Advanced Studies in Pure Mathematics 26, 2000
Analysis on Homogeneous Spaces and Representation Theory of Lie Groups
pp. 77–98

K-type Structure in the Principal Series of GL_3, I

Roger Howe[1]

§1. Introduction

It is about 50 years since the principal series were defined [GN], and about 45 since Harish-Chandra showed via his Subquotient Theorem [HC] that they are fundamental to understanding representations of semisimple Lie groups. However, the structure of the principal series, especially the composition structure at points of reducibility, still presents mysteries. In recent years, several authors ([HT], [Sa], [J], [Za], [L1], [L2], [HL], [Fu]), have shown that in many cases when the principal series (or, more loosely, degenerate principal series) is multiplicity-free under the action of a maximal compact subgroup, it is possible to understand their structure, including composition series and unitarity, fairly completely and explicitly. The goal of this paper is to begin an investigation of the detailed structure of some examples of principal series in which representations of the maximal compact subgroup appear with arbitrarily large multiplicities. Specifically, we investigate here the principal series of $GL_3(\mathbb{R})$. Our main finding is that each isotypic space for the maximal compact subgroup $K = 0_3$ has a unique distinguished basis compatible with the occurrence of finite-dimensional subrepresentations. Using this basis, we are able to see certain subquotients of the principal series in a manner entirely analogous to the investigations of multiplicity-free situations. In particular, in this basis we can display the full composition series when a finite-dimensional constituent occurs. However, from other considerations, we may see that at some points where the principal series is reducible, it is not at all evident from looking at our basis. This indicates a need for further investigation.

Received May 28, 1998.
Revised June 21, 1999.
[1]Research partially supported by NSF grant DMS-9622916. The author also is grateful for hospitality from the Mittag-Leffler Institute during May 1996.

§2. Notation and setup

We will study representations realized on concrete spaces of functions, and our interest will be in the infinitesimal action – how the Lie algebra $\mathfrak{gl}_3$ acts on these functions. We begin by recalling (for example, from [Ho1]) the action of GL_3 on functions on $\mathbb{R}^3$, which is a degenerate principal series. Let x_1, x_2, x_3 be the coordinates on $\mathbb{R}^3$. The Lie algebra $\mathfrak{gl}_3$ acts on functions in these variables by means of differential operators

$$(2.1) \qquad E_{ab} = x_a \frac{\partial}{\partial x_b} = x_a D_{x_b} \qquad 1 \le a, b \le 3.$$

In the sequel, we will usually write $\frac{\partial}{\partial x_b} = D_{x_b}$.

The maximal compact subgroup of GL_3 is O_3, which leaves invariant the usual (square of the) Euclidean length

$$(2.2) \qquad r^2 = x_1^2 + x_2^2 + x_3^2.$$

The Lie algebra $\mathfrak{so}_3$ of O_3 is spanned by the operators

$$(2.3) \qquad \hat{E}_{ab} = E_{ab} - E_{ba}.$$

It is convenient to use complexified coordinates that allow us to display the action of $\mathfrak{so}_3$ in a perspicuous manner. We set

$$(2.4) \qquad z = x_1 + ix_2 \qquad y = x_3.$$

Then

$$(2.5) \qquad r^2 = z\bar{z} + y^2,$$

and a basis for $\mathfrak{so}_3$ is provided by the operators

$$(2.6) \quad h = 2(zD_z - \bar{z}D_{\bar{z}}),\ e^+ = zD_y - 2yD_{\bar{z}},\ e^- = -(\bar{z}D_y - 2yD_z).$$

These satisfy the standard commutation relations

$$(2.7) \qquad [h, e^{\pm}] = \pm 2e^{\pm} \quad [e^+, e^-] = h.$$

We can organize the remainder of the Lie algebra of $\mathfrak{gl}_3$ into the central element

$$(2.8) \qquad \mathfrak{z} = zD_z + \bar{z}D_{\bar{z}} + yD_y = x_1D_{x_1} + x_2D_{x_2} + x_3D_{x_3}$$

and $\mathfrak{p}$, a five-dimensional space invariant under the adjoint action of $\mathfrak{so}_3$. The space $\mathfrak{p}$ has a basis consisting of eigenvectors for the operator h in $\mathfrak{so}_3$. These are

(2.9)
$$\begin{aligned} &p_4 = zD_{\bar z} \qquad p_2 = zD_y + 2yD_{\bar z} \\ &p_0 = -zD_z - \bar z D_{\bar z} + 2yD_y \quad p_{-2} = -(\bar z D_y + 2yD_z) \quad p_{-4} = \bar z D_z. \end{aligned}$$

The subscript on each p_j describes the ad h eigenvalue of p_j. The commutation relations of the p_j with $e^{\pm}$ are as follows

$$\begin{array}{lll} [e^+, p_4] = 0 & [e^+, p_2] = 4p_4 & [e^+, p_0] = 3p_2 \\ [e^+, p_{-2}] = 2p_0 & [e^+, p_{-4}] = p_{-2} & \\ [e^-, p_4] = p_2 & [e^-, p_2] = 2p_0 & [e^-, p_0] = 3p_{-2} \\ [e^-, p_{-2}] = 4p_{-4} & [e^-, p_{-4}] = 0. & \end{array} \tag{2.10}$$

The action of $\mathfrak{gl}_3$ on functions of x_1, x_2, x_3 preserves the functions of a given degree of homogeneity. These are the eigenspaces for the central element $\mathfrak{z}$ of formula (2.8). We can decompose the space of functions of given homogeneity into representations for $\mathfrak{so}_3$, each of which is characterized by a highest weight vector, which in this context means an eigenvector for h annihilated by e^+. The homogeneous highest weight vectors have the form

$$z^m r^{2\alpha} = \psi(m, \alpha) \tag{2.11}$$

The degree of homogeneity of $\psi(m, \alpha)$ is $\beta = m + 2\alpha$. Since $r^2 \geq 0$, we can allow α to be any complex number. On the other hand, m is a non-negative integer.

Let $\mathcal{S}^\beta$ denote the space of functions on $\mathbb{R}^3$ with a given degree β of homogeneity. The action of $\mathfrak{gl}_3$ on $\mathcal{S}^\beta$ may be described in terms of the effect of certain "transition operators" on the $\psi(m, \alpha)$. Transition operators were used in [L1], [L2],[HL] to analyze various degenerate principle series. In our situation, they are elements of the enveloping algebra $\mathcal{U}(\mathfrak{gl}_3)$ which take a highest weight vector for $\mathfrak{so}_3$, of weight m, to a highest weight vector of weight $m + d$, where $d = \pm 4, \pm 2$, or 0 – the weights of ad h on $\mathfrak{p}$. There is one transition operator for each value of d. The transition operators for $d = 4, 2, 0$ are given as follows.

$$\begin{aligned} T_4 &= p_4 \\ T_2 &= p_2 h - 4p_4 e^- \\ T_0 &= p_0 h(h-1) - 3p_2 e^-(h-1) + 6p_4 e^{-2}. \end{aligned} \tag{2.12}$$

We do not need to know T_{-2} and T_{-4}, because these transitions are related to T_2 and T_4 by duality (cf. [L1], [L2]).

As a prelude to our main computation, we compute the results of applying T_j to the functions $\psi(m,\alpha)$ of formula (2.11). It is very easy to compute

$$T_4\psi(m,\alpha) = \alpha\psi(m+2,\alpha-1). \tag{2.13}$$

With slightly more effort, we can check that

$$T_2\psi(m,\alpha) = 0 \tag{2.14}$$

Thus, there is no two-transition. (This could have been predicted ahead of time, since the highest weight vectors $\psi(m,\alpha)$ transform by $(-1)^m$ under the central element -1 in $GL_3(\mathbb{R})$, so the space $\mathcal{S}^\beta$ breaks up into two GL_3-invariant subspaces, one containing the $\psi(m,\alpha)$ with m even, the other containing the $\psi(m,\alpha)$ with m odd.) Finally we compute $T_0\psi(m,\alpha)$. Even in this very simple situation, this computation requires some exertion. We may organize it into subcalculations as follows.

