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On Permutation Statistics and Hecke Algebra Characters

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Abstract.

Irreducible characters of Hecke algebras of type A may be represented as refined counts of simple statistics on suitable subsets of permutations. Such formulas have been generalized to characters of other Coxeter groups and their Hecke algebras and to coinvariant algebras. In this paper we present several formulas, applications to combinatorial identities, and related problems. New results are given with proofs.

§1. Introduction

Combinatorial properties of the Kazhdan-Lusztig basis and the study of coinvariant algebras have recently led to the discovery of a new family of combinatorial character formulas. Cf. [APR1, APR2, HLR, Ra2, Ste, Ro2, Ro5]. Here we survey some of these formulas with emphasis on permutation statistics. The goal of this paper is to present existing formulas and to study their role in deriving combinatorial identities.

A permutation $\pi \in S_n$ is called *unimodal* if there exists $1 \leq m \leq n$, such that

$$\pi(1) < \pi(2) < \cdots < \pi(m) > \pi(m+1) > \cdots > \pi(n).$$

Denote the set of all unimodal permutations in S_n by U_n.

For a permutation $\pi \in S_n$ define

$$\ell(\pi) := \#\{i < j | \pi(i) > \pi(j)\}.$$

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$$descent(\pi) := \#\{i|\pi(i) > \pi(i+1)\}.$$

$$major(\pi) := \sum\nolimits_{\{i|\pi(i)>\pi(i+1)\}} i$$

By considering different representations of symmetric group characters we derive the following identity:

$$\sum\nolimits_{\pi\in S_n} \omega^{major(\pi)} t^{\ell(\pi)} = \sum\nolimits_{\pi\in U_n} (-1)^{descent(\pi)} t^{\ell(\pi)} \tag{1}$$

where ω is a root of unity of order n.

This identity is generalized using Hecke algebra characters:

(2)

$$\sum\nolimits_{\pi\in S_n} \omega^{major(\pi^{-1})} q^{descent(\pi)} t^{major(\pi)} = \sum\nolimits_{\pi\in U_n} (-q)^{descent(\pi)} t^{\ell(\pi)}$$

The rest of the paper is organized as follows. In the second section we give necessary background from representation theory of Hecke algebras, coinvariant algebras and permutation statistics. In the first part of Section 3 we present two different representations of the irreducible characters of the symmetric group as refined counts of descent number and major index of permutations. A new elementary proof is given. The second part of Section 3 contains formulas for characters of coinvariant algebras and applications. In Section 4 formulas for various characters of Hecke algebras are given with applications to permutation statistics identities. Section 5 concludes the paper with a brief sketch on other Coxeter groups and open problems.

§2. Background

2.1. Permutation Statistics

Let S_n be the symmetric group on n letters, and let $f_i : S_n \to Z_+$ $(1 \le i \le t)$ be (non-negative, integer valued) combinatorial parameters. Then one is interested in the *refined count* of permutations according to these parameters:

$$\sum_{\pi\in S_n} q_1^{f_1(\pi)} \cdots q_t^{f_t(\pi)}.$$

The study of permutation statistics started with Euler, who considered the number of descents. Netto, at the beginning of the century, considered the number of inversions, and MacMahon considered the major index. Multivariate refined counting was studied by [Ca, FS, GG] and many others.

In this section we define basic permutation statistics, which appear in the rest of the paper, and describe some of their properties. It should be noted that permutation statistics are connected to tableaux statistics via standard maps, such as Robinson-Schensted-Knuth correspondence and row (column) word tableaux (see below). For our purposes we prefer the permutation language.

The *length* of $\pi \in S_n$ is the number

$$\ell(\pi) := \#\{i < j | \pi(i) > \pi(j)\}.$$

This statistic is also called the *inversion number.* This number is well known to be equal to the minimal length of π as a product of simple reflections $s_i = (i, i+1)$. Using this approach the length is generalized naturally to arbitrary Coxeter groups.

The *descent number* of $\pi \in S_n$ is defined by

$$descent(\pi) := \#\{i | \pi(i) > \pi(i+1)\}.$$

This statistic has also a natural generalization to arbitrary Coxeter groups.

The *major index* of a permutation $\pi \in S_n$ is the sum (possibly zero)

$$major(\pi) := \sum\nolimits_{\{i | \pi(i) > \pi(i+1)\}} i$$

Generalization of this statistic to arbitrary Coxeter group is a challenging open problem. See Section 5.1.

A classical Theorem of MacMahon shows that the length function and the major index have the same generating function. The following well known identity refines this result [FS, Corollary 1].

Theorem 0.1

$$\sum\nolimits_{\pi \in S_n} q^{major(\pi)} t^{major(\pi^{-1})} = \sum\nolimits_{\pi \in S_n} q^{major(\pi)} t^{\ell(\pi^{-1})}.$$

The Robinson-Schensted correspondence is a bijection between permutations $\pi \in S_n$ and pairs of standard tableaux of same shape $(P(\pi), P(\pi^{-1}))$. See e.g. [Sa, Ch. 3.3]. A *Knuth class* in the symmetric group S_n is a set $\{\pi \in S_n | P(\pi) = Q\}$, where $P(\pi)$ is the left standard tableau corresponding to π under the Robinson-Schensted correspondence and Q is a fixed standard tableau. Define an *inverse Knuth class* as a set of the form $\mathcal{C}^{-1} = \{\pi^{-1} | \pi \in \mathcal{C}\}$, where $\mathcal{C}$ is a Knuth class in S_n.

