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A Recursion Formula of the Weighted Parabolic Kazhdan-Lusztig Polynomials

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Abstract.

In this article, we give a recursion formula of the weighted parabolic Kazhdan-Lusztig polynomials and describe a relationship between those polynomials and weighted Kazhdan-Lusztig polynomials introduced by G.Lusztig ([4]).

§1. Introduction

Our aim in this article is to give a recursion formula of the weighted parabolic Kazhdan-Lusztig polynomials introduced by H. Tagawa [5] as an extension of the parabolic Kazhdan-Lusztig polynomials and the weighted Kazhdan-Lusztig polynomials. Also, we describe a relationship between those polynomials and weighted Kazhdan-Lusztig polynomials, which is an extension of Deodhar's result on the parabolic Kazhdan-Lusztig polynomials and the Kazhdan-Lusztig polynomials (cf.[1]).

Let us give a brief review of known results. In 1982, G. Lusztig introduced the weighted Kazhdan-Lusztig polynomials, the special case of which has a representation theoretic interpretation (cf.[4]). Also, in 1987, V. Deodhar introduced two kinds of parabolic Kazhdan-Lusztig polynomials, one of which gives the dimensions of the intersection cohomology modules of Schubert varieties in G/P, where G is a Kac-Moody group and P is a "standard" parabolic subgroup of G (cf.[1]). Recently, H. Tagawa introduced the weighted parabolic Kazhdan-Lusztig polynomials and he obtained combinatorial formulas which were extensions of Deodhar's results on the parabolic Kazhdan-Lusztig polynomials (cf.[2]). But, unfortunately, the coefficients of the weighted parabolic Kazhdan-Lusztig polynomials are not always non-negative.

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This paper is organized as follows: In the next section, we recall the definition of the weighted parabolic R-polynomials and the weighted parabolic Kazhdan-Lusztig polynomials. Moreover, we show some interesting equalities used in the sequel. In Section 3, we give a recursion formula of the weighted parabolic Kazhdan-Lusztig polynomials which is an extension of Lusztig's result on the weighted Kazhdan-Lusztig polynomials (cf.[4]). In Section 4, we describe a relationship between weighted parabolic Kazhdan-Lusztig polynomials and weighted Kazhdan-Lusztig polynomials.

§2. Preliminaries and Notations

The purpose of this section is to define the weighted parabolic R-polynomials and the weighted parabolic Kazhdan-Lusztig polynomials. Throughout this article, (W, S) is an arbitrary Coxeter system, e is the unit element of W. Let $\mathbf{Z}$ be the set of integers, $\mathbf{N}$ the set of non-negative integers, and $\mathbf{P}$ the set of natural numbers.

First, we recall the definition of the Bruhat order.

Definition 2.1. We put $T := \{wsw^{-1}; s \in S,\ w \in W\}$. For $y, z \in W$, we denote $y <' z$ if and only if there exists an element t of T such that $\ell(tz) < \ell(z)$ and $y = tz$, where ℓ is the length function. Then the Bruhat order denoted by $\leq$ is defined as follows: For $x, w \in W$, $x \leq w$ if and only if there exists a sequence $x_0, x_1, \ldots, x_r$ in W such that $x = x_0 <' x_1 <' \cdots <' x_r = w$. We also use the notation $x \lessdot w$ if $x < w$ and $\ell(x) = \ell(w) - 1$.

The following is well known as the subword property. For $w \in W$, let $s_1 s_2 \cdots s_m$ be a reduced expression of w, i.e. $w = s_1 s_2 \cdots s_m$, $s_i \in S$ for all $i \in \{1, 2, \ldots, m\}$ and $\ell(w) = m$. For $x \in W$, $x \leq w$ if and only if there exists a sequence of natural numbers $i_1, i_2, \ldots, i_t$ such that $1 \leq i_1 < i_2 < \cdots < i_t \leq m$ and $x = s_{i_1} s_{i_2} \cdots s_{i_t}$. This expression of x is not reduced in general, i.e. it may happen that $\ell(x) < t$. However it is known that one can find a sequence of natural numbers $j_1, j_2, \ldots, j_k$ such that $1 \leq j_1 < j_2 < \cdots < j_k \leq m$, $x = s_{j_1} s_{j_2} \cdots s_{j_k}$ and $\ell(x) = k$.

From now on, the order on W is the Bruhat order. Next, we recall the definition of weights (cf.[4]).

Definition 2.2. Let Γ be an abelian group or a $\mathbf{Z}$-algebra of an abelian group with the unit element $\mathbf{e}$. φ is called a weight of W into Γ if and only if φ is a map of W into Γ satisfying the following conditions:

(i) $\varphi(e) = \mathbf{e}$,

(ii) $\varphi(s_1 s_2 \dots s_m) = \varphi(s_1)\varphi(s_2)\dots\varphi(s_m)$ for any reduced expression $s_1 s_2 \dots s_m$ in W.

(iii) $\varphi(s)$ is an invertible element in Γ for any $s \in S$.

In particular, any weight φ satisfies the following.

(ii)' For $s, t \in S$, if the order of st is odd, then $\varphi(s) = \varphi(t)$.

Conversely, a map $\widetilde{\varphi}$ of S into Γ satisfying (i), (ii)' and (iii) is uniquely extended to a weight of W into Γ.

From now on, Γ is an abelian group, $\mathbf{e}$ is the unit element of Γ, φ is a weight of W into Γ and we put $S = \{s_1, s_2, \dots, s_n\}$. For $w \in W$, we denote $\varphi(w)$ by $q_w^{\frac{1}{2}}$ and $(q_{s_1}^{\frac{1}{2}}, q_{s_2}^{\frac{1}{2}}, \dots, q_{s_n}^{\frac{1}{2}})$ by $\mathbf{q}$. Next, we recall the definition of the weighted Hecke algebras and the weighted R-polynomials (cf.[4]).