(2.15)

$$\begin{aligned}
&\text{(a)} & h(h-1)z^m &= 2m(2m-1)z^m \\
& & e^-(h-1)z^m &= 2m(2m-1)yz^{m-1} \\
& & e^{-2}z^m &= 4m(m-1)z^{m-2}y^2 - 2mz^{m-1}\bar{z} \\
&\text{(b)} & T_0z^m &= -2m(m+1)(2m+3)z^m \\
&\text{(c)} & p_0(r^2)^\alpha &= 2\alpha r^{2(\alpha-1)}(2y^2 - z\bar{z}) \\
& & p_2(r^2)^\alpha &= 4\alpha zyr^{2(\alpha-1)} \\
& & p_4(r^2)^\alpha &= \alpha z^2r^{2(\alpha-1)} \\
&\text{(d)} & T_0\psi(m,\alpha) &= T_0(z^m)r^{2\alpha} + h(h-1)(z^m)p_0(r^2)^\alpha \\
& & &\quad - 3e^-(h-1)(z^m)p_2(r^2)^\alpha + 6(e^-)^2(z^m)p_4(r^2)^\alpha \\
& & &= -2m(m+1)(2m+3)z^mr^{2\alpha} \\
& & &\quad + 4\alpha(-2m(m+1)z^mr^{2\alpha} \\
& & &= -2m(m+1)(2m+4\alpha+3)\psi(m,\alpha)
\end{aligned}$$

Formulas (2.13), (2.14) and (2.15) d) are essentially equivalent to formulas of [Ho1], specialized to $p=3$. Note that in the case at hand, the operator T_0 plays no role in determining reducibility, since all the K-types appear with multiplicity 1. However, it does prevent any GL_3-constituents other than the trivial representation from being unitary when the parameter β is off the unitary axis (which is given by $Re\ \beta = -3/2$). In other situations, including the full principal series, to be discussed in the next section, T_0 is the most interesting of the transition operators. (See also [HL], §5.)

We want to adapt these formulas also to the situation of $GL_3(\mathbb{R})$ acting on the dual space $(\mathbb{R}^3)^\star$ of $\mathbb{R}^3$. For dual variables on $(\mathbb{R}^3)^\star$ we will put tildes on the variables for $\mathbb{R}^3$. The action of $\mathfrak{gl}_3$ will appear as the negative transpose of the action on the original variables. Thus, the elements of $\mathfrak{so}_3$ will operate in just the same way as they did above, but the elements of $\mathfrak{p}$ will become the negatives of what they were. Thus, the formulas (2.13), (2.14) and (2.15) d) will only change by a sign.

We will write all this explicitly. The variables on $(\mathbb{R}^3)^\star$ will be $\tilde{x}_j, j = 1,2,3$. The action of $\mathfrak{gl}_3$ is then given by the operators

$$\tilde{E}_{ab} = -\tilde{x}_b D_{\tilde{x}_a}. \tag{2.16}$$

We introduce variables

$$\tilde{z} = \tilde{x}_1 + i\tilde{x}_2 \qquad \bar{\tilde{z}} = \tilde{x}_1 - i\tilde{x}_2 \qquad \tilde{y} = \tilde{x}_3.$$

With respect to these variables, the operators of $\mathfrak{so}_3$ and of $\mathfrak{p}$ are given by

(2.17)

$$h = 2(\tilde{z}D_{\tilde{z}} - \bar{\tilde{z}}D_{\bar{\tilde{z}}}), \qquad e^+ = \tilde{z}D_{\tilde{y}} - 2\tilde{y}D_{\bar{\tilde{z}}}, \qquad e^- = -(\bar{\tilde{z}}D_{\tilde{y}} - 2\tilde{y}D_{\tilde{z}})$$

and

(2.18)

$$\begin{aligned} &p_4 = -\tilde{z}D_{\bar{\tilde{z}}} \qquad p_2 = -(\tilde{z}D_{\tilde{y}} + 2\tilde{y}D_{\bar{\tilde{z}}}). \\ &p_0 = \tilde{z}D_{\tilde{z}} + \bar{\tilde{z}}D_{\bar{\tilde{z}}} - 2\tilde{y}D_{\tilde{y}} \qquad p_{-2} = \bar{\tilde{z}}D_{\tilde{y}} + 2\tilde{y}D_{\tilde{z}}, \qquad p_{-4} = -\bar{\tilde{z}}D_{\tilde{z}} \end{aligned}$$

The highest weight vectors are

$$\tilde{\psi}(m,\alpha) = \tilde{z}^m\tilde{r}^{2\alpha} \tag{2.19}$$

where of course

$$\tilde{r}^2 = \tilde{z}\bar{\tilde{z}} + \tilde{y}^2.$$

Evidently the effect of the operators T_j on the $\tilde{\psi}(m,\alpha)$ is the same as on the $\psi(m,\alpha)$, except for a sign (i.e., a factor of -1).

§3. The full principal series

We now combine the calculations of the preceding section to analyze the full principal series. Specifically, we consider the action of GL_3 on functions in all six variables z, z, y, and $\tilde{z}, \bar{\tilde{z}}, \tilde{y}$. The operators of $\mathfrak{gl}_3$ are then

$$
\begin{aligned}
&(3.1)\ a) && h = 2(zD_z + \tilde{z}D_{\tilde{z}} - \bar{z}D_{\bar{z}} - \bar{\tilde{z}}D_{\bar{\tilde{z}}}) \\
& && e^+ = zD_y + \tilde{z}D_{\tilde{y}} - 2yD_{\bar{z}} - 2\tilde{y}D_{\bar{\tilde{z}}} \\
& && e^- = -(\bar{z}D_y + \bar{\tilde{z}}D_{\tilde{y}} - 2yD_z - 2\tilde{y}D_{\tilde{z}}) \\
&\quad b) && p_4 = zD_{\bar{z}} - \tilde{z}D_{\bar{\tilde{z}}} \quad p_2 = zD_y - \tilde{z}D_{\tilde{y}} + 2yD_{\tilde{z}} - 2\tilde{y}D_{\bar{\tilde{z}}} \\
& && p_0 = -zD_z + \tilde{z}D_{\tilde{z}} - \bar{z}D_{\bar{z}} + \bar{\tilde{z}}D_{\bar{\tilde{z}}} + 2yD_y - 2\tilde{y}D_{\tilde{y}} \\
& && p_{-2} = -\bar{z}D_y + \bar{\tilde{z}}D_{\tilde{y}} - 2yD_z + 2\tilde{y}D_{\tilde{z}} \quad p_{-4} = \bar{z}D_z - \bar{\tilde{z}}D_{\tilde{z}}.
\end{aligned}
$$

The action of O_3 is just the diagonal action of O_3 on two copies of $\mathbb{R}^3$. Thus, according to Classical Invariant Theory (see, e.g., [Ho2] and the references therein), we know that the algebra of polynomials invariant under O_3 is generated by the three quadratic polynomials

$$
\begin{aligned}
(3.2) \qquad & r^2 = z\bar{z} + y^2 \quad \tilde{r}^2 = \tilde{z}\bar{\tilde{z}} + \tilde{y}^2 \text{ and} \\
& r\tilde{r} = \frac{1}{2}(z\bar{\tilde{z}} + \bar{z}\tilde{z}) + y\tilde{y} = x_1\tilde{x}_1 + x_2\tilde{x}_2 + x_3\tilde{x}_3
\end{aligned}
$$

The third polynomial $r\tilde{r}$ is in fact invariant under all of $GL_3(\mathbb{R})$, as one can check directly from formulas (3.1) and (3.2).

The version of the FFT of Classical Invariant Theory describing covariants [Ho2] says that the highest weight vectors for $\mathfrak{so}_3$ of a given weight $2m$ are sums of polynomials of the form

$$
(3.3) \qquad z^a \tilde{z}^b \begin{vmatrix} z & y \\ \tilde{z} & \tilde{y} \end{vmatrix}^{\varepsilon} Q(r^2, \tilde{r}^2, r\tilde{r})
$$

where Q is an arbitrary polynomial in three variables and

$$
a + b + \varepsilon = m.
$$

Here a and b are non-negative integers, and $\varepsilon = 0$ or 1.