The *descent set* of a permutation $\pi \in S_n$ is the set $\{i | \pi(i) > \pi(i+1)\}$.

Fact 0.2 *All permutations in an inverse Knuth class have a common descent set; So, have a common major index and a common descent number.*

It follows that the set of all permutations of a fixed descent number (major index) is a union of inverse Knuth classes. This fact is generalized to arbitrary Coxeter groups in the following way: The set of all elements of a fixed descent set is a union of Kazhdan-Lusztig left cells. See e.g. [Hu, Ch. 7.15].

Let $\mathcal{C}$ be a Knuth class in the symmetric group S_n. Denote by *descent*$(\mathcal{C}^{-1})$ the descent number of the permutations in $\mathcal{C}^{-1}$, and by *major*$(\mathcal{C}^{-1})$ the major index of the permutations in $\mathcal{C}^{-1}$.

The following hook formula of Stanley is very useful.

Theorem 0.3 *[ECII, Corollary 21.5]*
Let $\mathcal{C}$ be a Knuth class of shape λ. Then

$$\sum_{\pi\in\mathcal{C}} q^{major(\pi)} = q^{\sum_i \lambda_i(i-1)} \frac{\prod_{i=1}^{n}(q^i-1)}{\prod_{(i,j)\in\lambda}(q^{h_{ij}}-1)},$$

where $h_{i,j}$ are the hook lengths in the diagram of λ.

Any permutation may be considered as a sequence of positive integers. A less classical statistic on sequences (with repeats) is the *charge.* For definition see [Md, p. 242]. For permutation sequences the following folkloristic claim holds.

Claim 0.4 *For any permutation $\pi \in S_n$,*

$$charge(\pi) = major(w_0\pi^{-1}w_0),$$

where $w_0 = n, n-1, n-2, \ldots, 1$ is the longest permutation in S_n.

To verify this claim recall that the charge of a permutation π is the sum of (weighted) lengths of increasing subsequences of consequent digits in π. The claim is based on the fact that the length of the i-th increasing subsequence of consequent digits in π equals to the difference between the i-th and the $i-1$-th descents of $w_0\pi^{-1}w_0$. Therefore, by an elementary observation the sum in the charge equals to the sum of the descents of $w_0\pi^{-1}w_0$. I.e, to the major index of $w_0\pi^{-1}w_0$.

The Kostka-Foulkes polynomials, denoted by $K_{\lambda,\mu}(q)$, are defined as the entries of the transition matrix between the Schur polynomial basis of the symmetric functions and the Hall-Littlewood q-polynomials. For more details see [Md, Ch. III, 6]. The Kostka-Foulkes polynomials may be represented as refined counts of charge of semi-standard tableaux [LS]. The following is a special case of Lascoux-Schützenberger Theorem [Md, Ch. III, (6.5)]. Let T be a standard tableau. Denote by $\pi(T)$ the permutation obtained by reading T from right to left in consecutive rows. $\pi(T)$ is called the *row-word of* T.

Theorem 0.5

$$K_{\lambda,1^n}(q) = \sum_T q^{\text{charge}(\pi(T))},$$

where the sum is taken over all standard tableaux of shape λ.

2.2. Representations

2.2.1. *Hecke Algebras and their Cellular Representations*

Hecke algebras. Let W be a Coxeter group, with a set of simple reflections S. The associated *Hecke algebra* $\mathcal{H}_W(q)$ is defined over the polynomial ring $Z[q]$ as follows. $\mathcal{H}_W(q)$ is spanned over the basis $\{T_w | w \in W\}$, where multiplication is defined by

$$T_w T_v = T_{wv}, \text{ if } \ell(wv) = \ell(w) + \ell(v)$$

$$(T_s - 1)(T_s + q) = 0, \ \forall\ s \in S$$

Here $\ell(w)$ is the length of w.

It should be noted that the last relation is slightly non-standard; this is done in order to get more elegant q-analogues. In order to shift to the standard version, one should replace each T_i by $-T_i$.

The elements $T_s, s \in S$ generate $\mathcal{H}_W(q)$.

Denote the Hecke algebra of the symmetric group S_n by $\mathcal{H}_n(q)$, and denote $T_{s_i} \in \mathcal{H}_n(q)$ by T_i. Then $\mathcal{H}_n(q)$ is generated by $T_1, \ldots, T_{n-1}$with respect to the following relations

$$T_i T_j = T_j T_i, \quad \text{if } |i-j| > 1$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \le i < n-1$$

$$(T_i - 1)(T_i + q) = 0, \quad \forall i$$

Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition of n. The element $T_\mu \in \mathcal{H}_n(q)$ is defined by

$$T_\mu := T_1 T_2 \cdots T_{\mu_1 - 1} T_{\mu_1 + 1} \cdots\cdots T_{\mu_1 + \ldots \mu_t - 1}$$

I.e. T_μ is the subproduct of $T_1 T_2 \cdots T_{\mu_1 + \cdots + \mu_t - 1}$ omitting $T_{\mu_1 + \cdots + \mu_i}$ for $1 \le i < t$.