Definition 2.3. Let $\mathcal{H}_\varphi(W)$ be the free $\mathbf{Z}[\Gamma]$-module having the set $\{T'_w; w \in W\}$ as a basis and multiplication such that

$$T'_s T'_w = \begin{cases} T'_{sw} & \text{if } w < sw, \\ q_s T'_{sw} + (q_s - \mathbf{e})T'_w & \text{if } sw < w \end{cases}$$

for $w \in W$ and $s \in S$. We call $\mathcal{H}_\varphi(W)$ the weighted Hecke algebra (of W with respect to φ).

It is known that $\mathcal{H}_\varphi(W)$ is an associative algebra (see [3] Chapter 7 for more general theory). For $s \in S$, we can easily see that $(T'_s)^{-1} = (q_s^{-1} - \mathbf{e})T'_e + q_s^{-1} T'_s$.

Then, the weighted R-polynomial is defined as follows:

Definition 2.4. There exists a unique family of polynomials $\{R'_{x,w}(\mathbf{q}) \in \mathbf{Z}[\Gamma]; x, w \in W\}$ satisfying

$$\overline{T'_w} = q_w^{-1} \sum_{x \in W} (-1)^{\ell(x)+\ell(w)} R'_{x,w}(\mathbf{q}) T'_x \text{ for } w \in W,$$

where we put $\overline{T'_w} := T'^{-1}_{w^{-1}}$ for $w \in W$. We call these polynomials $R'_{x,w}(\mathbf{q})$ weighted R-polynomials of W.

Let J be a subset of S, W_J the subgroup of W generated by J and $W^J := \{y \in W; \ell(yz) = \ell(y) + \ell(z) \text{ for any } z \in W_J\}$. Then, it is well known that, for $w \in W$, there exist a unique element w^J in W^J and a unique element w_J in W_J such that $w = w^J w_J$ (cf.[3]).

Now, we can define weighted parabolic Hecke modules.

Definition 2.5. Let $A(\varphi)$ be the $\mathbf{Z}$-algebra of $\mathbf{Z}[\Gamma]$ generated by $\{q_s^{\frac{1}{2}}; s \in S\}$ and ψ a weight of W into $A(\varphi)$ with $\psi(s) = -\mathbf{e}$ or $\psi(s) = q_s$ for each $s \in S$. In the same way, for $w \in W$, we denote $\psi(w)$ by u_w. After this, for convenience, we denote $\mathbf{e}$ by 1. Also, for $s \in S$, we put $\widetilde{u}_s := q_s$ if $u_s = -1$ and $\widetilde{u}_s := -1$ if $u_s = q_s$. Note that the map $\widetilde{\psi}$ of W into $A(\varphi)$ defined as follows is also a weight.

$$\widetilde{\psi}(w) := \begin{cases} \mathbf{e} & \text{if } w{=}\mathrm{e}, \\ \widetilde{u}_{s_1}\widetilde{u}_{s_2}\cdots\widetilde{u}_{s_m} & \text{if } s_1s_2\ldots s_m \text{ is a reduced expression of } w. \end{cases}$$

Let $M^J_{\varphi,\psi}(W)$ be the free $\mathbf{Z}[\Gamma]$-module with basis $\{m'^J_w; w \in W^J\}$. For $s \in S$, we define $L'(s) \in \mathrm{Hom}_{\mathbf{Z}[\Gamma]}(M^J_{\varphi,\psi}(W))$ as follows:

$$L'(s)m'^J_w := \begin{cases} q_s m'^J_{sw} + (q_s - 1)m'^J_w & \text{if } sw < w, \\ m'^J_{sw} & \text{if } w < sw \in W^J, \\ u_s m'^J_w & \text{if } w < sw \notin W^J, \end{cases}$$

and linear extension.

Then, we call $M^J_{\varphi,\psi}(W)$ the weighted parabolic Hecke module (of W^J with respect to φ and ψ).

Let ρ'_J be a map from $\mathcal{H}_\varphi(W)$ to $M^J_{\varphi,\psi}(W)$ defined by

$$\rho'_J(\sum_{x\in W} a_x T'_x) := \sum_{x\in W} a_x u_{x_J} m'^J_{x^J},$$

where x^J and x_J are unique elements satisfying $x = x^J x_J$, $x^J \in W^J$ and $x_J \in W_J$. Then, the following is known (see [5]).

Lemma 2.6. ([5, Lemma 2.5])

(i) *ρ'_J is onto.*

(ii) *For $s \in S$ and $x \in W$, $L'(s)(\rho'_J(T'_x)) = \rho'_J(T'_sT'_x)$.*

(iii) *For $s \in S$, $L'(s)^2 = q_sL'(e) + (q_s - 1)L'(s)$, where $L'(e)$ is the identity map on $M^J_{\varphi,\psi}(W)$.*

(iv) *For $w \in W$ and $x \in W^J$, we can define*

$$T'_w \cdot m'^J_x := \begin{cases} m'^J_x \ \textit{if}\ w = e, \\ (L'(s_1)L'(s_2)\ldots L'(s_m))m'^J_x \\ \quad \textit{if } s_1s_2\ldots s_m \textit{ is a reduced expression of } w. \end{cases}$$

Namely, $M^J_{\varphi,\psi}(W)$ has an $\mathcal{H}_\varphi(W)$-module structure.

(v) *For* $w \in W$, $\rho'_J(T'_w) = T'_w \cdot m'^J_e$.

We define an operation $^-$ on $M^J_{\varphi,\psi}(W)$ as follows:

$$\overline{\sum_{\gamma\in\Gamma} b_\gamma \gamma} := \sum_{\gamma\in\Gamma} b_\gamma \gamma^{-1} \text{ for } \sum_{\gamma\in\Gamma} b_\gamma\gamma \in \mathbf{Z}[\Gamma],$$

$$\overline{m'^J_w} := T'^{-1}_{w^{-1}} \cdot m'^J_e \text{ for } w \in W^J,$$

$$\overline{\sum_{w\in W^J} a_w m'^J_w} := \sum_{w\in W^J} \overline{a_w}\,\overline{m'^J_w} \text{ for } \sum_{w\in W^J} a_w m'^J_w \in M^J_{\varphi,\psi}(W).$$

We can see that the operation $^-$ is an involution on $M^J_{\varphi,\psi}(W)$ by the following.