Consider the subvariety $\mathcal{N}$ of $\mathbb{R}^3 \oplus (\mathbb{R}^3)^\star$ defined by setting $r\tilde{r}$ equal to zero. For a point $p = \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} \right)$ to be in $\mathcal{N}$ means that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in the plane of vectors $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ defined by $v_1\tilde{x}_1 + v_2\tilde{x}_2 + v_3\tilde{x}_3 = 0$. Thus, an element of $GL_3(\mathbb{R})$ which fixes p will stabilize the line through

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and the plane orthogonal to $\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$ (which contains the line through $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$). Thus, the stabilizer of p is contained in a Borel subgroup of $GL_3(\mathbb{R})$ and contains the unipotent radical of the Borel. It follows that the space $\mathcal{S}^{d_1,d_2}(\mathcal{N})$ of functions on $\mathcal{N}$ which are homogeneous, of degrees d_1 and d_2 respectively, under dilations of $\mathbb{R}^3$ and under dilations of $(\mathbb{R}^3)^\star$, can be identified to a principal series representation of $GL_3(\mathbb{R})$. Furthermore, as d_1 and d_2 vary in $\mathbb{C}$, we obtain all possible principal series of $GL_3(\mathbb{R})$, up to twisting by a character. Thus, we will study the spaces $\mathcal{S}^{d_1,d_2}(\mathcal{N})$. Here we should note that the numbers d_1 and d_2 specify how functions behave under dilations of $\mathbb{R}^3$ or $(\mathbb{R}^3)^\star$ by <u>positive</u> numbers; they can also either change sign or remain fixed under dilation by (-1) on each space. Thus $\mathcal{S}^{d_1,d_2}(\mathcal{N})$ is actually a sum of four $GL_3(\mathbb{R})$-invariant subspaces, namely the eigenspaces under reflections through the origin in $\mathbb{R}^3$ and in $(\mathbb{R}^3)^\star$. This will guarantee that certain transitions are automatically zero, as in the case of formula (2.14).

If we require that $r\tilde{r} = 0$, and demand homogeneity on each of $\mathbb{R}^3$ and $(\mathbb{R}^3)^\star$, the functions (3.3) become limited to the cases

$$\psi(a,b,\varepsilon,\alpha,\beta) = z^a \tilde{z}^b \begin{vmatrix} z & y \\ \tilde{z} & \tilde{y} \end{vmatrix}^\varepsilon r^{2\alpha}\tilde{r}^{2\beta}, \tag{3.4}$$

where

$$a+\varepsilon+2\alpha = d_1 \qquad \text{and} \qquad b+\varepsilon+2\beta = d_2$$

are the degrees of homogeneity. Thus, our main goal is to describe the effect of the transition operators $T_j, j = 4,2,0$, on the functions $\psi(a,b,\varepsilon,\alpha,\beta)$.

The effect of $T_4 = p_4$ is easily computed. We verify that $T_4(z) = T_4(\tilde{z}) = T_4\left(\begin{vmatrix} z & y \\ \tilde{z} & \tilde{y} \end{vmatrix}\right) = 0$, while $T_4(r^2) = z^2$ and $T_4(\tilde{r}^2) = -\tilde{z}^2$. From these formulas we quickly deduce that

$$T_4\psi(a,b,\varepsilon,\alpha,\tilde{\alpha}) = \alpha\psi(a+2,b,\varepsilon,\alpha-1,\beta) - \beta\psi(a,b+2,\varepsilon,\alpha,\beta-1). \tag{3.5}$$

To find the result of applying T_2 or T_0 to $\psi(a,b,\varepsilon,\alpha,\beta)$ requires considerably more effort than needed for T_4. We state the answer here, and defer details of the calculation to §4.

(3.6)

(a) $$T_2(\psi(a,b,0,\alpha,\beta) = -8b\alpha\psi(a+1,b-1,1,\alpha-1,\beta) - 8a\beta\psi(a-1,b+1,1,\alpha,\beta-1)$$

(b) $$T_2(\psi(a,b,1,\alpha,\beta) = c_2\psi(a+1,b+1,0,\alpha,\beta) - 8b\alpha\psi(a+3,b-1,0,\alpha-1,\beta+1) - 8a\beta\psi(a-1,b+3,0,\alpha+1,\beta-1)$$

Here the coefficient c_2 is a function of a, b, a and β. Similarly, the coefficient c_0 below in formula (3.7) depends on $a, b, \varepsilon, \alpha$ and β.

(3.7)
$$T_0(\psi(a,b,\varepsilon,\alpha,\beta) = c_0\psi(a,b,\varepsilon,\alpha,\beta) + 24\alpha b(b-1)\psi(a+2,b-2,\varepsilon,\alpha-1,\beta+1) - 24\beta a(a-1)\psi(a-2,b+2,\varepsilon,\alpha+1,\beta-1).$$

To interpret these formulas, we associate to the highest weight vector $\psi(a,b,\varepsilon,\alpha,\beta)$ the point $\begin{bmatrix} a+\varepsilon \\ b+\varepsilon \\ \varepsilon \end{bmatrix}$ in $\mathbb{R}^3$. These points then fill out two quarter planes of lattice points – the positive quarter plane in the (x,y)-plane, and the quarter plane at height one above it, and shifted by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ – that is, the points of the upper layer must have both of their first two coordinates strictly positive. More formally, $\psi(a,b,\varepsilon,\alpha,\beta)$ are labeled by the set

$$\mathcal{L} = \left\{ \begin{bmatrix} p \\ q \\ r \end{bmatrix} : r = 0 \text{ or } 1; p \in \mathbb{Z}; q \in \mathbb{Z}; p - r \geq 0 \leq q - r \right\} \tag{3.8}$$

As we have mentioned, for a given bidegree (d_1, d_2) of homogeneity under $(\mathbb{R}^{+\times})^2$, a highest weight vector $\psi(a,b,\varepsilon,\alpha,\beta)$ may belong to one of four eigenspaces for the dilations by -1 of $\mathbb{R}^3$ and of $(\mathbb{R}^3)^\star$. It is not hard to check that $\psi(a,b,\varepsilon,\alpha,\beta)$ transforms by $(-1)^{a+\varepsilon}$ and $(-1)^{b+\varepsilon}$ respectively. Thus, the $\psi(a,b,\varepsilon,\alpha,\beta)$ belonging to a given full homogeneity class (e.g. a homogeneity class for $(\mathbb{R}^\times)^2$, not simply $(\mathbb{R}^{+\times})^2$) correspond to points $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} a+\varepsilon \\ b+\varepsilon \\ \varepsilon \end{bmatrix}$ such that p and q belong to a fixed congruence class modulo 2. In particular, both points $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$, with $r = 0$ or 1, label functions in the same full homogeneity class. Note that the h-weight of the eigenvector labeled by $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$ is $2(p+q-r)$.

Let us relabel

$$\psi(a, b, \varepsilon, \alpha, \beta) = \psi'(p, q, r, \alpha, \beta) \tag{3.9}$$

with $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$ in the set $\mathcal{L}$ of (3.8). Then the formulas (3.5) to (3.7) read

(3.10)

$$\begin{aligned} T_4(\psi'(p,q,r,\alpha,\beta)) &= \alpha\psi'(p+2,q,r,\alpha-1,\beta) \\ &\quad - \beta\psi'(p,q+2,r,\alpha,\beta-1) \\ T_2(\psi'(p,q,0,\alpha,\beta) &= \quad -8q\alpha\psi'(p+2,q,1,\alpha-1,\beta) \\ &\quad - 8p\beta\psi'(p,q+2,1,\alpha,\beta-1) \\ T_2(\psi'(p,q,1,\alpha,\beta) &= c_2\psi'(p,q,0,\alpha,\beta) \\ &\quad - 8(q-1)\alpha\psi'(p+2,q-2,0,\alpha-1,\beta+1) \\ &\quad - 8(p-1)\beta\psi'(p-2,q+2,0,\alpha+1,\beta-1) \\ T_0(\psi'(p,q,r,\alpha,\beta) &= c_0\psi'(p,q,r,\alpha,\beta) \\ &\quad + 24\alpha(q-r)(q-r-1)\psi'(p+2,q-2,r,\alpha-1,\beta+1) \\ &\quad - 24\beta(p-r)(p-r-1)\psi'(p-2,q+2,r,\alpha-1,\beta+1) \end{aligned}$$

Let us also note that the function $\psi'(p,q,r,\alpha,\beta)$ belongs to the space $\mathcal{S}^{d_1,d_2}(\mathcal{N})$, where

$$d_1 = p + 2\alpha \qquad d_2 = q + 2\beta. \tag{3.11}$$

Let us say that T_j *effects a transition* from $\psi'(p,q,r,\alpha,\beta)$ to $\psi'(p_1,q_1,r_1,\alpha_1,\beta_1)$ if $\psi'(p_1,q_1,r_1,\alpha_1,\beta_1)$ appears in the expression for $T_j(\psi'(p,q,r,\alpha,\beta))$ given in formula (3.10). By inspection of these formulas, we see that whenever T_j effects a transition from $\psi'(p,q,r,\alpha,\beta)$

to $\psi'(p_1, q_1, r_1, \alpha_1, \beta_1)$ with $p_1 > p$, the following facts hold true.