In their fundamental paper [KL], Kazhdan and Lusztig present a distinguished basis for every Hecke algebra, and construct a rich family of representations of Coxeter groups and their Hecke algebras. In the case of the symmetric group, this construction gives a decomposition of the Hecke algebra into irreducible representations.

A cornerstone for this theory is the concept of cellular structure. Any Coxeter group can be partitioned into subsets called *Kazhdan-Lusztig cells.* The action of the group on each of these cells gives rise to a so-called *Kazhdan-Lusztig representation.* Denote the associated Kazhdan-Lusztig representation of a right (left) Kazhdan-Lusztig cell, $\mathcal{C}$, by $\rho^{\mathcal{C}}$, and the corresponding character by $\chi^{\mathcal{C}}$.

In the symmetric group case the Kazhdan-Lusztig right cells are the Knuth classes, while the Kazhdan-Lusztig left cells are the inverse Knuth classes [KL, §5, proof of Proposition 1.4]. The representation associated with a Kazhdan-Lusztig right (left) cell is the S_n-irreducible representation S^λ, where λ is the shape of Q. We say that λ is the *shape of the cell (Knuth class)* $\mathcal{C}$.

2.2.2. *The Coinvariant Algebra*

The symmetric group S_n acts on the polynomial ring $P_n = Q[x_1, \ldots, x_n]$ by permuting the variables. The *coinvariant algebra* is the quotient P_n/I_n, where I_n is the ideal generated by the symmetric (S_n-invariant) polynomials without a constant term. The coinvariant algebra of a finite Weyl group W is defined similarly. The group algebra of W and its coinvariant algebra are isomorphic as W-modules [Hu, Ch. 3.6]. Early work of Borel showed how to identify the coinvariant algebra with the cohomology ring of G/B, where G is a simple Lie group and B is a Borel subgroup. Schubert polynomials, constructed in the seminal papers [BGG] and [De], form a distinguished basis for the coinvariant algebra. These polynomials correspond to the Schubert cells in $H^*(G/B)$.

The coinvariant algebra has a natural grading to homogenous components, induced from the grading of the polynomial ring by total degree. Denote by R^k the k-th homogeneous component of the coinvariant algebra. and by χ^k its corresponding character as a W-module.

Decomposition into irreducibles. The decomposition of the coinvariant algebra into irreducible representations involves major indices [KW].

Theorem 0.6 *The multiplicity of the S_n-irreducible representation S^λ in the k-th homogeneous component of the coinvariant algebra, R^k, is equal to the number of Knuth classes of shape λ and $major(\mathcal{C}^{-1}) = k$.*

An elegant proof of this theorem, using the principal specialization of Schur functions, was given by Stanley and developed in [Ste, Ga, Reu, Ch. 8.3]. In [Ro5] we applied Foata and Schützenberger's work on the major index, together with properties of Kazhdan-Lusztig representations, to derive this result. An analogous rule for decomposing the coinvariant algebras of classical Weyl groups of type B and other wreath products was given in [Ste]. This analogue involves a new interpretation of the major index in terms of Coxeter elements [AR1]. Decomposition of other quotient rings is described by Kostka-Foulkes polynomials [GP].

Hecke algebra action. Explicit deformations of the symmetric group action on the coinvariant algebra is presented in [APR1]. Two different Hecke algebra actions on the polynomial ring P_n are defined.

For the first action, each generator T_i acts on P_n as the linear operator

$$R_i(x_i^\alpha x_{i+1}^\beta m) := \begin{cases} q x_i^\beta x_{i+1}^\alpha m, & \text{if } \alpha \geq \beta \\ (1-q)x_i^\alpha x_{i+1}^\beta m + x_i^\beta x_{i+1}^\alpha m, & \text{if } \alpha < \beta \end{cases}$$

Here m is a monomial involving neither x_i nor x_{i+1}. The action of R_i is extended to the full polynomial ring by linearity. For the second action each generator T_i acts on P_n as the q-commutator

$$A_i := \partial_i X_i - q X_i \partial_i$$

where ∂_i is the divided difference operator $\partial_i := (x_i - x_{i+1})^{-1}(1 - s_i)$, and X_i is multiplication by x_i.

The symmetric functions are invariant under these two actions. Therefore, the actions on the homogeneous components of the coinvariant algebra form Hecke algebra representations. It should be noted that the two actions form equivalent representations. The associated characters may represented as refined counts over subsets of permutations. Surprisingly, the Kazhdan-Lusztig characters of the Hecke algebra may be represented as refined counts of exactly the same statistic over different summation sets. See Theorems 8 and 9 below.

§3. Symmetric Group Characters

3.1. Irreducible Characters

The Murnaghan-Nakayama classical rule represents the symmetric group characters as a signed enumeration of the so called rim-hook tableaux. Cf. [Sa, Ch. 4.10]. For a representation theoretical interpretation of refined counts of rim-hook tableaux see [LLT]. In this section we represent symmetric group characters as refined counts of permutations. These representations are convenient for generalizations to Hecke algebras, coinvariant algebras, and other groups and algebras.