Lemma 2.7. ([5, Lemma 2.6]) *Let* $x \in W^J$ *and* $s \in S$. *Then, we have*

$$\overline{m'^J_x} = \rho'_J(\overline{T'_x}), \quad \overline{T'_s \cdot m'^J_x} = \overline{T'_s} \cdot \overline{m'^J_x}, \quad \overline{\overline{m'^J_x}} = m'^J_x.$$

Here, we describe the following interesting formula.

Proposition 2.8. *For* $w \in W$,

$$q_w^{-1} \sum_{x\in W} (-1)^{\ell(x)+\ell(w)} u_x R'_{x,w}(\mathbf{q}) = u_w^{-1}. \tag{1}$$

Proof. By the definition of the weighted R-polynomials, we can easily find a recursion formula of those polynomials. So, by direct calculation and the recursion formula, we can show this proposition by induction on $\ell(w)$. <u>*q.e.d*</u>

As a corollary of Proposition 2.8, we see the following.

Corollary 2.9. *For* $X \in \mathcal{H}_\varphi(W)$,

$$\overline{\rho'_J(X)} = \rho'_J(\overline{X}).$$

Proof. First, for $w \in W_J$, by Proposition 2.8, we have

$$\rho'_J(\overline{T'_w}) = q_w^{-1} \sum_{x\in W_J} (-1)^{\ell(x)+\ell(w)} R'_{x,w}(\mathbf{q}) u_x m'^J_e = u_w^{-1} m'^J_e.$$

Hence, for $w \in W$, by Lemma 2.6 and Lemma 2.7,

$$\overline{\rho'_J(T'_w)} = \overline{u_{w_J} m'^J_{w^J}} = u^{-1}_{w_J}(\overline{T'_{w^J}} \cdot m'^J_e) = \overline{T'_{w^J}} \cdot (\rho'_J(\overline{T'_{w_J}})) = \rho'_J(\overline{T'_w}),$$

where w^J and w_J are unique elements satisfying $w = w^J w_J$, $w^J \in W^J$ and $w_J \in W_J$. Hence, by definitions of the operation and ρ'_J, Corollary 2.9 holds. q.e.d

From now on, we denote $(u_{s_1}, u_{s_2}, \ldots, u_{s_n})$ by $\mathbf{u}$ and $(\widetilde{u}_{s_1}, \widetilde{u}_{s_2}, \ldots, \widetilde{u}_{s_n})$ by $\widetilde{\mathbf{u}}$. By using this operation, we can define the weighted parabolic R-polynomials as follows:

Definition 2.10. There exists a unique family of polynomials $\{R'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}} \in \mathbf{Z}[\Gamma]; x, w \in W^J\}$ satisfying

$$\overline{m'^{J}_{w}} = q_w^{-1} \sum_{x \in W^J} (-1)^{\ell(x)+\ell(w)} R'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}} m'^{J}_{x} \text{ for } w \in W^J.$$

We call these polynomials $R'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}}$ weighted parabolic R-polynomials of W^J. For convenience, we put $R'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}} := 0$ if $x \notin W^J$ or $w \notin W^J$.

For example, the following equalities are known.

Proposition 2.11. ([5, Lemma 3.4, Proposition 3.9])
Let $x, w \in W^J$.

(i) $(-1)^{\ell(x)+\ell(w)} q_w q_x^{-1} \overline{R'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}}} = R'^{J}_{x,w}(\mathbf{q})_{\widetilde{\mathbf{u}}}$.

(ii) $\displaystyle\sum_{x \le y \le w} (-1)^{\ell(y)+\ell(w)} R'^{J}_{x,y}(\mathbf{q})_{\mathbf{u}} R'^{J}_{y,w}(\mathbf{q})_{\widetilde{\mathbf{u}}} = \delta_{x,w}$, *where $\delta_{x,w}$ is Kronecker delta.*

(iii) *Let $s \in S$ with $sw < w$.*

$$R'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}} = \begin{cases} R'^{J}_{sx,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } sx < x, \\ q_s R'^{J}_{sx,sw}(\mathbf{q})_{\mathbf{u}} + (q_s - 1) R'^{J}_{x,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } x < sx \in W^J, \\ \widetilde{u}_s R'^{J}_{x,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } x < sx \notin W^J. \end{cases}$$

A relationship between the weighted parabolic R-polynomials and the weighted R-polynomials is the following.

Proposition 2.12. ([5, Proposition 3.11, Lemma 3.12])

(i) $R'^{\phi}_{x,w}(\mathbf{q})_{\mathbf{u}} = R'_{x,w}(\mathbf{q})$ *for $x, w \in W$.*

(ii) $\displaystyle R'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}} = \sum_{y \in W_J} (-1)^{\ell(y)} u_y R'_{xy,w}(\mathbf{q})$ *for $x, w \in W^J$.*

We define some more notations.

Notation 2.13.

(i) Let r be the number of the different elements in $\{q_s; s \in S\}$, i.e. $r = \sharp\{q_s; s \in S\}$, and we put $\{q_{s_1}, q_{s_2}, \ldots, q_{s_r}\} = \{q_s; s \in S\}$, where $\sharp A$ is the cardinality of a set A. Put

$$\begin{aligned}\Gamma' &:= \{q_{s_1}^{\frac{n_1}{2}} q_{s_2}^{\frac{n_2}{2}} \cdots q_{s_r}^{\frac{n_r}{2}}; n_i \in \mathbf{Z} \text{ for } i \in [r]\},\\ \Gamma'' &:= \Gamma'^2 (= \{\gamma^2; \gamma \in \Gamma'\})\end{aligned}$$

where $[r] := \{1, 2, \ldots, r\}$.

(ii) For μ, $\gamma \in \Gamma''$, we denote $\mu \lhd \gamma$ if and only if there exist integers h_i and k_i with $h_i \leq k_i$, $\mu = q_{s_1}^{h_1} q_{s_2}^{h_2} \cdots q_{s_r}^{h_r}$ and $\gamma = q_{s_1}^{k_1} q_{s_2}^{k_2} \cdots q_{s_r}^{k_r}$.

In order to define the weighted parabolic Kazhdan-Lusztig polynomials, we define a total order on Γ' called a strong order.

Definition 2.14. We define a "strong order" on Γ' as a total order $<$ which satisfies the following conditions:

(i) For $\alpha, \beta, \gamma \in \Gamma'$, if $\alpha \leq \beta$, then $\alpha\gamma \leq \beta\gamma$.

(ii) For any $s \in S$, $\mathbf{e} < q_s^{\frac{1}{2}}$.

Example 2.15. If a weight φ of W into Γ satisfies that

$$q_{s_1}^{\frac{k_1}{2}} q_{s_2}^{\frac{k_2}{2}} \ldots q_{s_r}^{\frac{k_r}{2}} = \mathbf{e} \Leftrightarrow k_i = 0 \text{ for all } i \in [r].$$

Then, the lexicographic order with respect to $k_1, k_2, \ldots, k_r$ is a strong order on Γ'.

From now on, we assume that φ has a strong order on Γ' and we fix a strong order on Γ'. Put $\Gamma'_+ := \{\gamma \in \Gamma'; \mathbf{e} < \gamma\}$, $\Gamma'_- := \{\gamma \in \Gamma'; \gamma < \mathbf{e}\} (= (\Gamma'_+)^{-1})$ and $\Gamma''_+ := \{\gamma \in \Gamma''; \mathbf{e} \lhd \gamma\}$. Then, we can define weighted parabolic Kazhdan-Lusztig polynomials as follows:

Proposition 2.16. ([5, Proposition 4.4]) *There exists a unique family of polynomials* $\{P'^J_{x,w}(\mathbf{q})_\mathbf{u} \in \mathbf{Z}[\Gamma''_+]; x, w \in W^J\}$ *satisfying the following conditions:*

(i) $P'^J_{x,x}(\mathbf{q})_\mathbf{u} = 1$ *for all* $x \in W^J$.

(ii) $P'^J_{x,w}(\mathbf{q})_\mathbf{u} = 0$ *if* $x \not\leq w$.

(iii) $q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P'^J_{x,w}(\mathbf{q})_\mathbf{u} \in \mathbf{Z}[\Gamma'_-]$ *if* $x < w$.

(iv)

$$q_w q_x^{-1} \overline{P'^J_{x,w}(\mathbf{q})_\mathbf{u}} = \sum_{x \leq y \leq w, y \in W^J} R'^J_{x,y}(\mathbf{q})_\mathbf{u} P'^J_{y,w}(\mathbf{q})_\mathbf{u}.$$

We define the "uniquely" determined polynomials from Proposition 2.16 as the weighted parabolic Kazhdan-Lusztig polynomials with respect to the strong order $<$. Note that we can easily see that $P'^{\phi}_{x,w}(\mathbf{q})_{\mathbf{u}} = P'_{x,w}(\mathbf{q})$ for $x, w \in W$, here $P'_{x,w}(\mathbf{q})$ is the weighted Kazhdan-Lusztig polynomials defined in Section 4. From now on, for convenience, we put $P'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}} := 0$ if $x \notin W^J$ or $w \notin W^J$.

§3. A recursion formula

In this section, we define an extension of $\mu(x,w)$, which is the coefficient of $q^{\frac{\ell(w)-\ell(x)-1}{2}}$ in the Kazhdan-Lusztig polynomial $P_{x,w}(q)$, and get a recursion formula of the weighted parabolic Kazhdan-Lusztig polynomials.

Definition-Proposition 3.1. Let $s \in S$ and we put

$$c(s,\mathbf{u}) := \{x \in W^J; \begin{cases} sx < x \text{ or } sx \notin W^J & \text{if } u_s = q_s, \\ sx < x & \text{if } u_s = -1 \end{cases}\}.$$

Then, there exists a unique family of polynomials

$$\{M^{Js}_{x,w} \in \mathbf{Z}[\Gamma']; x, w \in W^J, x < w < sw, x \in c(s,\mathbf{u})\}$$

satisfying

$$\sum_{x \le y < w, y \in c(s,\mathbf{u})} P^{*J}_{x,y}(\mathbf{q})_{\tilde{\mathbf{u}}} M^{Js}_{y,w} - q_s^{\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} \in \mathbf{Z}[\Gamma'_-], \quad \overline{M^{Js}_{x,w}} = M^{Js}_{x,w},$$

where $P^{*J}_{x,w}(\mathbf{q})_{\mathbf{u}} := q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}}$ for $x, w \in W^J$.

This is easily obtained by direct calculation and induction on $\ell(w) - \ell(x)$ and the proof is therefore omitted. Then, a recursion formula of the weighted parabolic Kazhdan-Lusztig polynomials is described as follows:

Theorem 3.2.

(i) *Let $x, w \in W^J$ and $s \in S$ with $sw < w$. Then, we have*

$$P'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}} = \begin{cases} q_s P'^{J}_{x,sw}(\mathbf{q})_{\mathbf{u}} + P'^{J}_{sx,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } sx < x \\ P'^{J}_{x,sw}(\mathbf{q})_{\mathbf{u}} + q_s P'^{J}_{sx,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } x < sx \in W^J \\ (\tilde{u}_s + 1) P'^{J}_{x,sw}(\mathbf{q})_{\mathbf{u}} & \text{if } x < sx \notin W^J \end{cases}$$
$$- \sum_{x \le y < sw, y \in c(s,\tilde{\mathbf{u}})} q_y^{-\frac{1}{2}} q_w^{\frac{1}{2}} P'^{J}_{x,y}(\mathbf{q})_{\mathbf{u}} M^{Js}_{y,sw}.$$

(ii) *Let $x, w \in W^J$. If there exists $s \in S$ such that $sw < w$ and $sx \in W^J$, then we have*

$$P'^J_{x,w}(\mathbf{q})_{\mathbf{u}} = P'^J_{sx,w}(\mathbf{q})_{\mathbf{u}}.$$

Note that if $sw < w$, then $x \le w \Leftrightarrow sx \le w$.

(iii) *Let $x, w \in W^J$. If there exists $s \in S$ such that $sw < w$, $x < sx \notin W^J$ and $u_s = q_s$, then we have*

$$P'^J_{x,w}(\mathbf{q})_{\mathbf{u}} = 0.$$

Before the proof of this theorem, we show some lemmas and propositions.

Lemma 3.3. *Let $x, w \in W^J$ and $s \in S$ with $w < sw \notin W^J$ and $sx \in W^J$. Then, we have*

$$R'^J_{x,w}(\mathbf{q})_{\mathbf{u}} = \begin{cases} \widetilde{u}_s^{-1} R'^J_{sx,w}(\mathbf{q})_{\mathbf{u}} & \text{if } sx < x, \\ \widetilde{u}_s R'^J_{sx,w}(\mathbf{q})_{\mathbf{u}} & \text{if } x < sx. \end{cases}$$

Proof. First, by Lemma 2.6 and Lemma 2.7, we can easily see that

$$\overline{q_s^{-\frac{1}{2}}(L'(s) + L'(e))m'^J_w} = q_s^{-\frac{1}{2}}(L'(s) + L'(e))\overline{m'^J_w}. \tag{2}$$

Hence, by (2) and our assumption that $w < sw \notin W^J$,

$$u_s^{-1}\overline{m'^J_w} + \overline{m'^J_w} = q_s^{-1}L'(s)\overline{m'^J_w} + q_s^{-1}\overline{m'^J_w}.$$

Hence, we have

$$L'(s)\overline{m'^J_w} = \sum_{x \in W^J} q_w^{-1}(-1)^{\ell(w)+\ell(x)} u_s R'^J_{x,w}(\mathbf{q})_{\mathbf{u}} m'^J_x.$$

On the other hand, by the definition of $R'^J_{x,w}(\mathbf{q})_{\mathbf{u}}$, we can see

$$\begin{aligned} &L'(s)\overline{m'^J_w} \\ &= \sum_{sy<y\in W^J} q_w^{-1}(-1)^{\ell(w)+\ell(y)}((q_s-1)R'^J_{y,w}(\mathbf{q})_{\mathbf{u}} - R'^J_{sy,w}(\mathbf{q})_{\mathbf{u}})m'^J_y \\ &\quad - \sum_{y<sy\in W^J} q_w^{-1}(-1)^{\ell(w)+\ell(y)} q_s R'^J_{sy,w}(\mathbf{q})_{\mathbf{u}} m'^J_y \\ &\quad + \sum_{y<sy\notin W^J} q_w^{-1}(-1)^{\ell(w)+\ell(y)} u_s R'^J_{y,w}(\mathbf{q})_{\mathbf{u}} m'^J_y. \end{aligned}$$

Thus, we have

$$u_s R'^J_{x,w}(\mathbf{q})_\mathbf{u} = \begin{cases} (q_s - 1)R'^J_{x,w}(\mathbf{q})_\mathbf{u} - R'^J_{sx,w}(\mathbf{q})_\mathbf{u} & \text{if } sx < x, \\ -q_s R'^J_{sx,w}(\mathbf{q})_\mathbf{u} & \text{if } x < sx \in W^J, \\ u_s R'^J_{x,w}(\mathbf{q})_\mathbf{u} & \text{if } x < sx \notin W^J. \end{cases}$$

By using this equality, we can obtain this lemma. *q.e.d.*

Lemma 3.4. *Let $x, y, w \in W^J$ and $s \in S$. If $sx < x \lessdot w < sw \notin W^J$, $sx < y < w$ and $x \neq y$, then $y \notin W^J$.*

We can easily obtain this lemma by the subword property and the proof is therefore omitted.

Then, we can show the following.

Proposition 3.5. *Let $x, w \in W^J$, $s \in S$, $w < sw \notin W^J$, $sx \in W^J$ and $u_s = -1$. Then, we have*

$$P'^J_{x,w}(\mathbf{q})_\mathbf{u} = P'^J_{sx,w}(\mathbf{q})_\mathbf{u}. \tag{3}$$

Note that the above equality does not always hold in case $u_s = q_s$.

Proof. We may assume that $sx < x$. Case 1. $x \not\leq w$. In this case, we can easily see that $sx \not\leq w$. So, both sides of (3) are equal to 0. Case 2. $x \leq w$. In this case, we show this theorem by induction on $\ell(w) - \ell(x)$. In case $\ell(w) - \ell(x) = 1$. Note that we may not consider the case that $\ell(w) - \ell(x) = 0$ by our assumption in this proposition. Let $q_w q_x^{-1} = q_t$ $(t \in S)$ and $y \in W - \{x\}$ with $sx \lessdot y \lessdot w$. Then, by Lemma 3.4, $y \notin W^J$. So, by the fact that $R'^J_{x,w}(\mathbf{q})_\mathbf{u} = q_s - 1$ if $x \lessdot w$ and $q_w q_x^{-1} = q_s$, $P'^J_{x,w}(\mathbf{q})_\mathbf{u} = 1$ if $x \lessdot w$, we have

$$q_w q_{sx}^{-1} \overline{P'^J_{sx,w}(\mathbf{q})_\mathbf{u}} - P'^J_{sx,w}(\mathbf{q})_\mathbf{u} = q_s q_t - 1.$$

Hence,

$$\overline{P^{*J}_{sx,w}(\mathbf{q})_\mathbf{u}} - q_s^{\frac{1}{2}} q_t^{\frac{1}{2}} = P^{*J}_{sx,w}(\mathbf{q})_\mathbf{u} - q_s^{-\frac{1}{2}} q_t^{-\frac{1}{2}}. \tag{4}$$