(3.12)

(i) $p_1 = p + 2$.

(ii) $\alpha_1 = \alpha - 1$.

(iii) α is a factor of the coefficient of $\psi'(p_1, q_1, r_1, \alpha_1, \beta_1)$ in the expression (3.10) for $T_j(\psi'(p, q, r, \alpha, \beta))$.

From (3.11) and (3.12), we see that, if d_1 is a non-negative integer, and $p = d_1$, so that $\alpha = 0$, then all transitions from $\psi'(d_1, q, r, \alpha, \beta)$ to $\psi'(p_1, q_1, r_1, \alpha_1, \beta_1)$ with $p_1 > d_1$ vanish. It then further follows from condition (3.12) i) that the span of the $\psi'(p, q, r, \alpha, \beta)$ that belong to $\mathcal{S}^{d_1,d_2}(\mathcal{N})$ (that is, such that relations (3.11) hold), and such that $p \leq d_1$, and $p \equiv d_1(\text{mod}2)$, form a GL_3 submodule of $\mathcal{S}^{(d_1,d_2)}(\mathcal{N})$. This is the main part of our main result, which we now state.

(3.13) Theorem. *For (d_1, d_2) in $\mathbb{C}^2$, consider the family of vectors*

$$\mathcal{B}_{(d_1,d_2)} = \left\{ \psi'(p, q, r, \alpha, \beta) : \begin{bmatrix} p \\ q \\ r \end{bmatrix} \in \mathcal{L}; p, q, \alpha, \beta \textit{ satisfy (3.11)} \right\}.$$

The families $\mathcal{B}_{(d_1,d_2)}$ have the following properties.

a) *$\mathcal{B}_{(d_1,d_2)}$ is a basis for the $\mathfrak{so}_3$-highest weight vectors in $\mathcal{S}^{(d_1,d_2)}(\mathcal{N})$.*

b) *For any (d_1', d_2') in $\mathbb{C}^2$,*

$$\mathcal{B}_{(d_1',d_2')} = r^{2\gamma}\bar{r}^{2\delta}\mathcal{B}_{(d_1,d_2)},$$

where

$$\begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} d_1' \\ d_2' \end{bmatrix} - \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}\right).$$

c) *i) If d_1 is a non-negative integer, then the subset*

$$\mathcal{B}_{(\leq d_1,d_2)} = \left\{ \psi'_{(p,q,r,\alpha,\beta)} \in \mathcal{B}_{(d_1,d_2)} : p \leq d_1, p \equiv d_1(\textit{mod } 2) \right\}$$

is a basis for the $\mathfrak{so}_3$-highest weight vectors in a GL_3-submodule $\mathcal{S}^{(\leq d_1,d_2)}(\mathcal{N})$ of $\mathcal{S}^{(d_1,d_2)}(\mathcal{N})$.

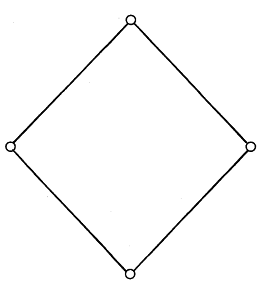
ii) *The analogous statement holds also for d_2.*
iii) *If d_1 and d_2 are both non-negative integers, then $\mathcal{B}_{(\leq d_1,d_2)} \cap \mathcal{B}_{(d_1,\leq d_2)}$ is a basis of $\mathfrak{so}_3$-highest weight vectors for the finite-dimensional GL_3-sub-module $\mathcal{S}^{(\leq d_1,d_2)} \cap \mathcal{S}^{(d_1,\leq d_2)}$ of $\mathcal{S}^{(d_1,d_2)}$.*

Furthermore, the properties a), b), and c) above uniquely characterize the vectors $\psi'(p,q,r,\alpha,\beta)$, up to scalar multiples.

Proof:. Statement a) is essentially the outcome of the discussion leading to the definition of the $\psi'(p,q,r,\alpha,\beta)$ (or their equivalents, the $\psi(a,b,\varepsilon,\alpha,\beta)$ of equation (3.4)). Statement b) is obvious from the definition of the $\psi'(p,q,r,\alpha,\beta)$. Statement c) summarizes the discussion leading up to the statement of the theorem.

It remains only to justify the final claim, that properties a), b) and c) uniquely specify the $\psi'(p,q,r,\alpha,\beta)$ up to scalar multiples. This may be seen by observing that, when d_1 and d_2 are both non-negative integers the GL_3-module $\mathcal{S}^{(\leq d_1,d_2)} \cap \mathcal{S}^{(d_1,\leq d_2)}$ contains unique (up to multiples) $\mathfrak{so}_3$-highest weight vectors of weights $2(d_1+d_2)$ and $2(d_1+d_2-1)$, and these are equal to $\psi'(d_1,d_2,r,0,0)$ for $r=0,1$. Thus any basis satisfying properties a), b) and c) must contain $\psi'(p,q,r,\alpha,\beta)$, up to multiples, for all $\begin{bmatrix} p \\ q \\ r \end{bmatrix}$ in $\mathcal{L}$, and all $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ in $\mathbb{C}^2$.

(3.14) Remarks: a) Theorem 3.13 shows that our bases $\mathcal{B}_{(d_1,d_2)}$ are adapted to display the submodule structure of $\mathcal{S}^{(d_1,d_2)}(\mathcal{N})$ whenever d_1 or d_2 is integral, and they are characterized by this property. In particular, when d_1 and d_2 are both non-negative integers, the submodules defined by $\mathcal{B}_{(\leq d_1,d_2)}$ and $\mathcal{B}_{(d_1,\leq d_2)}$ give rise to four submodules with Hasse diagram



inside the appropriate full homogeneity class in $\mathcal{S}^{(d_1,d_2)}(\mathcal{N})$. The bottom node corresponds to the finite-dimensional GL_3 submodule of

$\mathcal{S}^{(d_1,d_2)}(\mathcal{N})$. According to [Sp] (see also [Fo]), this is the full composition series for this particular principal series representation.

b) On the other hand, if neither d_1 nor d_2 is integral, but $d_1 - d_2$ is integral, then at least one of the full homogeneity classes in $\mathcal{S}^{(d_1,d_2)}(\mathcal{N})$ is reducible, but the $\mathfrak{so}_3$-highest weight vectors in the submodule are not spanned by a subset of $\mathcal{B}_{(d_1,d_2)}$. Thus, there can be no single basis for the $\mathfrak{so}_3$-highest weight vectors which reflects all submodule structure of the principal series of GL_3. The existence of bases adapted to these other submodules of the principal series is a question which we hope to investigate further.

c) Implicit in, and very material to, the calculations presented here, is the branching rule from GL_3 to O_3, describing how an irreducible finite-dimensional representation of GL_3 decomposes as an O_3-module. The algebra $\mathcal{R}(\mathcal{N})$ of regular functions on the variety $\mathcal{N}$ contains one copy of each irreducible representation of GL_3 with a highest weight vector of the form $\begin{bmatrix} d_1 \\ 0 \\ -d_2 \end{bmatrix}$, with d_1 and d_2 non-negative integers. More precisely, the irreducible representation with this highest weight is $\mathcal{R}(\mathcal{N}) \cap \mathcal{S}^{(d_1,d_2)}(\mathcal{N})$ – that is, it consists of the subspace of $\mathcal{R}(\mathcal{N})$ satisfying appropriate homogeneity conditions.