Definition. A sequence of positive integers $a = a_1, \ldots, a_n$ is *unimodal* if there exists $1 \le m \le n$, such that

$$a_1 < a_2 < \cdots < a_m > a_{m+1} > \cdots > a_n.$$

Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition of n. A sequence of n positive integers is *μ-unimodal* if the first μ_1 integers form a unimodal sequence, the next μ_2 integers form a unimodal sequence, and so on.

A permutation $\pi \in S_n$ is called a *μ-unimodal permutation* if the sequence $\pi(1), \ldots, \pi(n)$ is μ-unimodal. For example, $\pi = 174239856$ is $(4,3,2)$-unimodal, but not $(5,4)$-unimodal.

Denote the set of all μ-unimodal permutations in S_n by U_μ. Let λ and μ be partitions of n, and let χ_μ^λ be the S_n-character value of the irreducible representation S^λ at a conjugacy class of type μ. The following Theorem is a special case of [Ro2, Theorem 4].

Theorem 1.

$$\chi_\mu^\lambda = \sum\nolimits_{\pi \in C \cap U_\mu} (-1)^{descent(\pi)},$$

where the sum runs over all μ-unimodal permutations in a Knuth class C of shape λ.

For an elementary combinatorial proof see [Ro2, Proof of Theorem 6].

The following lemma is an immediate consequence of Stanley's hook formula (Theorem 0.3).

Lemma 2. *Let C be a Knuth class of shape λ. Then*

$$\sum\nolimits_{\pi \in C} \omega^{major(\pi)} = \begin{cases} (-1)^k, & \text{if } \lambda = (n-k, 1^k) \\ 0, & \text{otherwise} \end{cases},$$

where ω is a root of unity of order n.

Proof. By Theorem 0.3 for any Knuth class of shape $(n-k, 1^k)$

$$\sum\nolimits_{\pi \in C} \omega^{major(\pi)} = \prod_{i=1}^{k} \omega^i \frac{\omega^{n-i} - 1}{\omega^i - 1} = (-1)^k.$$

If λ is not a hook then combining the fact that the refined count is a polynomial together with Theorem 0.3 implies that

$$\sum_{\pi\in C} q^{major(\pi)} = \frac{q^n-1}{q^j-1}p(q),$$

for some $j < n$, where j is a divisor of n, and $p(q)$ is a polynomial in q. Hence,

$$\sum_{\pi\in C} \omega^{major(\pi)} = (1+\omega^j+\omega^{2j}+\cdots+\omega^{n-j})\cdot p(\omega) = 0\cdot p(\omega) = 0.$$

$\square$

Another combinatorial representation of irreducible characters follows from this lemma.

Theorem 3.

$$\chi^\lambda_{(n)} = \sum_{\pi\in C} \omega^{major(\pi)},$$

where the sum is taken over a Knuth class C of shape λ, and ω is a root of unity of order n.

Proof. It follows from Theorem 1 that $\chi^\lambda_{(n)} = (-1)^k$ if $\lambda = (n-k, 1^k)$ and zero otherwise. Combining this fact with Lemma 2 gives the desired result.

$\square$

Remark. Theorem 3 was first proved in a deep work of Stembridge. In this work it was shown that the summands in the right hand side are essentially the eigenvalues. See [Ste, Theorem 3.2].

The Kostka-Foulkes polynomials are defined as entries of transition matrices between bases of q-symmetric functions. See Section 2.2. Combining Theorem 3 with Theorem 0.5 we obtain

Corollary 4.

$$\chi^\lambda_{(n)} = K_{\lambda',1^n}(\omega),$$

where $K_{\lambda,\mu}(q)$ is the Kostka-Foulkes polynomial, ω is a root of unity of order n, and λ' is the conjugate partition of λ.

Proof. By Claim 0.4 and Theorem 0.5

$$K_{\lambda,1^n}(q) = \sum_T q^{charge(\pi(T))} = \sum_{\pi\in w_0 S_\lambda w_0} q^{major(\pi)}.$$

Here $S_\lambda := \{(\pi(T))^{-1} | T \text{ is a standard tableau of shape } \lambda\}$, where $\pi(T)$ is the row word of T.

But S_λ is a Knuth class of shape λ'. Moreover, for any Knuth class $\mathcal{C}$ of shape λ', $w_0\mathcal{C}$ and $\mathcal{C}w_0$ are Knuth classes of shape λ. Hence, $w_0 S_\lambda w_0$ is a Knuth class of shape λ'.

Theorem 3 completes the proof. □

3.2. Coinvariant Algebra Characters

Let χ_μ^k be the S_n-character value at a conjugacy class of type μ of the k-th homogeneous component of the coinvariant algebra of the symmetric group S_n. Denote the set $\{\pi \in S_n | \ell(\pi) = k\}$ by $L(k)$. An analogue of Theorem 1 is proved in [Ro5, Theorem 1].

Theorem 5.

$$\chi_\mu^k = \sum\nolimits_{\pi \in L(k) \cap U_\mu} (-1)^{descent(\pi)},$$

where the sum runs over all μ-unimodal permutations of length k.

In other words

Theorem 5'.

$$\sum\nolimits_k \chi_\mu^k t^k = \sum\nolimits_{\pi \in U_\mu} (-1)^{descent(\pi)} t^{\ell(\pi)}.$$

It follows from Theorem 3 that

Theorem 6.

$$\sum\nolimits_k \chi_{(n)}^k t^k = \sum\nolimits_{\pi \in S_n} \omega^{major(\pi)} t^{\ell(\pi)}.$$

Proof. Clearly,

$$\chi_{(n)}^k = \sum\nolimits_\lambda m_{\lambda,k} \chi_{(n)}^\lambda,$$

where $m_{\lambda,k}$ is the multiplicity of the irreducible representation S^λ in the k-th homogeneous component of the coinvariant algebra.

Theorem 0.6 asserts that $m_{\lambda,k}$ equals to the number of Knuth classes $\mathcal{C}$ with $major(\mathcal{C}^{-1}) = k$. It follows that

$$\sum\nolimits_k \chi_{(n)}^k t^k = \sum\nolimits_{\mathcal{C}} \chi_{(n)}^{\mathcal{C}} t^{major(\mathcal{C}^{-1})}.$$

Here $\chi^{\mathcal{C}}$ is the irreducible character χ^λ, where λ is the shape of $\mathcal{C}$.

By Theorem 3 the right hand side equals to

$$\sum\nolimits_{\mathcal{C}} \sum\nolimits_{\pi \in \mathcal{C}} \omega^{major(\pi)} t^{major(\mathcal{C}^{-1})} = \sum\nolimits_{\pi \in S_n} \omega^{major(\pi)} t^{major(\pi^{-1})}.$$

Theorem 0.1 completes the proof.

□

Comparing Theorem 5' to Theorem 6 implies the following identity.

Corollary 7.

$$\sum_{\pi\in S_n} \omega^{major(\pi)} t^{\ell(\pi)} = \sum_{\pi\in U_n} (-1)^{descent(\pi)} t^{\ell(\pi)},$$

where U_n is the set of all unimodal permutations in S_n.

§4. Hecke Algebra Characters

Hecke algebra characters provide q-analogues of the above results. The following formula for the irreducible characters is proved in [Ro2, Theorem 4] and in [Ra2]. Recall the definition of $T_\mu \in \mathcal{H}_n(q)$ from Section 2.2.1, and let $\chi_q^\lambda(T_\mu)$ be the $\mathcal{H}_n(q)$-character value of the irreducible representation corresponding to λ at the element $T_\mu \in \mathcal{H}_n(q)$. Then

Theorem 8.

$$\chi_q^\lambda(T_\mu) = \sum_{\pi\in\mathcal{C}\cap U_\mu} (-q)^{descent(\pi)},$$

where the sum runs over all μ-unimodal permutations in a Knuth class $\mathcal{C}$ of shape λ.

Action of Hecke algebra of type A on coinvariant algebras is described in Section 2.1.2. Let χ_q^k be the character of the Hecke algebra $\mathcal{H}_n(q)$, defined by the action on the k-th homogeneous component of the coinvariant algebra. The following analogue of Theorem 5 is proved in [APR1, Theorems 5.1 and 6.6].

Theorem 9.

$$\chi_q^k(T_\mu) = \sum_{\pi\in L(k)\cap U_\mu} (-q)^{descent(\pi)},$$

where $L(k)$ is the set of all permutations of length k in S_n.

Here is an alternative combinatorial description.

Theorem 10.

$$\chi_q^k(T_{(n)}) = \sum_{\{\pi\in S_n \mid major(\pi)=k\}} \omega^{major(\pi^{-1})} q^{descent(\pi)}.$$

Proof. In order to prove this corollary we need the following lemma.

Lemma 11.

$$\chi_q^\lambda(T_{(n)}) = \sum\nolimits_{\pi\in\mathcal{C}} \omega^{major(\pi)} q^{descent(\pi^{-1})},$$

where the sum is taken over a Knuth class $\mathcal{C}$ of shape λ.

Proof of Lemma 11. By Fact 0.2 all permutations in an inverse Knuth class $\mathcal{C}^{-1}$ have a common descent number, denoted by *descent*$(\mathcal{C}^{-1})$. Moreover, it is easy to verify that for any Knuth class $\mathcal{C}$ of hook shape $(n-k,1^k)$, *descent*$(\mathcal{C}^{-1}) = k$. Combining these facts with Lemma 2 we obtain

$$\sum\nolimits_{\pi\in\mathcal{C}} \omega^{major(\pi)} q^{descent(\pi^{-1})} = q^{descent(\mathcal{C}^{-1})} \sum\nolimits_{\pi\in\mathcal{C}} \omega^{major(\pi)} =$$

$$= \begin{cases} (-q)^k, & \text{if } \lambda = (n-k,1^k) \text{ for some } 0 \le k < n \\ 0, & \text{otherwise} \end{cases}.$$

On the other hand, by Theorem 8

$$\chi_q^\lambda(T_{(n)}) = \begin{cases} (-q)^k, & \text{if } \lambda = (n-k,1^k) \text{ for some } 0 \le k < n \\ 0, & \text{otherwise} \end{cases}.$$

□

Using Theorem 0.6, as in the proof of Theorem 6, together with Lemma 11 yields

$$\sum\nolimits_k \chi_q^k(T_{(n)}) t^k = \sum\nolimits_{\mathcal{C}} \chi^{\mathcal{C}}(T_{(n)}) t^{major(\mathcal{C}^{-1})} =$$
$$= \sum\nolimits_{\mathcal{C}} \sum\nolimits_{\pi\in\mathcal{C}} \omega^{major(\pi)} q^{descent(\pi^{-1})} t^{major(\mathcal{C}^{-1})} =$$
$$= \sum\nolimits_{\pi\in S_n} \omega^{major(\pi)} q^{descent(\pi^{-1})} t^{major(\pi^{-1})}.$$

We conclude that

$$\chi_q^k(T_{(n)}) = \sum\nolimits_{\{\pi\in S_n \mid major(\pi^{-1})=k\}} \omega^{major(\pi)} q^{descent(\pi^{-1})}.$$

□

Comparing Theorem 9 with Theorem 10 we obtain

Corollary 12. *With the above notations*

$$\sum\nolimits_{\pi\in S_n} \omega^{major(\pi^{-1})} q^{descent(\pi)} t^{major(\pi)} = \sum\nolimits_{\pi\in U_n} (-q)^{descent(\pi)} t^{\ell(\pi)}.$$

§5. Final Remarks and Open Problems

5.1. Other Weyl Groups

Let H be a parabolic subgroup of an arbitrary Coxeter group W, which is isomorphic to a direct product of symmetric groups. In the following definition we refer to cycle type and $weight^q_\mu$ of elements in H under this isomorphism.

Definition. Let μ be a cycle type of an element in H. For any element $w = r \cdot \pi \in W$, where $\pi \in H$ and r is the representative of minimal length of the left coset of wH in W, define

$$weight^q_\mu(w) := \begin{cases} (-q)^{descent(\pi)}, & \text{if } \pi \text{ is } \mu\text{-unimodal} \\ 0, & \text{otherwise} \end{cases}.$$

Note that $weight^q_\mu$ is independent of the choice of H, provided that H is isomorphic to a direct product of symmetric groups and that μ is the cycle type of some element in H.

Let $v_\mu \in H$ have a cycle type μ, and let T_μ be the element in the Hecke algebra $\mathcal{H}_W(q)$ indexed by v_μ.

Theorem 13. *[Ro1, Corollary 3] Let $\mathcal{C}$ be a finite Kazhdan-Lusztig right cell in an arbitrary Coxeter group W, and let $\chi^{\mathcal{C}}$ be its associated Hecke algebra character. Then*

$$\chi^{\mathcal{C}}(T_\mu) = \sum\nolimits_{w\in\mathcal{C}} weight^q_\mu(w).$$

A formally similar result for coinvariant algebras is proved in [APR2]. Let R^k be the k-th homogeneous component of the coinvariant algebra of W. Denote by χ^k the W-character of R^k. Let $v_\mu \in H$ have cycle type μ. Then

Theorem 14. *[APR2, Theorem 4] Let W be an arbitrary finite Weyl group. With the above notations*

$$\chi^k(v_\mu) = \sum\nolimits_{\{w\in W:\ell(w)=k\}} weight^1_\mu(w).$$

So, coinvariant algebra characters of an arbitrary finite Weyl group W and Kazhdan-Lusztig characters of these groups may be represented as sums of exactly the same weights, but over different summation sets. This curious analogy seems to deserve further study.

Unfortunately, we do not know of an explicit Hecke algebra action on homogeneous components of coinvariant algebras of general type.

Problem 1. *Define an action of the Hecke algebra of an arbitrary finite Weyl group on its coinvariant algebra, which produces a natural q-analogue of Theorem 14.*

When it comes to Theorem 3, analogues are known for classical Weyl groups and wreath products [Ste].

Problem 2. *Give an analogue of Theorem 3 for an arbitrary finite Coxeter group.*

Stembridge proved that Theorem 3 describes the S_n-character $\chi^{\lambda}_{(n)}$ as a sum of the eigenvalues of the full cycle at the irreducible representation S^{λ}. Theorem 13 describes Kazhdan-Lusztig characters of a general Hecke algebra. Eigenvalues are described by Geck and Michel.

Theorem 15. *[GM, Proposition 1.3] Let W be a finite Coxeter group, and let ρ be an irreducible representation of its Hecke algebra $\mathcal{H}_W(q)$. Let $w \in W$ be an element of minimal length in some conjugacy class in W. Let ω be a root of unity of order d, where d is the order of w. Then there exist integers m_i and rational numbers r_i, such that the eigenvalues of $\rho(T_w), T_w \in \mathcal{H}_W(q)$, are*

$$\omega^{m_i} q^{r_i}, \qquad (1 \le i \le dim\rho).$$

Unfortunately, the problem of determining the integers m_i is not solved in general. In case of classical Weyl groups and related wreath products, these integers are determined by a generalized major index [Ste, Theorem 5.1]. In this perspective, the problem of determining the integers m_i (and so, solving Problem 2) is strongly related to the problem of defining major index on arbitrary Coxeter groups. Partial results appear in [Reil-2, FC1-3, Stei ,Ste, AR1].

A closely related problem is the following: Recall that the sets of permutations of a fixed major index are unions of Kazhdan-Lusztig cells. This fact, together with Garsia-Gessel refined count of the major index of shuffles [GG], implies an extremely simple combinatorial rule for restricting the coinvariant algebra of type A to parabolic subgroups [Ro4]. This rule is an exact analogue of the Barbasch-Vogan rule for restricting Kazhdan-Lusztig representations of arbitrary Weyl groups. No such a rule is known for coinvariant algebras of other Weyl groups.

Geck and Michel give an algorithm for calculating the exponents r_i in Theorem 15 [GM, §4.3]. In light of Corollary 4, a combinatorial understanding of this algorithm may be helpful in the study of deformations of Kostka-Foulkes polynomials. See e.g. [GH].

5.2. Other Representation Theory Interpretations

Stanley's hook formula for refined counts of major index over standard tableaux of shape λ (Theorem 0.3) is identical with Olsson's hook formula, giving the dimensions of unipotent representations of $GL_n(q)$, the general linear group over a finite field [O]. Refined counts of charge on semi-standard tableaux of shape λ gives Kostka-Foulkes polynomials [LS]. These polynomials are equal to unipotent characters of $GL_n(q)$ [Lu3, Sh1, Ch. 2.7]. Therefore the charge essentially refines the major index, both as permutation statistics and in the representation theoretic interpretation. The charge gives also the eigenvalues of conjugacy classes of type $r^{n/r}$. Unfortunately, eigenvalues of conjugacy classes of general type are not given by charge. See [Ste].

Finally, it should be mentioned that Hecke algebra bitraces may be represented as refined counts of nonnegative integer matrices [HLR]. This result follows from Theorem 8. The problem of representing characters of Brauer and Birman-Wenzl algebras as refined counts is a promising open problem [Ra3].

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References

[APR1] R. M. Adin, A. Postnikov, and Y. Roichman, *Hecke algebra actions on the coinvariant algebra.* Journal of Algebra, to appear.

[APR2] R. M. Adin, A. Postnikov, and Y. Roichman, *On characters of Weyl groups*, Discrete Math., to appear.

[AR1] R. M. Adin and Y. Roichman, *A Flag major index for signed permutations*, Proc. 9th Conf. Formal Power Series and Alg. Comb., Universitat Politéchnica de Catalunya, Barcelona, 1999, 10–17.

[AR2] R. M. Adin and Y. Roichman, *Descent Functions and random Young tableaux.* Combinatorics, Probability & Comp. to appear.

[BGG] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, *Schubert cells and cohomology of Schubert spaces G/P.* Usp. Mat. Nauk. 28 (1973), 3–26.

[Ca] L. Carlitz, *q-Bernoulli and Eulerian numbers.* Trans. Amer. Math. Soc. **76** (1954), 332–350.

[CF1] R. J. Clarke and D. Foata, *Eulerian calculus, I: Univariable statistics*, European J. Combin. **15** (1994), 345–362.

[CF2] R. J. Clarke and D. Foata, *Eulerian calculus, II: An extension of Han's fundamental transformation,* European J. Combin. **16** (1995), 221–252.

[CF3] R. J. Clarke and D. Foata, *Eulerian calculus. III. The ubiquitous Cauchy formula.* European J. Combin. 16 (1995), 329–355.

[De] M. Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion.* Invent. Math. 21 (1973), 287–301.

[F] D. Foata, *On the Netto inversion number of a sequence*, Proc. Amer. Math. Soc. **19** (1968), 236–240.

[FS] D. Foata and M. P. Schützenberger, *Major index and inversion number of permutations*, Math. Nachr. **83** (1978), 143–159.

[Fu] J Fulman, *The distribution of descents in fixed conjugacy classes of the symmetric groups*, J. Comb. Theory Ser. A 84 (1998), 171–180.

[Ga] A. M. Garsia, *Combinatorics of the free Lie algebra and the symmetric group*, in: Analysis, et cetera..., Jürgen Moser Festschrift, P. H. Rabinowitz and E. Zehnder (eds.), Academic Press, New York, 1990, pp. 309–382.

[GG] A. M. Garsia and I. Gessel, *Permutation statistics and partitions.* Adv. in Math. **31** (1979), 288–305.

[GH] A. M. Garsia and M. Haiman, *Some natural bigraded S_n-modules and q,t-Kostka coefficients*, The Foata Festschrift, Electron. J. Combin. **3** (1996), no. 2.

[GP] A. M. Garsia and C. Procesi, *On certain graded S_n-modules and the q-Kostka polynomials*, Adv. in Math. **94** (1992), 82–138.

[GM] M. Geck and J. Michel, *'Good' elements of finite Coxeter groups and representations of Iwahori-Hecke algebras*, Proc. London Math. Soc. 74 (1997), 275–305.

[Hai] M. Haiman, *Conjectures on the quotient ring by diagonal invariants*, J. Algebraic Combin. **3** (1994), 17–76.

[HLR] T. Halverson, R. Leduc and A. Ram, *Iwahori-Hecke algebras of type A, bitraces and symmetric functions*, Internat. Math. Res. Notices 1997, no. 9, 401–416.

[HR] T. Halverson and A. Ram, *Murnaghan-Nakayama rules for characters of Iwahori-Hecke algebras of classical type*, Trans. Amer. Math. Soc. 348 (1996), no. 10, 3967–3995.

[Hu] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Univ. Press, 1992.

[KL] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[KW] W. Kraskiewicz and J. Weyman, *Algebra of coinvariants and the action of Coxeter elements*, Math. Inst. Copernicus Univ. Chopina, Poland, preprint, 1987.

[LS] A. Lascoux and M. P. Schützenberger, *Sur une conjecture de H. O. Foulkes*, C.R. Acad. Sci. Paris 268A (1978), 323–324.

[LLT] A. Lascoux, B. Leclerc and J. Y. Thibon, *Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties*, J. Math. Phys. 38 (1997), 1041–1068.

[Lu1] G. LUSZTIG, *Characters of Reductive Groups Over a Finite Field*, Annals of Math. Stud. no. 107, Princeton Univ. Press, Princeton, NJ, 1984.

[Lu2] G. LUSZTIG, *Left cells in Weyl groups*, Lect. Notes in Math. 1024, Springer-Verlag, Berlin/New York, 1983, pp. 99–111.

[Lu3] G. LUSZTIG, *Green polynomials and singularities of unipotent classes*, Adv. in Math. **42** (1981), 169–178.

[Md] I. G. MACDONALD, *Symmetric Functions and Hall Polynomials*, second edition, Oxford Math. Monographs, Oxford Univ. Press, Oxford, 1995.

[MM] P. A. MACMAHON, *Combinatory Analysis I-II*, Cambridge Univ. Press, London/New-York, 1916. (Reprinted by Chelsea, New-York, 1960.)

[O] J. B. OLSSON, *On blocks of $GL(n,q)$*, Trans. Amer. Math. Soc. 222 (1976), 143–156.

[Ra1] A. RAM, *A Frobenius formula for the characters of Hecke algebras*, Invent. Math. **106** (1991), 461–488.

[Ra2] A. RAM, *An elementary proof of Roichman's rule for irreducible characters of Iwahori-Hecke algebras of type A*, In: Mathematical essays in honor of Gian-Carlo Rota, Progr. Math., 161, Birkhäuser, Boston, 1998, 335–342.

[Ra3] A. RAM, personal communication.

[Raw] D. RAWLINGS, *Generalized Worpitzki identities with applications to permutation enumeration*, European J. Combin. **2** (1981), 67–78.

[Rei1] V. REINER, *Signed permutation statistics*, European J. Combin. **14** (1993), 553–567.

[Rei2] V. REINER, *Signed permutation statistics and cycle type*, European J. Combin. **14** (1993), 569–579.

[Reu] C. REUTENAUER, Free Lie Algebras, London Math. Soc. Monographs (New Series) no. 7, Oxford Univ. Press, 1993.

[Ro1] Y. ROICHMAN, *Upper bounds on characters of the symmetric groups*, Invent. Math. **125** (1996), 451–485.

[Ro2] Y. ROICHMAN, *A recursive rule for Kazhdan-Lusztig characters*, Adv. in Math. **129** (1997), 24–45.

[Ro3] Y. ROICHMAN, *Some combinatorial properties of the coinvariant algebra*, in: Proceedings of FPSAC-98, Fields Inst., Toronto, 1998, pp. 529–537.

[Ro4] Y. ROICHMAN, *Major index of shuffles and restriction of representations*, preprint, 1997.

[Ro5] Y. ROICHMAN, *Schubert polynomials, Kazhdan-Lusztig basis and characters*, Discrete Math. **217** (2000), 353–365.

[Sa] B. E. SAGAN, *The Symmetric Group:Representations,Combinatorial Algorithms & Symmetric Functions.* Wadworth & Brooks/Cole, 1991.

[Sh] SHI JIAN-YI, *Kazhdan-Lusztig Cells in Certain Affine Weyl Groups*, Lect. Notes in Math. no. 1179, Springer-Verlag, 1986.

[St] R. P. STANLEY, *Some aspects of groups acting on finite posets*, J. Combin. Theory Ser. A **32** (1982), 132–161.

[ECI] R. P. STANLEY, *Enumerative Combinatorics (Vol. 1)*, Cambridge Univ. Press, 1998.

[ECII] R. P. STANLEY, *Enumerative Combinatorics (Vol. 2)*, Cambridge Univ. Press, to appear.

[Stei] E. STEINGRIMSSON, *Permutation statistics of indexed permutations*, European J. Combin. **15** (1994), 187–205.

[Ste] J. STEMBRIDGE, *On the eigenvalues of representations of reflection groups and wreath products.* Pacific J. of Math. 140, 1989, 359–396.

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