Then, the left hand side of (4) is an element in $\mathbf{Z}[\Gamma'_+]$ and the right hand side of (4) is an element in $\mathbf{Z}[\Gamma'_-]$. So, by the fact that $\mathbf{Z}[\Gamma'_+] \cap \mathbf{Z}[\Gamma'_-] = \{0\}$, we have

$$P'^J_{sx,w}(\mathbf{q})_\mathbf{u} = 1.$$

On the other hand, since $\ell(w) - \ell(x) = 1$,

$$P'^J_{x,w}(\mathbf{q})_\mathbf{u} = 1.$$

We suppose that (3) holds when $\ell(w) - \ell(x) < k$ ($k \geq 2$) and we will show this one in case $\ell(w) - \ell(x) = k$. For $y \in W^J$ with $sy < y$, by Proposition 2.11-(iii), we have

$$q_s R'^J_{x,y}(\mathbf{q})_\mathbf{u} - R'^J_{sx,y}(\mathbf{q})_\mathbf{u} = R'^J_{sx,sy}(\mathbf{q})_\mathbf{u} - q_s R'^J_{x,sy}(\mathbf{q})_\mathbf{u}.$$

Hence, by our inductive hypothesis, we have

$$\begin{aligned}&\sum_{sy<y\in W^J} (q_s R'^J_{x,y}(\mathbf{q})_\mathbf{u} - R'^J_{sx,y}(\mathbf{q})_\mathbf{u}) P'^J_{y,w}(\mathbf{q})_\mathbf{u}\\ &= P'^J_{x,w}(\mathbf{q})_\mathbf{u} - P'^J_{sx,w}(\mathbf{q})_\mathbf{u}\\ &- \sum_{z<sz\in W^J} (q_s R'^J_{x,z}(\mathbf{q})_\mathbf{u} - R'^J_{sx,z}(\mathbf{q})_\mathbf{u}) P'^J_{z,w}(\mathbf{q})_\mathbf{u}.\end{aligned}$$

So, we have

$$\begin{aligned}&\sum_{sy<y\in W^J \text{ or } y<sy\in W^J} (q_s R'^J_{x,y}(\mathbf{q})_\mathbf{u} - R'^J_{sx,y}(\mathbf{q})_\mathbf{u}) P'^J_{y,w}(\mathbf{q})_\mathbf{u}\\ &= P'^J_{x,w}(\mathbf{q})_\mathbf{u} - P'^J_{sx,w}(\mathbf{q})_\mathbf{u}.\end{aligned}$$

On the other hand, by Lemma 3.3,

$$\sum_{y\in W^J, y<sy\notin W^J} (q_s R'^J_{x,y}(\mathbf{q})_\mathbf{u} - R'^J_{sx,y}(\mathbf{q})_\mathbf{u}) P'^J_{y,w}(\mathbf{q})_\mathbf{u} = 0.$$

Thus, by the above equalities,

$$\sum_{y\in W^J} (q_s R'^J_{x,y}(\mathbf{q})_\mathbf{u} - R'^J_{sx,y}(\mathbf{q})_\mathbf{u}) P'^J_{y,w}(\mathbf{q})_\mathbf{u} = P'^J_{x,w}(\mathbf{q})_\mathbf{u} - P'^J_{sx,w}(\mathbf{q})_\mathbf{u}.$$

Hence, by Proposition 2.16-(iv), we have

$$q_s^{\frac{1}{2}} \overline{P^{*J}_{x,w}(\mathbf{q})_\mathbf{u}} - \overline{P^{*J}_{sx,w}(\mathbf{q})_\mathbf{u}} = q_s^{-\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_\mathbf{u} - P^{*J}_{sx,w}(\mathbf{q})_\mathbf{u}.$$

So, we can see

$$q_s^{-\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_\mathbf{u} - P^{*J}_{sx,w}(\mathbf{q})_\mathbf{u} = 0.$$

This completes the proof of Proposition 3.5. <u>*q.e.d*</u>

By Proposition 2.11, we can easily obtain the following.

Definition-Proposition 3.6. *For $w \in W^J$, we put*

$$\begin{aligned}C'^J_w &:= q_w^{-\frac{1}{2}} \sum_{x\leq w} P'^J_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} m'^J_x,\\ D'^J_w &:= \sum_{x\in W^J} (-1)^{\ell(x)+\ell(w)} q_w^{\frac{1}{2}} q_x^{-1} \overline{P'^J_{x,w}(\mathbf{q})_\mathbf{u}} m'^J_x.\end{aligned}$$

Then, we have

$$\overline{C'^J_w} = C'^J_w, \quad \overline{D'^J_w} = D'^J_w.$$

Then, as a corollary of Proposition 3.5, we can see the following.

Corollary 3.7. *Let $w \in W^J$, $s \in S$, $w < sw \notin W^J$ and $u_s = q_s$. Then, we have*

$$L'(s)C'^J_w = q_s C'^J_w.$$

The following lemma is easily obtained by direct calculation.

Lemma 3.8. *Let $w \in W^J$ and $s \in S$.*

(i) *If $w < sw$, we put*

$$q_s^{-\frac{1}{2}}(L'(s) + L'(e))C'^J_w - C'^J_{sw} - \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y = \sum_{x\in W^J} f_x \widetilde{m'}^J_x,$$

where $\widetilde{m'}^J_x := q_x^{-\frac{1}{2}} m'^J_x$ for $x \in W^J$. Then, we have

$$f_x = \begin{cases} q_s^{\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} + P^{*J}_{sx,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } sx < x \\ q_s^{-\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} + P^{*J}_{sx,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } x < sx \in W^J \\ q_s^{-\frac{1}{2}}(u_s+1) P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } x < sx \notin W^J \end{cases}$$
$$- P^{*J}_{x,sw}(\mathbf{q})_{\tilde{\mathbf{u}}} - \sum_{x\le y<w, y\in c(s,\mathbf{u})} P^{*J}_{x,y}(\mathbf{q})_{\tilde{\mathbf{u}}} M^{Js}_{y,w}.$$

(ii) *If $sw < w$, we put*

$$(q_s^{-\frac{1}{2}} L'(s) - q_s^{\frac{1}{2}} L'(e))C'^J_w = \sum_{x\in W^J} g_x \widetilde{m'}^J_x.$$

Then, we have

$$g_x = \begin{cases} P^{*J}_{sx,w}(\mathbf{q})_{\tilde{\mathbf{u}}} - q_s^{-\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } sx < x, \\ P^{*J}_{sx,w}(\mathbf{q})_{\tilde{\mathbf{u}}} - q_s^{\frac{1}{2}} P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } x < sx \in W^J, \\ q_s^{-\frac{1}{2}}(u_s - q_s) P^{*J}_{x,w}(\mathbf{q})_{\tilde{\mathbf{u}}} & \text{if } x < sx \notin W^J. \end{cases}$$

Then, we have the following.

Proposition 3.9. *For $w \in W^J$ and $s \in S$, we have*

$$q_s^{-\frac{1}{2}} L'(s) C'^J_w = \begin{cases} -q_s^{-\frac{1}{2}} C'^J_w + C'^J_{sw} + \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y & \text{if } w < sw \in W^J, \\ q_s^{\frac{1}{2}} C'^J_w & \text{if } sw < w. \end{cases}$$

Proof. We show this proposition by induction on $\ell(w)$. We can easily see that Proposition 3.9 holds in case $\ell(w) = 0$. So, we suppose that Proposition 3.9 holds when $\ell(w) < k$ $(k \geq 1)$ and we will show this one in case $\ell(w) = k$. Case 1. $w < sw \in W^J$. We put

$$q_s^{-\frac{1}{2}}(L'(s) + L'(e))C'^J_w - C'^J_{sw} - \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y = \sum_{x\in W^J} f_x \widetilde{m'}^J_x.$$

Note that $f_x = 0$ if $\ell(x) > \ell(sw)$. First, by Lemma 3.8, Definition-Proposition 3.1 and Corollary 3.7, we can see that $f_x \in \mathbf{Z}[\Gamma'_-]$. Next, we show that $f_x = 0$ for all $x \in W^J$. By Proposition 3.6 and the equality that $\overline{M^{Js}_{y,w}} = M^{Js}_{y,w}$, we can obtain

$$\begin{aligned} &\overline{q_s^{-\frac{1}{2}} L'(s) C'^J_w + q_s^{-\frac{1}{2}} C'^J_w - C'^J_{sw} - \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y} \\ &= q_s^{-\frac{1}{2}}(L'(s) + L'(e))C'^J_w - C'^J_{sw} - \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y. \end{aligned}$$

So, we have

$$\sum_{x\in W^J} f_x \widetilde{m'}^J_x = \sum_{x,y\in W^J, y\leq x} \overline{f_x} q_x^{-\frac{1}{2}} q_y^{\frac{1}{2}} (-1)^{\ell(x)+\ell(y)} R'^J_{y,x}(\mathbf{q})_{\mathbf{u}} \widetilde{m'}^J_y. \tag{5}$$

We suppose that there exists $x \in W^J$ satisfying $f_x \neq 0$. Let x_0 be an element in W^J such that $f_{x_0} \neq 0$ and $f_x = 0$ for any $x \in W^J$ with $\ell(x) > \ell(x_0)$. Then, we see that the coefficient of $\widetilde{m'}^J_{x_0}$ in the right hand side of (5) is $\overline{f_{x_0}}$. Hence, we have $f_{x_0} = \overline{f_{x_0}} \neq 0$. This contradicts that $f_{x_0} \in \mathbf{Z}[\Gamma_-]$. So, we have

$$f_x = 0 \text{ for } \forall x \in W^J$$

and we obtain

$$q_s^{-\frac{1}{2}} L'(s) C'^J_w = -q_s^{-\frac{1}{2}} C'^J_w + C'^J_{sw} + \sum_{y<w, y\in c(s,\mathbf{u})} M^{Js}_{y,w} C'^J_y.$$

Case 2. $sw < w$. By our inductive hypothesis, we can use

$$C'^J_w = q_s^{-\frac{1}{2}}(L'(s) + L'(e))C'^J_{sw} - \sum_{y<sw, y\in c(s,\mathbf{u})} M^{Js}_{y,sw} C'^J_y.$$

So, by Proposition 3.7, Lemma 2.6 and our inductive hypothesis, we can see that

$$q_s^{-\frac{1}{2}} L'(s) C'^J_w = q_s^{\frac{1}{2}} C'^J_w.$$

Therefore, this completes the proof of Proposition 3.9 q.e.d

At last, we can prove our main theorem.

Proof of Theorem 3.2. By Proposition 3.9 and Lemma 3.8-(i), we can easily see (i). Also, (ii) and (iii) are easily obtained by Proposition 3.9 and Lemma 3.8-(ii). q.e.d

§4. A relationship with weighted K-L polynomials

The purpose of this section is to show a relationship between weighted parabolic Kazhdan-Lusztig polynomials and weighted Kazhdan-Lusztig polynomials, which is an extension of Deodhar's result on a relationship between parabolic Kazhdan-Lusztig polynomials and Kazhdan-Lusztig polynomials ([1]). First, we recall the definition of the weighted Kazhdan-Lusztig polynomials.

Definition-Proposition 4.1. ([4]) There exists a unique family of polynomials $\{P'_{x,w}(\mathbf{q}) \in \mathbf{Z}[\Gamma''_+]; x, w \in W\}$ satisfying the following conditions:

(i) $P'_{x,x}(\mathbf{q}) = 1$ for all $x \in W$.
(ii) $P'_{x,w}(\mathbf{q}) = 0$ if $x \not\leq w$.
(iii) $q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} P'_{x,w}(\mathbf{q}) \in \mathbf{Z}[\Gamma'_-]$ if $x < w$.
(iv)
$$q_w q_x^{-1} \overline{P'_{x,w}(\mathbf{q})} = \sum_{x \leq y \leq w} R'_{x,y}(\mathbf{q}) P'_{y,w}(\mathbf{q}).$$

As the beginning of this section, we show the following.

Lemma 4.2. *Let $w \in W$. We put*

$$D'_w = \sum_{x \leq w} (-1)^{\ell(x)+\ell(w)} q_w^{\frac{1}{2}} q_x^{-1} \overline{P'_{x,w}(\mathbf{q})} T'_x.$$

(i) $\overline{D'_w} = D'_w$.

(ii) $\rho'_J(D'_w) = \sum_{x \in W^J} (-1)^{\ell(x)+\ell(w)} q_w^{\frac{1}{2}} q_x^{-1} \left(\sum_{y \in W_J} \widetilde{u}_y^{-1} \overline{P'_{xy,w}(\mathbf{q})} \right) m_x'^J.$

Proof. We can easily obtain this lemma by the direct calculation and the definition of the weighted Kazhdan-Lusztig polynomials. Note that $(-1)^{\ell(x)} q_x^{-1} u_y = \widetilde{u}_y^{-1}$. q.e.d

Then, we have the following.

Theorem 4.3. *Let $x, w \in W^J$.*

(i) *If $\widetilde{u}_y q_y^{-\frac{1}{2}} \in \mathbf{Z}[\Gamma'_-]$ for all $y \in W_J$ satisfying $xy \leq w$,*

$$P'^J_{x,w}(\mathbf{q})_{\mathbf{u}} = \sum_{y \in W_J} \widetilde{u}_y P'_{xy,w}(\mathbf{q}).$$

In particular, if $u_s = q_s$ for $\forall s \in S$,

$$P'^J_{x,w}(\mathbf{q})_{\mathbf{u}} = \sum_{y \in W_J} (-1)^{\ell(y)} P'_{xy,w}(\mathbf{q}).$$

(ii) *If $u_s = -1$ for all $s \in S$ and $\sharp W_J < +\infty$,*

$$P'^J_{x,w}(\mathbf{q})_{\mathbf{u}} = P'_{xz_0,wz_0}(\mathbf{q}),$$

where z_0 is the longest element in W_J.

Proof. (i) For $x, w \in W^J$, we put

$$G_{x,w} := \sum_{y \in W_J} \widetilde{u}_y P'_{xy,w}(\mathbf{q}) \in \mathbf{Z}[\Gamma''_+].$$

Then, we will show that a family of polynomials $\{G_{x,w}; x, w \in W^J\}$ satisfies conditions (i), (ii), (iii) and (iv) in Proposition 2.16. Let $x, w \in W^J$. By the fact that $\widetilde{u}_e^{-1} = 1$ and $P'_{x,x}(\mathbf{q}) = 1$, we have $G_{x,x} = 1$. So, (i) holds. If $x \not\leq w$, for $y \in W_J$, we can easily see that $xy \not\leq w$ by the subword property. Hence, (ii) holds. If $x < w$, by our assumption that $\widetilde{u}_y q_y^{-\frac{1}{2}} \in \mathbf{Z}[\Gamma'_-]$ for all $y \in W_J$ satisfying $xy \leq w$, we have

$$q_w^{-\frac{1}{2}} q_x^{\frac{1}{2}} G_{x,w} = \sum_{y \in W_J} \widetilde{u}_y q_y^{-\frac{1}{2}} q_w^{-\frac{1}{2}} q_{xy}^{\frac{1}{2}} P'_{xy,w}(\mathbf{q}) \in \mathbf{Z}[\Gamma'_-].$$

Hence, (iii) holds. By Lemma 4.2-(ii), we can see

$$\overline{\rho'_J(D'_w)} = \sum_{y \in W^J} (-1)^{\ell(y)+\ell(w)} q_w^{-\frac{1}{2}} \Big(\sum_{x \in W^J} R'^J_{y,x}(\mathbf{q})_{\mathbf{u}} G_{x,w} \Big) m'^J_y.$$

On the other hand, by Corollary 2.9 and Lemma 4.2-(i), we have

$$\overline{\rho'_J(D'_w)} = \rho'_J(D'_w).$$

Hence, we have

$$\begin{aligned} &\sum_{x \in W^J} (-1)^{\ell(x)+\ell(w)} q_w^{\frac{1}{2}} q_x^{-1} \overline{G_{x,w}} m'^J_x \\ &= \sum_{x \in W^J} (-1)^{\ell(x)+\ell(w)} q_w^{-\frac{1}{2}} \Big(\sum_{y \in W^J} R'^J_{x,y}(\mathbf{q})_{\mathbf{u}} G_{y,w} \Big) m'^J_x. \end{aligned}$$

Thus, we obtain

$$q_w q_x^{-1}\overline{G_{x,w}} = \sum_{y\in W^J} R'^{J}_{x,y}(\mathbf{q})_{\mathbf{u}} G_{y,w}$$

and (iv) holds. Therefore, by the uniqueness of the weighted parabolic Kazhdan-Lusztig polynomials, we have

$$P'^{J}_{x,w}(\mathbf{q})_{\mathbf{u}} = G_{x,w} = \sum_{y\in W_J} \widetilde{u}_y P'_{xy,w}(\mathbf{q}).$$

(ii) First, we can easily see that $P'_{x,w}(\mathbf{q}) = P'_{x^{-1},w^{-1}}(\mathbf{q})$ for $x, w \in W$. Moreover, it is shown by Lusztig [4] that $P'_{x,w}(\mathbf{q}) = P'_{sx,w}(\mathbf{q})$ for $x, w \in W$ and $s \in S$ satisfying $x \leq w$, $sx < x$, $sw < w$. So, we have

$$P'_{xy,wz_0}(\mathbf{q}) = P'_{xz_0,wz_0}(\mathbf{q}) \text{ for } \forall x, w \in W^J \text{ and } \forall y \in W_J.$$

Hence, by Lemma 4.2-(ii), we have

$$\begin{aligned}&\rho'_J(D'_{wz_0})\\ &= (-1)^{\ell(z_0)} q_{z_0}^{\frac{1}{2}} \sum_{y\in W_J} \widetilde{u}_y^{-1} \sum_{x\in W^J} (-1)^{\ell(x)+\ell(w)} q_w^{\frac{1}{2}} q_x^{-1} \overline{P'_{xz_0,wz_0}(\mathbf{q})} m'^{J}_x.\end{aligned}$$

Hence, by almost the same method to (i), we can obtain (ii). Note that $\overline{q_{z_0}^{\frac{1}{2}} \sum_{y\in W_J} q_y^{-1}} = q_{z_0}^{\frac{1}{2}} \sum_{y\in W_J} q_y^{-1}$. *q.e.d.*

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