Up to twisting by powers of the determinant character of GL_3, the representations of GL_3 occurring in $\mathcal{R}(\mathcal{N})$ constitute all finite dimensional irreducible representations. Therefore, to understand how an irreducible representation of GL_3 decomposes when restricted to O_3, it suffices to describe the subalgebra $\mathcal{R}(\mathcal{N})^{e^+}$ of functions annihilated by the operator e^+ of formula (3.1) – the algebra of $\mathfrak{so}_3$-highest weight vectors in $\mathcal{R}(\mathcal{N})$. Of course, our description should include the information given by the bigrading of $\mathcal{R}(\mathcal{N})$ resulting from the fact that $\mathcal{N}$ is invariant under dilations both of $\mathbb{R}^3$ and of $(\mathbb{R}^3)^\star$.

The argument which yields formula (3.4) implies that $\mathcal{R}(\mathcal{N})$ is generated by the functions

$$z, \tilde{z}, r^2, \tilde{r}^2, \delta.$$

Here δ is given by formula (4.1).

Since each of these functions is bi-homogeneous, with degrees $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively, the monomials in them will also be bi-homogeneous. As for the relations between these generators, we know how to express δ^2 in terms of $z, \tilde{z}, r^2$ and $\tilde{r}^2$ (formula (4.8)). On the other hand, it is not difficult to check that these four generators are algebraically independent. (For example, by consideration of bidegree and $\mathfrak{so}_3$-weight, one can see that the only possibility for a relation would be that $z^2\tilde{r}^2$ and $\tilde{z}^2r^2$ be multiples of each other; on the other hand, this is easily checked not to be so). Hence $\mathcal{R}(\mathcal{N})^{e^+}$ is generated by the five functions used to create the bases $\mathcal{B}_{(d_1,d_2)}$, subject to the single relation (4.8). From this, the computation of the O_3 structure of an irreducible GL_3 representation is a straightforward matter. (This result is of course well-known, see for example [Zl].)

§4. Calculation of T_2 and T_0

For purposes of calculating T_2 and T_0 applied to the functions $\psi(a,b,\varepsilon,\alpha,\beta)$ of formula (3.4), we first calculate simpler operators applied to simpler functions. The following formulas each take only a few lines of computation to verify. To simplify our notation very slightly, we put

$$\begin{vmatrix} z & y \\ \tilde{z} & \tilde{y} \end{vmatrix} = \delta \tag{4.1}$$

We then calculate

(4.2)

$$\begin{array}{lll}
\text{a)} & h(z^a\tilde{z}^b) & = 2(a+b)z^a\tilde{z}^b \qquad h(\delta) = 2\delta \quad h(r^{2\alpha}\tilde{r}^{2\beta}) = 0 \\
\text{b)} & e^-(z^a\tilde{z}^b) & = 2(ay\tilde{z} + b\tilde{y}z)z^{a-1}\tilde{z}^{b-1} \\
 & e^-(\delta) & = -\begin{vmatrix} z & \bar{z} \\ \tilde{z} & \bar{\tilde{z}} \end{vmatrix} \qquad e^-(r^{2\alpha}\tilde{r}^{2\beta}) = 0 \\
\text{c)} & e^{-2}(z^a\tilde{z}^b) & = (4a(a-1)y^2\tilde{z}^2 + 4b(b-1)\tilde{y}^2\tilde{z} + 8aby\tilde{y}z\tilde{z} \\
 & & \quad -2a\bar{z}z\tilde{z}^2 - 2b\bar{\tilde{z}}z^2\tilde{z})z^{a-2}\tilde{z}^{b-2} \\
 & e^{-2}\delta & = -2\begin{vmatrix} y & \bar{z} \\ \tilde{y} & \bar{\tilde{z}} \end{vmatrix} \\
\text{d)} & p_4(z^a\tilde{z}^b) & = 0 \qquad p_4(\delta) = 0 \\
 & p_4(r^{2\alpha}\tilde{r}^{2\beta}) & = (\alpha z^2\tilde{r}^2 - \beta\tilde{z}^2r^2)r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)} \\
\text{e)} & p_2(z^a\tilde{z}^b) & = 0 \qquad p_2(\delta) = -2z\tilde{z} \\
 & p_2(r^{2\alpha}\tilde{r}^{2\beta}) & = (4\alpha zy\tilde{r}^2 - 4\beta\tilde{z}\tilde{y}r^2)r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)} \\
\text{f)} & p_0(z^a\tilde{z}^b) & = (-a+b)z^a\tilde{z}^b \qquad p_0(\delta) = -3(z\tilde{y} + \tilde{z}y) \\
 & p_0(r^{2\alpha}\tilde{r}^{2\beta}) & = 2(\alpha(-z\bar{z} + 2y^2)\tilde{r}^2 + \beta(\tilde{z}\bar{\tilde{z}} - 2\tilde{y}^2)r^2)r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)}
\end{array}$$

To compute T_2 and T_0 acting on $\psi(a,b,\varepsilon,\alpha,\beta)$ we begin with the facts, easily verified using (4.2) a), b), and e), that

$$T_2(z^a\tilde{z}^b) = 0 \text{ and } T_2(r^{2\alpha}\tilde{r}^{2\beta}) = 0.$$

We combine these by using the identity

(4.3)
$$\begin{aligned} &T_2(\phi\eta) = T_2(\phi)\eta + \phi T_2(\eta) \\ &+ p_2(\phi)h(\eta) + h(\phi)p_2(\eta) - 4p_4(\phi)e^-(\eta) - 4e^-(\phi)p_4(\eta) \end{aligned}$$

for any two functions ϕ and η. If we take $\phi = z^a\tilde{z}^b$ and $\eta = r^{2\alpha}\tilde{r}^{2\beta}$, then using (4.2) and the vanishing of $T_2(\phi)$ and $T_2(\eta)$, we see that (4.3) reduces to

$$T_2(\psi(a,b,0,\alpha,\beta)) = h(z^a\tilde{z}^b)p_2(r^{2\alpha}r^{2\beta}) - 4e^-(z^a\tilde{z}^b)p_4(r^{2\alpha}\tilde{r}^{2\beta})$$

Again using (4.2), we can readily deduce that

(4.4)
$$\begin{aligned} T_2(\psi(a,b,0,\alpha,\beta)) = &-8b\alpha\psi(a+1,b-1,1,\alpha-1,\beta) \\ &- 8a\beta\psi(a-1,b+1,1,\alpha,\beta-1) \end{aligned}$$

To compute $T_2\psi(a,b,1,\alpha,\beta)$, we proceed similarly. First we verify that

(4.5)
$$T_2(\delta) = -12z\tilde{z}$$

Next, applying (4.3) with $\phi = z^a\tilde{z}^b$ and $\eta = \delta$, we get

(4.6)
$$\begin{aligned} T_2(\psi(a,b,1,0,0)) &= z^a\tilde{z}^bT_2(\delta) + h(z^a\tilde{z}^b)p_2(\delta) \\ &= -12z^{a+1}\tilde{z}^{b+1} + 2(a+b)z^a\tilde{z}^b(-2z\tilde{z}) \\ &= -4(a+b+3)\psi(a+1,b+1,0,0,0). \end{aligned}$$

We do another partial calculation with $\phi = \delta$ and $\eta = r^{2\alpha}\tilde{r}^{2\beta}$. This gives

(4.7)

$$\begin{aligned}
T_2(\psi(0,0,1,\alpha,\beta)) &= T_2(\delta)r^{2\alpha}\tilde{r}^{2\beta} + h(\delta)p_2(r^{2\alpha}\tilde{r}^{2\beta}) - 4e^-(\delta)p_4(r^{2\alpha}\tilde{r}^{2\beta}) \\
&= -12\psi(1,1,0,\alpha,\beta) \\
&\quad + 2(z\tilde{y} - \tilde{z}y)4(\alpha zy\tilde{r}^2 - \beta\tilde{z}\tilde{y}r^2)r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)} \\
&\quad - 4(-z\bar{\tilde{z}} + \bar{z}\tilde{z})(\alpha z^2\tilde{r}^2 - \beta\tilde{z}^2r^2)r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)} \\
&= -12\psi(1,1,0,\alpha,\beta) \\
&\quad + 4r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)}\big[\alpha\tilde{r}^2(2z^2y\tilde{y} - 2z\tilde{z}y^2 + z^3\bar{\tilde{z}} - z^2\bar{z}\tilde{z}) \\
&\qquad\qquad -\beta r^2(2z\tilde{z}\tilde{y}^2 - 2\tilde{z}^2y\tilde{y} + z\tilde{z}^2\bar{\tilde{z}} - \bar{z}\tilde{z}^3)\big] \\
&= -12\psi(1,1,0,\alpha,\beta) \\
&\quad + 4r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)}\big[\alpha\tilde{r}^2(2z^2(r\tilde{r}) - 2z\tilde{z}r^2) \\
&\qquad\qquad -\beta r^2(-2\tilde{z}^2(r\tilde{r}) + 2z\tilde{z}\tilde{r}^2)\big] \\
&= -12\psi(1,1,0,\alpha,\beta) \\
&\quad + 4r^{2\alpha}\tilde{r}^{2\beta}(-2\alpha z\tilde{z} - 2\beta z\tilde{z}) \\
&\quad + 8(r\tilde{r})r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)}(2\alpha z^2\tilde{r}^2 + 2\beta\tilde{z}^2r^2) \\
&\equiv -4(3+2\alpha+2\beta)\psi(1,1,0,\alpha,\beta)(\mathrm{mod}\ r\tilde{r}).
\end{aligned}$$

Since we are only concerned with functions on the variety $\mathcal{N}$, defined by the vanishing of $r\tilde{r}$, this last expression tells us $T_2(\psi(0,0,1,\alpha,\beta))$.

We now combine the above partial calculations to calculate $T_2(\psi(a,b,1,\alpha,\beta))$ in general. We again use (4.3), this time with $\psi = z^a\tilde{z}^b\delta$ and $\eta = r^{2\alpha}\tilde{r}^{2\beta}$, and then we exploit the fact that h and e^- are derivations, so that they satisfy a product rule. We find

$$\begin{aligned}
T_2(\psi(a,b,1,\alpha,\beta)) &= T_2(z^a\tilde{z}^b\delta)r^{2\alpha}\tilde{r}^{2\beta} - h(z^a\tilde{z}^b\delta)p_2(r^{2\alpha}\tilde{r}^{2\beta}) \\
&\qquad - 4e^-(z^a\tilde{z}^b\delta)p_4(r^{2\alpha}\tilde{r}^{2\beta}) \\
&= -4(a+b+3)\psi(a+1,b+1,0,\alpha,\beta) \\
&\quad + (h(z^a\tilde{z}^b)\delta + z^a\tilde{z}^bh(\delta)p_2(r^{2\alpha}\tilde{r}^{2\beta}) \\
&\quad - 4(e^-(z^a\tilde{z}^b)\delta + z^a\tilde{z}^be^-(\delta))p_4(r^{2\alpha}\tilde{r}^{2\beta}) \\
&= -4(a+b+3)\psi(a+1,b+1,0,\alpha,\beta) \\
&\quad + \delta\left[h(z^a\tilde{z}^b)p_2(r^{2\alpha}\tilde{r}^{2\beta}) - 4e^-(z^a\tilde{z}^b)p_4(r^{2\alpha}\tilde{r}^{2\beta})\right] \\
&\quad + z^a\tilde{z}^b\left[h(\delta)p_2(r^{2\alpha}\tilde{r}^{2\beta}) - 4e^-(\delta)p_4(r^{2\alpha}\tilde{r}^{2\beta})\right].
\end{aligned}$$

From our partial calculations (4.4) and (4.7), we know the values of the expressions in square brackets. Substituting these gives

$$\begin{aligned} T_2(\psi(a,b,1,\alpha,\beta)) = &- 4(a+b+3)\psi(a+1,b+1,0,a,b) \\ &+ \delta\,[-8b\alpha\psi(a+1,b-1,1,\alpha-1,\beta) \\ &\qquad -8a\beta\psi(a-1,b+1,1,\alpha,\beta-1)] \\ &+ z^a\tilde{z}^b\,[-8(\alpha+\beta)\psi(1,1,0,\alpha,\beta)] \\ = &- 4(a+b+2\alpha+2\beta+3)\psi(a+1,b+1,0,\alpha,\beta) \\ &- 8\delta\,[b\alpha\psi(a+1,b-1,1,\alpha-1,\beta) \\ &\qquad +a\beta\psi(a-1,b+1,1,\alpha,\beta-1)]\,. \end{aligned}$$

The last term in this expression contains the factor δ^2. To finish evaluating it, we need to express δ^2 in terms of the generators $z, \tilde{z}, r^2, \tilde{r}^2, r\tilde{r}$ and δ. The desired expression is easily verified to be

$$\delta^2 = z^2\tilde{r}^2 + \tilde{z}^2r^2 - 2z\tilde{z}(r\tilde{r}) \tag{4.8}$$

Plugging this in to our formula above (and remembering that for our purposes $r\tilde{r} = 0$), we finally find that

(4.9)

$$\begin{aligned} T_2(\psi(a,b,1,\alpha,\beta)) = &- 4(a+b+2\alpha+2\beta+3)\psi(a+1,b+1,0,\alpha,\beta) \\ &- 8b\alpha\psi(a+3,b-1,0,\alpha-1,\beta+1) \\ &- 8a\beta\psi(a-1,b+3,0,\alpha+1,\beta-1) \\ &- 8(b\alpha+a\beta)\psi(a+1,b+1,0,\alpha,\beta) \\ = &- 4(a+b+2\alpha+2\beta+2(b\alpha+a\beta)+3) \\ &\quad \times \psi(a+1,b+1,0,\alpha,\beta) \\ &- 8b\alpha\psi(a+3,b-1,0,\alpha-1,\beta+1) \\ &- 8a\beta\psi(a-1,b+3,0,\alpha+1,\beta-1). \end{aligned}$$

We now turn to calculating the effect of T_0. We first verify that

$$T_0(z^a\tilde{z}^b) = -2(a-b)(2(a+b)+3)(a+b+1)z^a\tilde{z}^b \tag{4.10}$$

and

$$T_0(\delta) = 0. \tag{4.11}$$

Next, we record an identity analogous to (4.3). Again we consider a product $\phi\eta$. This time, we assume that η is O_3-invariant, so that it is annihilated by h and by e^-. Under this assumption, we may verify that

(4.12)

$$T_0(\phi\eta) = T_0(\phi)\eta + h(h-1)(\phi)p_0(\eta) \\ - 3e^-(h-1)(\phi)p_2(\eta) + 6e^{-2}(\phi)p_4(\eta).$$

Taking $\phi = z^a \tilde{z}^b$ and $\eta = r^{2\alpha}\tilde{r}^{2\beta}$, evaluating the last three terms in the right hand side of (4.12) using the formulas (4.2), combining terms appropriately, and remembering that $r\tilde{r} = 0$ on $\mathcal{N}$ we find the formula

$$
\begin{aligned}
T_0(\psi(a,b,0,\alpha,\beta)) &= -2(a-b)(2(a+b)+3)(a+b+1)\psi(a,b,0,\alpha,\beta) \\
&\quad + 4(a+b)(2(a+b)-1)[\alpha(-z\bar{z}+2y^2)\tilde{r}^2+\beta(\tilde{z}\bar{\tilde{z}}-2\tilde{y}^2)r^2] \\
&\quad \times z^a\tilde{z}^b r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)} \\
&\quad - 24(2(a+b)-1)(ay\tilde{z}+b\tilde{y}z)(\alpha zy\tilde{r}^2-\beta\tilde{z}\tilde{y}r^2) \\
&\quad \times z^{a-1}\tilde{z}^{b-1}r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)} \\
&\quad + 6[4a(a-1)y^2\tilde{z}^2+4b(b-1)\tilde{y}^2z^2+8aby\tilde{y}z\tilde{z}-2az\bar{z}\tilde{z}^2-2bz^2\tilde{z}\bar{\tilde{z}}] \\
&\quad \times (\alpha z^2\tilde{r}^2-\beta\tilde{z}^2r^2)z^{a-2}\tilde{z}^{b-2}r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)} \\
&= 2(b-a)(2(a+b)+3)(a+b+1)\psi(a,b,0,\alpha,\beta) \\
&\quad + 4\alpha z^{a-2}\tilde{z}^{b-2}r^{2(\alpha-1)}\tilde{r}^{2\beta} \\
&\quad \times \big[(a+b)(2(a+b)-1)(-z^3\bar{z}\tilde{z}^2+2z^2\tilde{z}^2y^2) \\
&\qquad - 6(2(a+b)-1)(az^2\tilde{z}^2y^2+bz^3\tilde{z}y\tilde{y}) \\
&\qquad + 3(2a(a-1)z^2\tilde{z}^2y^2+2b(b-1)z^4\tilde{y}^2 \\
&\qquad +4abz^3\tilde{z}y\tilde{y}-az^3\bar{z}\tilde{z}^2-bz^4\tilde{z}\bar{\tilde{z}})\big] \\
&\quad + 4\beta z^{a-2}\tilde{z}^{b-2}r^{2\alpha}\tilde{r}^{(\beta-1)} \\
&\quad \times \big[(a+b)(2(a+b)-1)(z^2\tilde{z}^3\bar{\tilde{z}}-2z^2\tilde{z}^2\tilde{y}^2) \\
&\qquad + 6(2(a+b)-1)(az\tilde{z}^3y\tilde{y}+bz^2\tilde{z}^2\tilde{y}^2) \\
&\qquad - 3(2a(a-1)\tilde{z}^4y^2+2b(b-1)z^2\tilde{z}^2\tilde{y}^2 \\
&\qquad +4abz\tilde{z}^3y\tilde{y}-az\bar{z}\tilde{z}^4-bz^2\tilde{z}^3\bar{\tilde{z}})\big] \\
&= 2(b-a)(2(a+b)+3)(a+b+1)\psi(a,b,0,\alpha,\beta) \\
&\quad + 4\alpha\psi(a-2,b-2,0,\alpha-1,\beta) \\
&\quad \times \big[-2a^2z^2\tilde{z}^2r^2-4abz^2\tilde{z}^2r^2 \\
&\qquad +b^2(4z^2\tilde{z}^2r^2+6z^4\tilde{r}^2-12z^3\tilde{z}(r\tilde{r})) \\
&\qquad -2az^2\tilde{z}^2r^2+b(-2z^2\tilde{z}^2r^2-6z^4\tilde{r}^2+6z^3\tilde{z}(r\tilde{r}))\big] \\
&\quad + 4\beta\psi(a-2,b-2,0,\alpha,\beta-1) \\
&\quad \times \big[a^2(-4z^2\tilde{z}^2\tilde{r}^2-6\tilde{z}^4r^2+12z\tilde{z}^3(r\tilde{r})) \\
&\qquad + 4abz^2\tilde{z}^2\tilde{r}^2+2b^2z^2\tilde{z}^2r^2 \\
&\qquad +a(6\tilde{z}^4r^2+2z^2\tilde{z}^2\tilde{r}^2-6z\tilde{z}^3(r\tilde{r}))+2bz^2\tilde{z}^2\tilde{r}^2\big]
\end{aligned}
$$

If we now collect all terms involving the same monomials in $z, \tilde{z}, r^2$ and $\tilde{r}^2$, and drop all terms with a factor of $r\tilde{r}$ in them, we arrive at the result

(4.13)
$$\begin{aligned} T_0((\psi(a,b,0,\alpha,\beta)) &= c_{00}(a,b,\alpha,\beta)\psi(a,b,0,\alpha,\beta) \\ &\quad + 24\alpha b(b-1)\psi(a+2,b-2,0,\alpha-1,\beta+1) \\ &\quad - 24\beta a(a-1)\psi(a-2,b+2,0,\alpha+1,\beta-1), \end{aligned}$$

where

$$\begin{aligned} c_{00} &= 2(b-a)(2(a+b)+3)(a+b+1) \\ &\quad - 8\alpha((a+b)(a+b+1)-3b^2) \\ &\quad + 8\beta((a+b)(a+b+1)-3a^2). \end{aligned}$$

We now take up the case of $T_0(\psi(a,b,1,\alpha,\beta))$. As with T_2, we first calculate $T_0(\psi(0,0,1,\alpha,\beta) = T_0(\delta r^{2\alpha}\tilde{r}^{2\beta})$. We can use (4.12) with $\phi = \delta$ and $\eta = r^{2\alpha}\tilde{r}^{2\beta}$. Using (4.11) and (4.2), we find that

(4.14)
$$\begin{aligned} T_0(\delta r^{2\alpha}\tilde{r}^{2\beta}) &= 2\delta p_0(r^{2\alpha}\tilde{r}^{2\beta}) - 3e^{-}(\delta)p_2(r^{2\alpha}\tilde{r}^{2\beta}) + 6e^{-2}(\delta)p_4(r^{2\alpha}\tilde{r}^{2\beta}) \\ &= 8(\alpha-\beta)\delta r^{2\alpha}\tilde{r}^{2\beta}. \end{aligned}$$

To apply formula (4.12), we also need to know $T_0(z^a z^b\delta)$. We calculate the terms

(4.15)
$$\begin{aligned} p_0(z^a\tilde{z}^b\delta) &= (b-a)z^a\tilde{z}^b\delta - 3z^a\tilde{z}^b(z\tilde{y}+\tilde{z}y) \\ e^{-}(z^a\tilde{z}^b\delta) &= 2ayz^{a-1}\tilde{z}^b\delta + 2b\tilde{y}z^a\tilde{z}^{b-1}\delta + z^a\tilde{z}^b(\bar{z}\tilde{z} - z\bar{\tilde{z}}) \\ p_2e^{-}(z^a\tilde{z}^b\delta) &= 2(a-b)z^a\tilde{z}^b\delta - 2(2a-1)z^a\tilde{z}^{b+1}y - 2(2b-1)bz^{a+1}\tilde{z}^b\tilde{y} \\ e^{-2}(z^a\tilde{z}^b\delta) &= \big[4a(a-1)y^2z^{a-2}\tilde{z}^b + 4b(b-1)\tilde{y}^2z^a\tilde{z}^{b-2} \\ &\qquad +8aby\tilde{y}z^{a-1}\tilde{z}^{b-1} - 2az\bar{z}z^{a-1}\tilde{z}^b - 2b\bar{\tilde{z}}z^a\tilde{z}^{b-1}\big]\,\delta \\ &\quad + 4(ayz^{a-1}\tilde{z}^b + b\tilde{y}z^a\tilde{z}^{b-1})(\bar{z}\tilde{z} - z\bar{\tilde{z}}) + 2z^a\tilde{z}^b(\bar{z}\tilde{y} - \bar{\tilde{z}}y) \\ p_4e^{-2}(z^a\tilde{z}^b\delta) &= 2(b-a)z^a\tilde{z}^b\delta + 2(4a+1)z^a\tilde{z}^{b+1}y + 2(4b+1)z^{a+1}\tilde{z}^b\tilde{y} \end{aligned}$$

Putting these formulas together gives

(4.16) $T_0(\psi(a,b,1,0,0)) = 2(b-a)(2a+2b+5)(a+b+5)\psi(a,b,1,0,0)$

Finally, to compute $T_0(\psi(a,b,1,\alpha,\beta))$, we use formula (4.12) with $\phi = z^a\tilde{z}^b\delta$ and $\eta = r^{2\alpha}\tilde{r}^{2\beta}$. We can give some useful extra structure to the calculations by observing that, if $\phi = \phi_1\phi_2$, then

(4.17)
$$\begin{aligned} h(h-1)(\phi_1\phi_2) &= (h(h-1)(\phi_1))\phi_2 + \phi_1(h(h-1)(\phi_2)) \\ &\quad + 2h(\phi_1)h(\phi_2) \\ e^-(h-1)(\phi_1\phi_2) &= (e^-(h-1)(\phi_1))\phi_2 + \phi_1(e^-(h-1)(\phi_2)) \\ &\quad e^-(\phi_1)h(\phi_2) + h(\phi_1)e^-(\phi_2) \\ e^{-2}(\phi_1\phi_2) &= e^{-2}(\phi_1)\phi_2 + \phi_1 e^{-2}(\phi_2) + 2e^-(\phi_1)e^-(\phi_2) \end{aligned}$$

Using these formulas with $\phi_1 = z^a\tilde{z}^b$, and $\phi_2 = \delta$, and using formula (4.12), we calculate that

(4.18)
$$\begin{aligned} &T_0(\psi(a,b,1,\alpha,\beta)) = T_0(\psi(a,b,1,0,0))r^{2\alpha}\tilde{r}^{2\beta} \\ &\quad +h(h-1)(z^a\tilde{z}^b\delta)p_0(r^{2\alpha}\tilde{r}^{2\beta}) - 3e^-(h-1)(z^a\tilde{z}^b\delta)p_2(r^{2\alpha}\tilde{r}^{2\beta}) \\ &\quad\ +6e^{-2}(z^a\tilde{z}^b\delta)p_4(r^{2\alpha}\tilde{r}^{2\beta}) \\ &= T_0(\psi(a,b,1,0,0))r^{2\alpha}\tilde{r}^{2\beta} \\ &\quad + \big[h(h-1)(z^a\tilde{z}^b)p_0(r^{2\alpha}\tilde{r}^{2\beta}) - 3e^-(h-1)(z^a\tilde{z}^b)p_2(r^{2\alpha}\tilde{r}^{2\beta}) \\ &\quad\ +6e^{-2}(z^a\tilde{z}^b)p_4(r^{2\alpha}\tilde{r}^{2\beta})\big]\delta \\ &\quad + \big[h(h-1)(\delta)p_0(r^{2\alpha}\tilde{r}^{2\beta}) - 3e^-(h-1)(\delta)p_2(r^{2\alpha}\tilde{r}^{2\beta}) \\ &\quad\ +6e^{-2}(\delta)p_4(r^{2\alpha}\tilde{r}^{2\beta})\big]z^a\tilde{z}^b \\ &\quad + \big[2h(z^a\tilde{z}^b)h(\delta)p_0(r^{2\alpha}\tilde{r}^{2\beta}) \\ &\quad\ -3(e^-(z^a\tilde{z}^b)h(\delta) + h(z^a\tilde{z}^b)e^-(\delta))p_2(r^{2\alpha}\tilde{r}^{2\beta}) \\ &\quad +12(e^-(z^a\tilde{z}^b)e^-(\delta)p_4(r^{2\alpha}\tilde{r}^{2\beta})\big] \\ &= T_0(\psi(a,b,1,0,0)r^{2\alpha}\tilde{r}^{2\beta} \\ &\quad +(T_0(\psi(a,b,0,\alpha,\beta)) - T_0(z^a\tilde{z}^b)r^{2\alpha}\tilde{r}^{2\beta})\delta \\ &\quad +(T_0(\psi(0,0,1,\alpha,\beta)) - T_0(\delta)r^{2\alpha}\tilde{r}^{2\beta})z^a\tilde{z}^b \\ &\quad + \big[2h(z^a\tilde{z}^b)h(\delta)p_0(r^{2\alpha}\tilde{r}^{2\beta}) \\ &\quad\ -3(e^-(z^a\tilde{z}^b)h(\delta) + h(z^a\tilde{z}^b)e^-(\delta))p_2(r^{2\alpha}\tilde{r}^{2\beta}) \\ &\quad\ + 12(e^-(z^a\tilde{z}^b)e^-(\delta)p_4(r^{2\alpha}\tilde{r}^{2\beta})\big] \end{aligned}$$

The first three terms in the final expression above are evaluated by formulas (4.16), (4.13) and (4.14) respectively. We now concentrate on evaluating the last term. Plugging in the formulas (4.2) into the last term gives

(4.19)

$$\begin{aligned}
&8(a+b)z^a\tilde{z}^b\cdot\delta\cdot 2\cdot\left[\alpha(-z\tilde{z}+2y^2)\tilde{r}^2+\beta(\tilde{z}\bar{\tilde{z}}-2\tilde{y}^2)r^2\right]r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)}\\
&\quad-3\left[2(a+b)z^a\tilde{z}^b(\bar{z}\tilde{z}-z\bar{\tilde{z}})+4(ay\tilde{z}+b\tilde{y}z)z^{a-1}\tilde{z}^{b-1}\delta\right]\\
&\qquad\times 4\cdot(\alpha zy\tilde{r}^2-\beta\tilde{z}\tilde{y}r^2)\\
&\quad+12\cdot 2(ay\tilde{z}+b\tilde{y}z)z^{a-1}\tilde{z}^{b-1}(\bar{z}\tilde{z}-z\bar{\tilde{z}})(\alpha z^2\tilde{r}^2-\beta\tilde{z}^2r^2)r^{2(\alpha-1)}\tilde{r}^{2(\beta-1)}.
\end{aligned}$$

Organizing this into terms involving α and terms involving β, canceling terms, combining terms, and remembering that $r\tilde{r}=0$, we can boil expression (4.19) down to

$$16(\alpha(2b-a)+\beta(b-2a))\psi(a,b,1,\alpha,\beta). \tag{4.20}$$

Finally, combining formula (4.20) with (4.16), (4.13) and (4.14), and plugging these into formula (4.18), we arrive at the equation

(4.21)

$$\begin{aligned}
T_0(\psi(a,b,1,\alpha,\beta)) &= [2(b-a)(2a+2b+5)(a+b+5)+8(\alpha-\beta)\\
&\quad-8\alpha((a+b)(a+b+1)-3b^2)\\
&\quad+\ 8\beta((a+b)(a+b+1)-3a^2)\\
&\quad+16(\alpha(2b-a)+\beta(b-2\alpha))]\\
&\qquad\times\psi(a,b,1,\alpha,\beta)\\
&\quad+24\alpha b(b-1)\psi(a+2,b-2,1,\alpha-1,\beta+1)\\
&\quad-24\beta a(a-1)\psi(a-2,b+2,1,\alpha+1,\beta-1)\\
&=[2(b-a)(2a+2b+5)(a+b+5)\\
&\quad+8\alpha(1-3a+3b-a^2-2ab+2b^2)\\
&\quad+8\beta(-1-3a+3b-2a^2+2ab+b^2)]\\
&\qquad\times\psi(a,b,1,\alpha,\beta)\\
&\quad+24\alpha b(b-1)\psi(a+2,b-2,1,\alpha-1,\beta+1)\\
&\quad-24\beta a(a-1)\psi(a-2,b+2,1,\alpha+1,\beta-1).
\end{aligned}$$

Acknowledgment: The author thanks the referee for a meticulous reading of the paper.

References

[Fo] A. Fomin, Quasisimple representations of $SL(3,\mathbb{R})$, Fun. Anal. & Appl., **9** (1976), 237–243.

[Fu] T. Fujimura, On Some Degenerate Principal Series Representations of $O(p,2)$, preprint.

[GN] I. Gelfand and M. Naimark, "Unitäre Darstellungen Klassischer Gruppen", Akademie Verlag, 1947.

[HC] Harish-Chandra, Representations of semisimple Lie groups, II, III, T.A.M.S., **76** (1954), 234–253.

[Ho1] R. Howe, Some simple examples in the representation theory of semisimple Lie groups, Jber.d. Dt. Math.-Verein., **96** (1995), 131–159.

[Ho2] R. Howe, Remarks on classical invariant theory, T.A.M.S., **313** (1989), 539–570.

[HL] R. Howe and S-T. Lee, Degenerate principal series representations of $GL_n(\mathbb{C})$ and $GL_n(\mathbb{R})$, J. Fun. Anal., **166** (1999), 244–309.

[HT] R. Howe and E-C Tan, Homogeneous function on light cones, B.A.M.S. (New Series), **28** (1993), 1–74.

[J] K. Johnson, