}(y_1, y_2, \ldots, y_{n_1})).\end{aligned}$$

The following theorem determines the supercentralizer of $\Psi(\hat{\mathcal{B}}_k)$ in $\mathrm{End}(W)$ and describes the characters of simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-modules appearing in W.

Theorem 4.1. (1) *The two superalgebras $\Psi(\hat{\mathcal{B}}_k)$ and $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$ act on W as the mutual supercentralizers of each other*:

$$\text{(4.4)} \quad \mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}(W) = \Psi(\hat{\mathcal{B}}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) = \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}).$$

(2) *The simple $\hat{\mathcal{B}}_k$-module $W_{\lambda,\mu}$ $((\lambda, \mu) \in (DP^2)_k)$ occurs in W if and only if $l(\lambda) \leq n_0$ and $l(\mu) \leq n_1$. Moreover we have*

$$\text{(4.5)} \qquad W \cong_{\hat{\mathcal{B}}_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} \bigoplus_{\substack{(\lambda,\mu) \in (DP^2)_k \\ l(\lambda) \leq n_0, l(\mu) \leq n_1}} W_{\lambda,\mu} \circ U_{\lambda,\mu}$$

where $U_{\lambda,\mu}$ denotes a simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-module.

(3) *We have $U_{\lambda,\mu} \cong_{\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}} U_\lambda \circ U_\mu$, where U_λ (resp. U_μ) denotes the simple $\mathcal{U}_{n_0}$ (resp. $\mathcal{U}_{n_1}$)-module corresponding to the simple $\mathcal{B}_{|\lambda|}$ (resp. $\mathcal{B}_{|\mu|}$)-module W_λ (resp. W_μ) in Sergeev's duality* (*cf. Theorem A*).

(4) *The character values of* $\mathrm{Ch}[U_{\lambda,\mu}]$ *are given as follows*:

(4.6)
$$\begin{aligned}&\mathrm{Ch}[U_{\lambda,\mu}](x_1, x_2, \ldots, x_{n_0}, y_1, y_2, \ldots, y_{n_1}) \\ &\quad = (\sqrt{2})^{d(\lambda,\mu) - l(\lambda) - l(\mu)} Q_\lambda(x_1, x_2, \ldots, x_{n_0}) Q_\mu(y_1, y_2, \ldots, y_{n_1})\end{aligned}$$

where $d\colon (DP^2)_k \to \mathbb{Z}_2$ *denotes a map defined by* $d(\lambda,\mu) = 0$ *(resp.* $d(\lambda,\mu) = 1$*) if* $l(\lambda)+l(\mu)$ *is even (resp.* $l(\lambda)+l(\mu)$ *is odd).*

Proof. First we will show the second equality of (4.4). Then the first equality also follows from the double supercentralizer theorem (abbreviated as DSCT) for semisimple superalgebras (cf. [11, Th. 2.1]).

By Theorem A (1), we have $\mathrm{End}^{\cdot}_{\Psi(\vartheta(\mathcal{B}_k))}(W) \supset \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$, since $\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$ is a subsuperalgebra of $\Theta(\mathcal{U}_n)$. Hence $\Theta(X \otimes Y)$ commutes with $\Psi(\vartheta(\mathcal{B}_k))$ for any $X \in \mathfrak{q}(n_0)$, $Y \in \mathfrak{q}(n_1)$. By direct calculations, it can be shown that $\Theta(X\otimes Y)$ and $\Psi(1\otimes\tau')$ also commute. Since $\hat{\mathcal{B}}_k$ is generated as an algebra by the elements $\vartheta(\tau_i)$, $1 \le i \le k$, the elements $\vartheta(\sigma_j)$, $1 \le j \le k-1$, and $1 \otimes \tau'$, we have $\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \supset \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$. We need only to show that

$$\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \subset \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}). \tag{4.7}$$

We have $\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \subset \mathrm{End}^{\cdot}_{\Psi(\vartheta(\mathcal{B}_k))}(W) = \Theta(\mathcal{U}_n)$ by Theorem A (1). It can be easily checked that $\Theta(\mathcal{U}_n) \subset Q(n) \dot{\otimes} \cdots \dot{\otimes} Q(n)$, where $Q(n)$ denotes the underlying vector space of $\mathfrak{q}(n)$ (or the superalgebra it forms), so that we have $\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \subset Q(n) \dot{\otimes} \cdots \dot{\otimes} Q(n)$. We identify $\mathrm{End}(W)$ with $\overbrace{\mathrm{End}(V) \dot{\otimes} \cdots \dot{\otimes} \mathrm{End}(V)}^{k}$ by defining the action of $f_1 \otimes f_2 \otimes \cdots \otimes f_k \in \mathrm{End}(V)^{\otimes k}$ on W by

$$\begin{aligned}&(f_1 \otimes f_2 \otimes \cdots \otimes f_k)(v_1 \otimes v_2 \otimes \cdots \otimes v_k)\\ &\quad = (-1)^{\overline{f_2}\cdot\overline{v_1} + \overline{f_3}\cdot(\overline{v_1}+\overline{v_2}) + \cdots + \overline{f_k}\cdot(\overline{v_1}+\cdots+\overline{v_{k-1}})} f_1 v_1 \otimes f_2 v_2 \otimes \cdots \otimes f_k v_k\end{aligned}$$

for all homogeneous elements $f_j \in \mathrm{End}(V)$ and $v_j \in V$, $1 \le j \le k$. Define a representation $\theta\colon \mathbb{C}[\mathfrak{S}_k] \to \mathrm{End}(\mathrm{End}(W))$ of $\mathbb{C}[\mathfrak{S}_k]$ by

$$\begin{aligned}&\theta(\sigma_i)(f_1 \otimes \cdots \otimes f_i \otimes f_{i+1} \otimes \cdots \otimes f_k)\\ &\quad = (-1)^{\overline{f_i}\cdot\overline{f_{i+1}}}(f_1 \otimes \cdots \otimes f_{i+1} \otimes f_i \otimes \cdots \otimes f_k)\end{aligned}$$

for all $1 \le i \le k-1$ and homogeneous elements f_j, $1 \le j \le k$, of $\mathrm{End}(V)$. Moreover, define elements T_i, $1 \le i \le k$, of $\mathrm{End}(\mathrm{End}(W))$ by

$$T_i(f_1 \otimes \cdots \otimes f_k) = f_1 \otimes \cdots \otimes Q f_i Q \otimes \cdots \otimes f_k$$

for all $f_j \in \mathrm{End}(V)$, $1 \le j \le k$. Furthermore put

$$S = \frac{1}{n!}\sum_{w \in \mathfrak{S}_k} \theta(w), \quad T = \prod_{i=1}^{k}\left(\frac{1}{2}(\mathrm{Id}_{\mathrm{End}(W)} + T_i)\right).$$

Note that, since $T_i \in \mathrm{End}^0(\mathrm{End}(W))$ for all i, the factors in the definition of T commute. If $f \in \mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W)$, then it follows that $S(f) = f$ and $\frac{1}{2}(\mathrm{Id}_{\mathrm{End}(W)} + T_i)(f) = f$, $1 \leq i \leq k$, since $\theta(\sigma)(f) = \Psi(\vartheta(\sigma)) \circ f \circ \Psi(\vartheta(\sigma))^{-1}$ and $T_i(f) = \Psi(1 \otimes \tau_i') \circ f \circ \Psi(1 \otimes \tau_i')$. Therefore, any element f of $\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W)$ can be expressed as a linear combination of elements of the form

$$ST(f_1 \otimes \cdots \otimes f_k)$$

with $f_j \in Q(n)$, $1 \leq j \leq k$. Since

$$T(f_1 \otimes \cdots \otimes f_k) = \left(\frac{1}{2}\right)^k (f_1 + Qf_1Q) \otimes \cdots \otimes (f_k + Qf_kQ)$$

and $f + QfQ$ belongs to $Q(n_0) \oplus Q(n_1)$ for any $f \in Q(n)$, we have

$$T\left(Q(n) \dot{\otimes} \cdots \dot{\otimes} Q(n)\right) \subset (Q(n_0) \oplus Q(n_1)) \dot{\otimes} \cdots \dot{\otimes} (Q(n_0) \oplus Q(n_1)).$$

Hence it follows that

$$\mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) \subset S\big((Q(n_0) \oplus Q(n_1)) \dot{\otimes} \cdots \dot{\otimes} (Q(n_0) \oplus Q(n_1))\big).$$

By induction on k, it can be shown that

$$S\big(\overbrace{(Q(n_0) \oplus Q(n_1)) \dot{\otimes} \cdots \dot{\otimes} (Q(n_0) \oplus Q(n_1))}^{k}\big)$$

is generated as an algebra by elements of the form $S(X \otimes 1 \otimes \cdots \otimes 1) = \frac{1}{n}\Theta(X)$ with $X \in \mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$. Therefore (4.7) follows.

Next we will show (2) and (3) simultaneously. Since V is a sum of the natural representations X and Y of $\mathfrak{q}(n_0)$ and $\mathfrak{q}(n_1)$ respectively: $V = X \oplus Y$, where $\mathbf{dim}\, X = (n_0, n_0)$, $\mathbf{dim}\, Y = (n_1, n_1)$, W can be decomposed into a sum of tensor powers of X and Y. Since a $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-submodule of W of the form $\cdots \otimes X \otimes Y \otimes \cdots$ is isomorphic to that of the form $\cdots \otimes Y \otimes X \otimes \cdots$, we have

$$W \cong_{\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}} \bigoplus_{k'=0}^{k} \left(\overbrace{X \otimes \cdots \otimes X}^{k'} \otimes \overbrace{Y \otimes \cdots \otimes Y}^{k-k'} \right)^{\oplus \binom{k}{k'}}.$$

From Theorem A (2), we have

$$(4.8)\qquad X^{\otimes k'} \cong_{\mathcal{B}_{k'}\dot{\otimes}\mathcal{U}_{n_0}} \bigoplus_{\substack{\lambda\in DP_{k'}\\ l(\lambda)\leq n_0}} W_\lambda \mathbin{\dot\circ} U_\lambda,$$

$$Y^{\otimes k-k'} \cong_{\mathcal{B}_{k-k'}\dot{\otimes}\mathcal{U}_{n_1}} \bigoplus_{\substack{\mu\in DP_{k-k'}\\ l(\mu)\leq n_1}} W_\mu \mathbin{\dot\circ} U_\mu.$$

Therefore, it follows that simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-modules which occur in W are of the form $U_\lambda \mathbin{\dot\circ} U_\mu$, $(\lambda,\mu) \in (DP^2)_k$, and that $U_\lambda \mathbin{\dot\circ} U_\mu$ occurs in W if and only if $l(\lambda) \leq n_0$ and $l(\mu) \leq n_1$. By (4.7) and DSCT, W can be decomposed into a sum of non-isomorphic simple $\hat{\mathcal{B}}_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-modules. In order to determine the simple $\hat{\mathcal{B}}_k$-module which is paired with the simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-module $U_\lambda \mathbin{\dot\circ} U_\mu$, we consider the $\mathcal{B}_{k'} \dot{\otimes} \mathcal{B}_{k-k'}$-submodule $\overbrace{X\otimes\cdots\otimes X}^{k'}\otimes\overbrace{Y\otimes\cdots\otimes Y}^{k-k'}$ of W. Since $\tau_i' \in \hat{\mathcal{B}}_{k'}$, $1 \leq i \leq k'$ (resp. $\tau_j' \in \hat{\mathcal{B}}_{k-k'}$, $1 \leq j \leq k-k'$), acts on $X^{\otimes k'}$ (resp. $Y^{\otimes k-k'}$) as $\mathrm{Id}_{X^{\otimes k'}}$ (resp. $-\mathrm{Id}_{Y^{\otimes k-k'}}$), the $\mathcal{B}_{k'}$ (resp. $\mathcal{B}_{k-k'}$)-submodule W_λ (resp. W_μ) of $X^{\otimes k'}$ (resp. $Y^{\otimes k-k'}$) can be regarded as a $\hat{\mathcal{B}}_{k'}$ (resp. $\hat{\mathcal{B}}_{k-k'}$)-module and is isomorphic to $W_{\lambda,\phi}$ (resp. $W_{\phi,\mu}$). From (4.8), a simple $\hat{\mathcal{B}}_k$-submodule of W which corresponds to $U_\lambda \mathbin{\dot\circ} U_\mu$ contains $W_{\lambda,\phi}\otimes W_{\phi,\mu}$ as a $\hat{\mathcal{B}}_{k'} \dot{\otimes} \hat{\mathcal{B}}_{k-k'}$-submodule. This condition forces this simple $\hat{\mathcal{B}}_k$-module to be isomorphic to $W_{\lambda,\mu}$. Consequently, the result (2) and (3) follow.

The result (4) immediately follows from Theorem A (3) and the fact that

$$\begin{aligned}&\mathrm{Ch}[U \mathbin{\dot\circ} U'](x_1,\dots,x_{n_0},y_1,\dots,y_{n_1})\\ &= \begin{cases} \mathrm{Ch}[U](x_1,\dots,x_{n_0})\,\mathrm{Ch}[U'](y_1,\dots,y_{n_1}) & \text{if } U \text{ or } U' \text{ is of type } M,\\ \dfrac{1}{2}\,\mathrm{Ch}[U](x_1,\dots,x_{n_0})\,\mathrm{Ch}[U'](y_1,\dots,y_{n_1}) & \text{if } U,\ U' \text{ are of type } Q.\end{cases}\end{aligned}$$

Q.E.D.

By Theorem 1.1, (1.2), Theorem 4.1 (3) and Theorem A, the simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-module $U_{\lambda,\mu}$ is of type M (resp. of type Q) if $l(\lambda)+l(\mu)$ is even (resp. odd). If $l(\lambda)+l(\mu)$ is odd, then fix a non-zero element $u_{\lambda,\mu}$ of $\mathrm{End}^1_{\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1}}(U_{\lambda,\mu})$.

We can rewrite (4.5) using the isomorphism $W_{\lambda,\mu} \cong X_k \mathbin{\dot\circ} V_{\lambda,\mu}$ as

$\hat{\mathcal{B}}_k$-modules. We have

$$W \cong \bigoplus_{(\lambda,\mu)\in(DP^2)_k} X_k \mathbin{\dot\circ} V_{\lambda,\mu} \mathbin{\dot\circ} U_{\lambda,\mu}.$$

Note that, if U, V and W are simple modules for superalgebras A, B and C respectively, then both $(U \mathbin{\dot\circ} V) \mathbin{\dot\circ} W$ and $U \mathbin{\dot\circ} (V \mathbin{\dot\circ} W)$ denote the unique (up to isomorphism) simple $(A \otimes B \otimes C)$-module occurring in $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$, so that, up to isomorphism, the operation $\dot\circ$ is associative. There are three cases where the $\mathcal{C}_k \otimes \mathcal{B}'_k \otimes (\mathcal{U}_{n_0} \dot\otimes \mathcal{U}_{n_1})$-module $X_k \mathbin{\dot\circ} V_{\lambda,\mu} \mathbin{\dot\circ} U_{\lambda,\mu}$ is different from the supertensor product $X_k \otimes V_{\lambda,\mu} \otimes U_{\lambda,\mu}$.

(1) If k is even and $(\lambda,\mu) \in (DP^2)_k^-$, then X_k, $V_{\lambda,\mu}$, $U_{\lambda,\mu}$ are of type M, Q, Q respectively. We have

$$X_k \mathbin{\dot\circ} V_{\lambda,\mu} \mathbin{\dot\circ} U_{\lambda,\mu} = X_k \otimes (V_{\lambda,\mu} \mathbin{\dot\circ} U_{\lambda,\mu})$$

where $V_{\lambda,\mu} \mathbin{\dot\circ} U_{\lambda,\mu}$ is one of the two eigenspaces of $x_{\lambda,\mu} \otimes u_{\lambda,\mu}$.

(2) If k is odd and $(\lambda,\mu) \in (DP^2)_k^+$, then X_k, $V_{\lambda,\mu}$, $U_{\lambda,\mu}$ are of type Q, M, Q respectively. We have

$$X_k \mathbin{\dot\circ} V_{\lambda,\mu} \mathbin{\dot\circ} U_{\lambda,\mu} = (X_k \otimes V_{\lambda,\mu}) \mathbin{\dot\circ} U_{\lambda,\mu}$$

where $(X_k \otimes V_{\lambda,\mu}) \mathbin{\dot\circ} U_{\lambda,\mu}$ is one of the two eigenspaces of $(z_k \otimes 1) \otimes u_{\lambda,\mu}$.

(3) If k is odd and $(\lambda,\mu) \in (DP^2)_k^-$, then X_k, $V_{\lambda,\mu}$, $U_{\lambda,\mu}$ are of type Q, Q, M respectively. We have

$$X_k \mathbin{\dot\circ} V_{\lambda,\mu} \mathbin{\dot\circ} U_{\lambda,\mu} = (X_k \mathbin{\dot\circ} V_{\lambda,\mu}) \otimes U_{\lambda,\mu}$$

where $X_k \mathbin{\dot\circ} V_{\lambda,\mu}$ is one of the two eigenspaces of $z_k \otimes x_{\lambda,\mu}$.

Put $r = \lfloor k/2 \rfloor$ and $\zeta_i = \sqrt{-1}\xi_{2i-1}\xi_{2i} \in \mathcal{C}_k$ for $1 \le i \le r$. Then the elements $\Psi(\zeta_i \otimes 1)$, $1 \le i \le r$, are commuting involutions of $\Psi((\mathcal{C}_k)_0 \otimes 1) \subset \Psi((\hat{\mathcal{B}}_k)_0) = \mathrm{End}^0_{\Theta(\mathcal{U}_{n_0} \dot\otimes \mathcal{U}_{n_1})}(W)$. For each $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r) \in \mathbb{Z}_2^r$, put $W^\varepsilon = \{w \in W \mid \Psi(\zeta_i \otimes 1)(w) = (-1)^{\varepsilon_i} w \quad (1 \le i \le r)\}$. Then we have $W = \bigoplus_{\varepsilon\in\mathbb{Z}_2^r} W^\varepsilon$. Since $\zeta_i \otimes 1$ commutes with $1 \dot\otimes \mathcal{B}'_k$ for each $1 \le i \le r$, W^ε is a $\mathcal{B}'_k \otimes (\mathcal{U}_{n_0} \dot\otimes \mathcal{U}_{n_1})$-module.

Theorem 4.2. *For each $\varepsilon \in \mathbb{Z}_2^r$, the submodule W^ε is decomposed as a multiplicity-free sum of simple $\mathcal{B}'_k \dot\otimes (\mathcal{U}_{n_0} \dot\otimes \mathcal{U}_{n_1})$-modules as follows:*

$$W^\varepsilon \cong_{\mathcal{B}'_k \dot\otimes (\mathcal{U}_{n_0} \dot\otimes \mathcal{U}_{n_1})} \bigoplus_{(\lambda,\mu)\in(DP^2)_k} V_{\lambda,\mu} \mathbin{\dot\circ} U_{\lambda,\mu}. \tag{4.9}$$

In the above decomposition, the simple $\mathcal{B}'_k$-modules are paired with the simple $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-modules in a bijective manner. More precisely, we have the following results.

(1) *Assume that k is even. Then the simple $\mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-modules $V_{\lambda,\mu} \circ U_{\lambda,\mu}$ in W^ε are all of type M. Furthermore we have*
(4.10)

$$\mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}(W^\varepsilon) = \Psi(\mathcal{B}'_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon) = \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}).$$

(2) *Assume that k is odd. Then the simple $\mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-modules $V_{\lambda,\mu} \circ U_{\lambda,\mu}$ in W^ε are all of type Q. Furthermore we have*
(4.11)

$$\mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})}(W^\varepsilon) \cong \mathcal{C}_1 \dot{\otimes} \Psi(\mathcal{B}'_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon) \cong \mathcal{C}_1 \dot{\otimes} \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}).$$

Proof. For each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{Z}_2^r$, put $X_k^\varepsilon = \{\xi \in X_k \mid \zeta_i \xi = (-1)^{\varepsilon_i} \xi \quad (1 \le i \le r)\}$. Then we have $X_k = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} X_k^\varepsilon$.

(1) Assume that k is even. Note that X_k^ε is one-dimensional. Let ξ^ε be a base of X_k^ε, namely $X_k^\varepsilon = \mathbb{C}\xi^\varepsilon$. Since the elements ζ_i are of degree 0, ξ^ε is a homogeneous element of X_k. Hence we have $X_k^\varepsilon \otimes (V_{\lambda,\mu} \circ U_{\lambda,\mu}) \cong_{\mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} V_{\lambda,\mu} \circ U_{\lambda,\mu}$.

If $(\lambda, \mu) \in (DP^2)_k^+$, then we have

$$\begin{aligned} X_k \circ V_{\lambda,\mu} \circ U_{\lambda,\mu} = X_k \otimes V_{\lambda,\mu} \otimes U_{\lambda,\mu} &= \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} X_k^\varepsilon \otimes V_{\lambda,\mu} \otimes U_{\lambda,\mu} \\ &= \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} X_k^\varepsilon \otimes (V_{\lambda,\mu} \circ U_{\lambda,\mu}). \end{aligned}$$

If $(\lambda, \mu) \in (DP^2)_k^-$, then we have

$$X_k \circ V_{\lambda,\mu} \circ U_{\lambda,\mu} = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} X_k^\varepsilon \otimes (V_{\lambda,\mu} \circ U_{\lambda,\mu})$$

since the elements ζ_i, $1 \le i \le r$, and $1 \otimes x_{\lambda,\mu} \otimes u_{\lambda,\mu}$ commute. Consequently we have

$$\begin{aligned} W^\varepsilon &\cong_{\mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} X_k^\varepsilon \otimes \left(\bigoplus_{(\lambda,\mu) \in (DP^2)_k} V_{\lambda,\mu} \circ U_{\lambda,\mu} \right) \\ &\cong_{\mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} \bigoplus_{(\lambda,\mu) \in (DP^2)_k} V_{\lambda,\mu} \circ U_{\lambda,\mu} \end{aligned}$$

Therefore (4.9) follows. By Theorem 1.1 and (1.2), the simple modules $V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu}$ appearing in the above decomposition are of type M.

First we will show the second equality in (4.10). Then the first equality follows from DSCT. Since W^ε is a $\mathcal{B}'_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-module, we have

$$\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon} \subset \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon).$$

By DSCT, (4.5) and (4.9) (already proved for this case), we have

$$\dim \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon) = \dim \mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W)$$

since both equal $\sum_{(\lambda,\mu)\in(DP^2)_k^+} (\dim U_{\lambda,\mu})^2 + \sum_{(\lambda,\mu)\in(DP^2)_k^-} \frac{1}{2}(\dim U_{\lambda,\mu})^2$. By Theorem 4.1 (1), we have $\dim \mathrm{End}^{\cdot}_{\Psi(\hat{\mathcal{B}}_k)}(W) = \dim \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$. Define a linear map $\mathfrak{p}_\varepsilon \colon \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}) \to \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon}$ by $\mathfrak{p}_\varepsilon(f) = f|_{W^\varepsilon}$ for $f \in \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$. It is clear that $\mathfrak{p}_\varepsilon$ is surjective. We claim that $\mathfrak{p}_\varepsilon$ is injective. Assume that $f \in \ker \mathfrak{p}_\varepsilon$, namely $f|_{W^\varepsilon} = 0 \in \mathrm{End}(W^\varepsilon)$. Since f and the elements ξ_{2j-1} commute, and a subgroup of $(\mathcal{C}_k)^\times$ generated by the elements ξ_{2j-1}, $1 \le j \le r$, transitively act on $\{W^{\varepsilon'}\,;\, \varepsilon' \in \mathbb{Z}_2^r\}$ as follows:

$$\xi_{2j-1} W^{(\varepsilon_1,\ldots,\varepsilon_r)} = W^{(\varepsilon_1,\ldots,\varepsilon_j+1,\ldots,\varepsilon_r)} \quad (1 \le^{\forall} j \le r)$$

it follows that $f|_{W^{\varepsilon'}} = 0$ for all $\varepsilon' \in \mathbb{Z}_2^r$. Therefore $f = 0$ in $\mathrm{End}(W)$. Hence $\mathfrak{p}_\varepsilon$ is injective. Consequently we have $\dim \Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon} = \dim \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$. It follows that $\Theta(\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})|_{W^\varepsilon} = \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$, as required.

(2) Assume that k is odd. Note that X_k^ε is 2-dimensional. Then $X_k^\varepsilon = \mathbb{C}\xi^\varepsilon \oplus \mathbb{C} z_k \xi^\varepsilon$.

If $(\lambda,\mu) \in (DP^2)_k^+$, then $V_{\lambda,\mu} \dot{\circ} X_k = V_{\lambda,\mu} \otimes X_k$ and we regard the $\mathcal{B}'_k \dot{\otimes} \mathcal{C}_k$-module $V_{\lambda,\mu} \otimes X_k$ as a $\mathcal{C}_k \dot{\otimes} \mathcal{B}'_k$-module via $\omega_{\mathcal{C}_k,\mathcal{B}'_k}$, where $\omega_{\mathcal{C}_k,\mathcal{B}'_k} \colon \mathcal{C}_k \dot{\otimes} \mathcal{B}'_k \to \mathcal{B}'_k \dot{\otimes} \mathcal{C}_k$ denotes an isomorphism of superalgebras determined by $\omega_{\mathcal{C}_k,\mathcal{B}'_k}(a \otimes b) = (-1)^{\bar{a}\cdot\bar{b}} b \otimes a$ for all homogeneous elements $a \in \mathcal{C}_k$ and $b \in \mathcal{B}'_k$. An isomorphism $\theta \colon X_k \otimes V_{\lambda,\mu} \xrightarrow{\sim} V_{\lambda,\mu} \otimes X_k$ is defined by $\theta(\xi \otimes v) = (-1)^{\bar{\xi}\cdot\bar{v}} v \otimes \xi$ for all homogeneous elements $\xi \in X_k$ and $v \in V_{\lambda,\mu}$. Since $\theta \circ (z_k \otimes 1) = (1 \otimes z_k) \circ \theta$, we have

$$\begin{aligned} X_k \dot{\circ} V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu} &\cong_{\hat{\mathcal{B}}_k \dot{\otimes} \mathcal{U}_n} (V_{\lambda,\mu} \otimes X_k) \dot{\circ} U_{\lambda,\mu} \\ &\cong_{\hat{\mathcal{B}}_k \dot{\otimes} \mathcal{U}_n} V_{\lambda,\mu} \otimes (X_k \dot{\circ} U_{\lambda,\mu}) \end{aligned}$$

where $X_k \dot{\circ} U_{\lambda,\mu}$ denotes one of the two eigenspaces of $z_k \otimes u_{\lambda,\mu}$. Since the elements ζ_i and $z_k \otimes u_{\lambda,\mu}$ commute, we have

$$X_k \dot{\circ} U_{\lambda,\mu} = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} X_k^\varepsilon \dot{\circ} U_{\lambda,\mu}$$

where $X_k^\varepsilon \dot{\circ} U_{\lambda,\mu}$ denotes one of the two eigenspaces of $z_k|_{X_k^\varepsilon} \otimes u_{\lambda,\mu}$. Since $X_k^\varepsilon \dot{\circ} U_{\lambda,\mu}$ is a $\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}$-submodule of $X_k \dot{\circ} U_{\lambda,\mu} \cong_{\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}} U_{\lambda,\mu}^{\oplus 2^r}$ and $\dim(X_k^\varepsilon \dot{\circ} U_{\lambda,\mu}) = \dim U_{\lambda,\mu}$, it follows that $X_k^\varepsilon \dot{\circ} U_{\lambda,\mu} \cong_{\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1}} U_{\lambda,\mu}$.

If $(\lambda, \mu) \in (DP^2)_k^-$, then we have

$$X_k \dot{\circ} V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu} = (X_k \dot{\circ} V_{\lambda,\mu}) \otimes U_{\lambda,\mu} = \bigoplus_{\varepsilon \in \mathbb{Z}_2^r} (X_k^\varepsilon \dot{\circ} V_{\lambda,\mu}) \otimes U_{\lambda,\mu}$$

since the elements ζ_i, $1 \leq i \leq r$, and $z_k \otimes x_{\lambda,\mu} \otimes 1$ commute, where $X_k^\varepsilon \dot{\circ} V_{\lambda,\mu}$ denotes one of the two eigenspaces of $z_k|_{X_k^\varepsilon} \otimes x_{\lambda,\mu}$. Since $X_k^\varepsilon \dot{\circ} V_{\lambda,\mu}$ is a $\mathcal{B}'_k$-submodule of $X_k \dot{\circ} V_{\lambda,\mu} \cong_{\mathcal{B}'_k} V_{\lambda,\mu}^{\oplus 2^r}$ and $\dim(X_k^\varepsilon \dot{\circ} V_{\lambda,\mu}) = \dim V_{\lambda,\mu}$, it follows that $X_k^\varepsilon \dot{\circ} V_{\lambda,\mu} \cong_{\mathcal{B}'_k} V_{\lambda,\mu}$.

Consequently we have

$$W^\varepsilon \cong \bigoplus_{(\lambda,\mu) \in (DP^2)_k} V_{\lambda,\mu} \otimes U_{\lambda,\mu}.$$

By Theorem 1.1 and (1.2), the simple modules $V_{\lambda,\mu} \otimes U_{\lambda,\mu}$ appearing in the above decomposition are of type Q and we have $V_{\lambda,\mu} \otimes U_{\lambda,\mu} = V_{\lambda,\mu} \dot{\circ} U_{\lambda,\mu}$. Therefore, (4.9) and the former statement of (2) follow.

The supercentralizer $\mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$ contains an invertible element $\Psi(\xi_k) \in \Psi(\mathcal{C}_k)$. The subsuperalgebra of $\mathrm{End}^{\cdot}_{\Psi(\mathcal{B}'_k)}(W^\varepsilon)$ generated by $\Psi(\xi_k)$ is isomorphic to $\mathcal{C}_1$. By the arguments similar to the proof of (4.10), the result (4.11) follows from DSCT (cf. [11, Cor. 2.2]). Q.E.D.

Let us mention a relation between the branching rule of the $\mathfrak{q}(n)$-modules to $\mathfrak{q}(n_0) \oplus \mathfrak{q}(n_1)$ and that of the $\hat{\mathcal{B}}_k$-modules to $\mathcal{B}_k$ (or that of the $\mathcal{B}'_k$-modules to $\mathcal{A}_k$).

Let A be a superalgebra and let B be a subsuperalgebra of A. If V is an A-module, then we can restrict it to a B-module, which we write as $V \downarrow_B^A$. Moreover, we write $[V : U]_A$ (or simply write $[V : U]$) for the multiplicity of a simple A-module U in an A-module V.

Corollary 4.3. *Put*

(4.12)
$$m^{\lambda}_{\mu,\nu} = [U_\lambda \downarrow^{\mathcal{U}_n}_{\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1}} : U_{\mu,\nu}],$$

(4.13)
$$m'^{\lambda}_{\mu,\nu} = [W_{\mu,\nu} \downarrow^{\hat{\mathcal{B}}_k}_{\mathcal{B}_k} : W_\lambda] \quad \left(\text{resp.}[V_{\mu,\nu} \downarrow^{\mathcal{B}'_k}_{\mathcal{A}_k} : V_\lambda]\right).$$

Then we have

(4.14)
$$m'^{\lambda}_{\mu,\nu} = \begin{cases} \frac{1}{2}m^{\lambda}_{\mu,\nu} & \text{if both } U_{\mu,\nu} \text{ and } W_{\mu,\nu} \text{ (resp. } V_{\mu,\nu}) \text{ are of type } M \\ & \quad \text{and both } U_\lambda \text{ and } W_\lambda \text{ (resp. } V_\lambda) \text{ are of type } Q, \\ 2m^{\lambda}_{\mu,\nu} & \text{if both } U_{\mu,\nu} \text{ and } W_{\mu,\nu} \text{ (resp. } V_{\mu,\nu}) \text{ are of type } Q \\ & \quad \text{and both } U_\lambda \text{ and } W_\lambda \text{ (resp. } V_\lambda) \text{ are of type } M, \\ m^{\lambda}_{\mu,\nu} & \text{otherwise.} \end{cases}$$

Proof. Put

$$W' = W_\lambda \,\dot{\circ}\, U_{\mu,\nu}, \quad W_1 = W \downarrow^{\mathcal{B}_k\dot{\otimes}\mathcal{U}_n}_{\mathcal{B}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}, \quad W_2 = W \downarrow^{\hat{\mathcal{B}}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}_{\mathcal{B}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}.$$

Since $W_1 \cong W_2$, we have $[W_1 : W'] = [W_2 : W']$. Moreover, put

$$W'_1 = (W_\lambda \,\dot{\circ}\, U_\lambda) \downarrow^{\mathcal{B}_k\dot{\otimes}\mathcal{U}_n}_{\mathcal{B}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}, \; W'_2 = (W_{\mu,\nu} \,\dot{\circ}\, U_{\mu,\nu}) \downarrow^{\hat{\mathcal{B}}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}_{\mathcal{B}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}.$$

From (4.5) and (A.2), we have $[W_1 : W'] = [W'_1 : W']$ and $[W_2 : W'] = [W'_2 : W']$. Using (4.12) and (4.13), we have

$$(W_\lambda \otimes U_\lambda) \downarrow^{\mathcal{B}_k\dot{\otimes}\mathcal{U}_n}_{\mathcal{B}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})} \cong \bigoplus_{(\mu,\nu)\in(DP^2)_k} (W_\lambda \otimes U_{\mu,\nu})^{\oplus m^{\lambda}_{\mu,\nu}},$$

$$(W_{\mu,\nu} \otimes U_{\mu,\nu}) \downarrow^{\hat{\mathcal{B}}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}_{\mathcal{B}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})} \cong \bigoplus_{\lambda\in DP_k} (W_\lambda \otimes U_{\mu,\nu})^{\oplus m'^{\lambda}_{\mu,\nu}}.$$

By Theorem 1.1 and (1.2), the above modules $(W_\lambda \otimes U_\lambda) \downarrow^{\mathcal{B}_k\dot{\otimes}\mathcal{U}_n}_{\mathcal{B}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}$, $(W_{\mu,\nu} \otimes U_{\mu,\nu}) \downarrow^{\hat{\mathcal{B}}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}_{\mathcal{B}_k\dot{\otimes}(\mathcal{U}_{n_0}\dot{\otimes}\mathcal{U}_{n_1})}$, $W_\lambda \otimes U_{\mu,\nu}$ are sums of two copies of W'_1, W'_2, W' if W_λ is of type Q, $W_{\mu,\nu}$ is of type Q, both W_λ and $U_{\mu,\nu}$ are of type Q, respectively. Note that U_λ (resp. $U_{\mu,\nu}$) is of the same type as W_λ (resp. $W_{\mu,\nu}$). Therefore we have

$$[W_1':W'] = \begin{cases} \frac{1}{2}m_{\mu,\nu}^{\lambda} & \text{if } W_\lambda \text{ is of type } Q \text{ and } W_{\mu,\nu} \text{ is of type } M, \\ m_{\mu,\nu}^{\lambda} & \text{otherwise,} \end{cases}$$

$$[W_2':W'] = \begin{cases} \frac{1}{2}m'^{\lambda}_{\mu,\nu} & \text{if } W_\lambda \text{ is of type } M \text{ and } W_{\mu,\nu} \text{ is of type } Q, \\ m'^{\lambda}_{\mu,\nu} & \text{otherwise.} \end{cases}$$

Comparing the above two equations, we obtain the result (4.14).

Next, using (4.9) and (B.1), we consider the multiplicities of the simple $\mathcal{A}_k \otimes (\mathcal{U}_{n_0} \otimes \mathcal{U}_{n_1})$-module $V_\lambda \mathbin{\dot\circ} U_{\lambda,\mu}$ in $W^\varepsilon \downarrow^{\mathcal{A}_k \dot\otimes \mathcal{U}_n}_{\mathcal{A}_k \otimes (\mathcal{U}_{n_0} \dot\otimes \mathcal{U}_{n_1})}$ and $W^\varepsilon \downarrow^{\mathcal{B}_k' \dot\otimes (\mathcal{U}_{n_0} \dot\otimes \mathcal{U}_{n_1})}_{\mathcal{A}_k \dot\otimes (\mathcal{U}_{n_0} \dot\otimes \mathcal{U}_{n_1})}$ respectively. Then (4.14) similarly follows. Q.E.D.

Let H_k' be the subgroup of $(\mathcal{B}_k')^\times$ generated by -1, τ', $\gamma_1, \ldots, \gamma_{k-1}$. Then H_k' is a double cover (a central extension with a $\mathbb{Z}_2$ kernel) of H_k. Let $w^{\kappa,\nu}$ denote the element of H_k' defined by

$$\begin{aligned} w^{\kappa,\nu} &= w_1 w_2 \cdots w_l w_1' w_2' \cdots w_{l'}' \quad (l = l(\kappa), l' = l(\nu)), \\ w_i &= \gamma_{a+1}\gamma_{a+2}\cdots\gamma_{a+\kappa_i-1} \quad (a = \kappa_1 + \cdots + \kappa_{i-1}), \\ w_i' &= \gamma_{b+1}\gamma_{b+2}\cdots\gamma_{b+\nu_i-1}\tau'_{b+\nu_i} \quad (b = |\kappa| + \nu_1 + \cdots + \nu_{i-1}). \end{aligned}$$

Note that the image of $w^{\kappa,\nu}$ in H_k is a representative of the conjugacy class of H_k indexed by (κ,ν).

Define a map $\varepsilon\colon (DP^2)_k \to \mathbb{Z}_2$ by $\varepsilon(\lambda,\mu) = 0$ (resp. $\varepsilon(\lambda,\mu) = 1$) if $(\lambda,\mu) \in (DP^2)_k^+$ (resp. $(\lambda,\mu) \in (DP^2)_k^-$).

We describe a formula for the character values of simple $\mathcal{B}_k'$-modules.

Corollary 4.4. *We have*

(4.15)
$$\begin{aligned} &2^{\frac{l(\kappa)+l(\nu)}{2}} p_\kappa(x,y) p_\nu(x,-y) \\ &\quad = \sum_{(\lambda,\mu)\in(DP^2)_k} \mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu}) 2^{\frac{-l(\lambda)-l(\mu)-\varepsilon(\lambda,\mu)}{2}} Q_\lambda(x) Q_\mu(y) \end{aligned}$$

for all $(\kappa,\nu) \in (OP^2)_k$*, where* $p_\kappa(x,y) = p_\kappa(x_1,x_2,\ldots,y_1,y_2,\ldots)$ *and* $p_\nu(x,-y) = p_\nu(x_1,x_2,\ldots,-y_1,-y_2,\ldots)$ *and* $\mathrm{Ch}[V_{\lambda,\mu}]$ *denotes the character of* $V_{\lambda,\mu}$*, namely* $\mathrm{Ch}[V_{\lambda,\mu}](w) = \mathrm{tr}(w_{V_{\lambda,\mu}})$ *for* $w \in \mathcal{B}_k'$ *where* $w_{V_{\lambda,\mu}}$ *denotes the action of* $w \in \mathcal{B}_k'$ *on* $V_{\lambda,\mu}$.

Proof. As we have noted in the preceding paragraph to Theorem 4.1, any $\mathcal{B}'_k \otimes (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})$-submodule W' of W can be regarded as a $\mathcal{B}'_k$-module with a commuting polynomial representation $\theta_{W'}$ of $GL(n_0,\mathbb{C}) \times GL(n_1,\mathbb{C})$. Here we extend our notation in Theorem 4.1 to let $\mathrm{Ch}[W'](x \otimes g)$ denote the trace $\mathrm{tr}(x_{W'} \circ \theta_{W'}(g))$ for $x \in \mathcal{B}'_k$ and $g \in GL(n_0,\mathbb{C}) \times GL(n_1,\mathbb{C})$, where $x_{W'}$ denotes the action of $x \in \mathcal{B}'_k$ on W'.

For any ε, $\varepsilon' \in \mathbb{Z}_2^r$, we have $W^\varepsilon \cong_{\mathcal{B}'_k \dot{\otimes} (\mathcal{U}_{n_0} \dot{\otimes} \mathcal{U}_{n_1})} W^{\varepsilon'}$. Hence, for $(\kappa,\nu) \in (OP^2)_k$ and $E = \mathrm{diag}(x_1,\ldots,x_{n_0},y_1,\ldots,y_{n_1}) \in GL(n,\mathbb{C})$, we have

$$\mathrm{Ch}[W^\varepsilon]\,(w^{\kappa,\nu} \otimes E) = 2^{-r}\,\mathrm{Ch}[W]\,((1 \otimes w^{\kappa,\nu}) \otimes E) \tag{4.16}$$

where $1 \otimes w^{\kappa,\nu} \in \mathcal{C}_k \dot{\otimes} \mathcal{B}'_k = \hat{\mathcal{B}}_k$. We calculate the right hand side using the embedding $\vartheta\colon \mathcal{B}_k \hookrightarrow \hat{\mathcal{B}}_k$ (cf. (3.3)), namely $1 \otimes \gamma_j = \vartheta(\frac{1}{\sqrt{2}}(\tau_j - \tau_{j+1})\sigma_j)$. Put $k' = |\kappa|$ and $l = l(\kappa)$. Then $k - k' = |\nu|$. Moreover put $W' = V^{\otimes k'}$ and $W'' = V^{\otimes k-k'}$. We have $w^{\kappa,\nu} = w^{\kappa,\phi} w^{\phi,\nu}$, where $w^{\kappa,\phi} \in \mathcal{B}'_{k'}$, $w^{\phi,\nu} \in \mathcal{B}'_{k-k'}$. Define a representations of $\hat{\mathcal{B}}_{k'}$ on W' (resp. a representation of $\hat{\mathcal{B}}_{k-k'}$ on W'') by the same manner as the representation Ψ of $\hat{\mathcal{B}}_k$ in W. Then the action of $1 \otimes w^{\kappa,\phi}$ (resp. the action of $1 \otimes w^{\phi,\nu}$) on W can be expressed as (the action of $1 \otimes w^{\kappa,\phi}$ on W') $\otimes \overbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}^{k-k'}$ (resp. $\overbrace{\mathrm{id} \otimes \cdots \otimes \mathrm{id}}^{k'}$ (the action of $1 \otimes w^{\phi,\nu}$ on W'')). Hence we have

$$\begin{aligned} &\mathrm{Ch}[W]\,((1 \otimes w^{\kappa,\nu}) \otimes E) \\ &= \mathrm{Ch}[W']\,\big((1 \otimes w^{\kappa,\phi}) \otimes E\big)\,\mathrm{Ch}[W'']\,\big((1 \otimes w^{\phi,\nu}) \otimes E\big). \end{aligned} \tag{4.17}$$

The element $1 \otimes w^{\kappa,\phi}$ of $\hat{\mathcal{B}}_{k'}$ is a product of $k' - l$ elements $1 \otimes \gamma_j = \vartheta(\frac{1}{\sqrt{2}}(\tau_j - \tau_{j+1})\sigma_j)$. This product can be rearranged into the following form:

(constant) $\times$ (a product of the elements $\vartheta(\tau_p) - \vartheta(\tau_q)$)
$\times$ (a product of the elements $\vartheta(\sigma_j)$).

The product of the elements $\vartheta(\sigma_j)$ equals $\vartheta(\sigma^{\kappa,\phi})$. Expanding the product of $\vartheta(\tau_p) - \vartheta(\tau_q)$ into a sum of $2^{k'-l}$ elements, we have

$$1 \otimes w^{\kappa,\phi} = \left(\frac{1}{\sqrt{2}}\right)^{k'-l} \times \sum (\text{a product of the elements } \vartheta(\tau_p)) \times \vartheta(\sigma^{\kappa,\phi})$$

where $\sigma^{\kappa,\phi} = g_1 g_2 \cdots g_l$, $g_i = \sigma_{a+1}\sigma_{a+2}\cdots\sigma_{a+\nu_i-1}$ $(a = \sum_{j=1}^{i-1}\kappa_j)$. Then all terms in the summation are conjugate to $\vartheta(\sigma^{\kappa,\phi})$ in $\vartheta((\mathcal{B}_k)^\times)$. Therefore we have

$$\begin{aligned}\mathrm{Ch}[W']\left((1\otimes w^{\kappa,\phi})\otimes E\right) &= 2^{k'-l}(\sqrt{2})^{l-k'}\,\mathrm{Ch}[W']\left(\vartheta(\sigma^{\kappa,\phi})\otimes E\right)\\ &= (\sqrt{2})^{k'+l}p_\kappa(x_1,x_2,\ldots,y_1,y_2,\ldots).\end{aligned}$$

Put $l' = l(\nu)$. Similarly we have

$$\begin{aligned}\mathrm{Ch}[W'']\left((1\otimes w^{\phi,\nu})\otimes E\right) &= 2^{k-k'-l'}(\sqrt{2})^{l'-k+k'}\,\mathrm{Ch}[W''](\vartheta(\sigma'^{\phi,\nu})\otimes E)\\ &= (\sqrt{2})^{k-k'+l'}p_\nu(x_1,x_2,\ldots,-y_1,-y_2,\ldots)\end{aligned}$$

where $\sigma'^{\phi,\nu} = g'_1 g'_2\cdots g'_{l'}$, $g'_i = \sigma_{b+1}\sigma_{b+2}\ldots\sigma_{b+\nu_i-1}\tau'_{b+\nu_i}$ $(b = \sum_{j=1}^{i-1}\nu_j)$. By (4.16) and (4.17), we have

$$\begin{aligned}&\mathrm{Ch}[W^\varepsilon]\left((1\otimes w^{\kappa,\nu})\otimes E\right)\\ &= \begin{cases}(\sqrt{2})^{l+l'}p_\kappa(x_1,x_2,\ldots,y_1,y_2,\ldots)p_\nu(x_1,x_2,\ldots,-y_1,-y_2,\ldots) & \text{if } k \text{ is even,}\\ (\sqrt{2})^{l+l'+1}p_\kappa(x_1,x_2,\ldots,y_1,y_2,\ldots)p_\nu(x_1,x_2,\ldots,-y_1,-y_2,\ldots) & \text{if } k \text{ is odd.}\end{cases}\end{aligned}$$

On the other hand, by (4.6) and (4.9), if k is even, then we have

$$\begin{aligned}&\mathrm{Ch}[\bigoplus_{(\lambda,\mu)\in(DP^2)_k} V_{\lambda,\mu}\,\dot{\circ}\,U_{\lambda,\mu}]\,(w^{\kappa,\nu}\otimes E)\\ &= \sum_{(\lambda,\mu)\in(DP^2)_k}\mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu})\\ &\qquad\times(\sqrt{2})^{-\varepsilon(\lambda,\mu)-l(\lambda)-l(\mu)}Q_\lambda(x_1,\ldots,x_{n_0})Q_\mu(y_1,\ldots,y_{n_1}),\end{aligned}$$

and if k is odd, then we have

$$\begin{aligned}&\mathrm{Ch}[\bigoplus_{(\lambda,\mu)\in(DP^2)_k} V_{\lambda,\mu}\,\dot{\circ}\,U_{\lambda,\mu}]\,(w^{\kappa,\nu}\otimes E)\\ &= \sqrt{2}\sum_{(\lambda,\mu)\in(DP^2)_k}\mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu})\\ &\qquad\times(\sqrt{2})^{-\varepsilon(\lambda,\mu)-l(\lambda)-l(\mu)}Q_\lambda(x_1,\ldots,x_{n_0})Q_\mu(y_1,\ldots,y_{n_1}).\end{aligned}$$

Since these hold for all n_0 and n_1, the result follows. Q.E.D.

We review Stembridge's formula for the character values of simple $\mathcal{B}'_k$-modules, in a form adapted to the simple modules in the $\mathbb{Z}_2$-graded sense.

Theorem 4.5. (cf. [10, Lem. 7.5]) *We have*

$$2^{\frac{3(l(\kappa)+l(\nu))}{2}} p_\kappa(x) p_\nu(y) = \sum_{(\lambda,\mu)\in(DP^2)_k} \mathrm{Ch}[V_{\lambda,\mu}](w^{\kappa,\nu}) 2^{\frac{-l(\lambda)-l(\mu)-\varepsilon(\lambda,\mu)}{2}} Q_\lambda(x,y) Q_\mu(x,-y)$$

for all $(\kappa,\nu) \in (OP^2)_k$, *where* $Q_\lambda(x,y) = Q_\lambda(x_1,x_2,\dots,y_1,y_2,\dots)$ *and* $Q_\mu(x,-y) = Q_\mu(x_1,x_2,\dots,-y_1,-y_2,\dots)$.

The formula (4.15) is different from Stembridge's formula. Let us mention a relationship between the two formulas. Define an algebra endomorphism ι of $\Omega_x \otimes \Omega_y$ by $\iota(f\otimes 1) = f(x,y) = f(x_1,x_2,\dots,y_1,y_2,\dots)$ and $\iota(1\otimes g) = g(x,-y) = g(x_1,x_2,\dots,-y_1,-y_2,\dots)$ (since the y-part belongs to Ω_y, this "naïve notation" actually coincides with $g(x-y)$ in the Λ-ring notation). Note that $\{Q_\lambda(x,y)Q_\mu(x,-y) \mid (\lambda,\mu) \in (DP^2)\}$ is a basis of $\Omega_x\otimes\Omega_y$ (cf. [10, Th. 7.1, Lem. 7.5]). Since $\iota(Q_\lambda(x)Q_\mu(y)) = Q_\lambda(x,y)Q_\mu(x,-y)$, it follows that ι is an automorphism. Moreover, since $\iota(p_r(x,y)) = 2p_r(x)$ and $\iota(p_r(x,-y)) = 2p_r(y)$ for any odd r, it follows that the image of (4.15) under ι coincides with Stembridge's formula.

Appendix

A. Sergeev's duality. We review Sergeev's duality relation between $\mathcal{B}_k$ and $\mathcal{U}_n$ using DSCT. Define a map $d\colon DP_k \to \mathbb{Z}_2$ by $d(\lambda) = 0$ (resp. $d(\lambda) = 1$) if $l(\lambda)$ is even (resp. $l(\lambda)$ is odd).

Theorem A. [8] (1) *The two superalgebras* $\Psi(\mathcal{B}_k)$ *and* $\mathcal{U}_n$ *act on* W *as mutual supercentralizers of each other*:

$$\text{(A.1)} \qquad \mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_n)}(W) = \Psi(\mathcal{B}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{B}_k)}(W) = \Theta(\mathcal{U}_n).$$

(2) *The simple* $\mathcal{B}_k$*-module* W_λ ($\lambda \in DP_k$) *occurs in* W *if and only if* $l(\lambda)) \le n$. *Then we have*

$$\text{(A.2)} \qquad W \cong_{\mathcal{B}_k \dot\otimes \mathcal{U}_n} \bigoplus_{\substack{\lambda\in DP_k \\ l(\lambda)\le n}} W_\lambda \,\dot\circ\, U_\lambda$$

where U_λ *denotes a simple* $\mathcal{U}_n$*-module corresponding to* W_λ *in* W *in the sense of DSCT.*

(3) *The character values of* $\mathrm{Ch}[U_\lambda]$ *are given as follows:*

$$\mathrm{Ch}[U_\lambda](x_1, x_2, \ldots, x_n) = (\sqrt{2})^{d(\lambda)-l(\lambda)} Q_\lambda(x_1, x_2, \ldots, x_n). \tag{A.3}$$

B. A duality of $\mathcal{A}_k$ and $\mathfrak{q}(n)$. We established a duality relation between $\mathcal{A}_k$ and $\mathcal{U}_n$ on the same space W^ε as in Theorem 4.2.

Theorem B. [11, Th. 4.1] *The submodule* W^ε *is decomposed as a multiplicity-free sum of simple* $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$*-modules as follows:*

$$W^\varepsilon \cong_{\mathcal{A}_k \dot{\otimes} \mathcal{U}_n} \bigoplus_{\substack{\lambda \in DP_k \\ l(\lambda) \le n}} V_\lambda \dot{\circ} U_\lambda. \tag{B.1}$$

(1) *Assume that* k *is even. Then the simple* $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$*-modules* $V_\lambda \dot{\circ} U_\lambda$ *in* W^ε *are of type* M*. Furthermore we have*

$$\mathrm{End}^{\cdot}_{\Theta(\mathcal{U}_n)}(W^\varepsilon) = \Psi(\mathcal{A}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) = \Theta(\mathcal{U}_n). \tag{B.2}$$

(2) *Assume that* k *is odd. Then the simple* $\mathcal{A}_k \dot{\otimes} \mathcal{U}_n$*-modules* $V_\lambda \dot{\circ} U_\lambda$ *in* W^ε *are of type* Q*. Furthermore we have*

$$\mathrm{End}^{\cdot}_{\Theta(U_n)}(W^\varepsilon) \cong \mathcal{C}_1 \dot{\otimes} \Psi(\mathcal{A}_k), \quad \mathrm{End}^{\cdot}_{\Psi(\mathcal{A}_k)}(W^\varepsilon) \cong \mathcal{C}_1 \dot{\otimes} \Theta(\mathcal{U}_n). \tag{B.3}$$

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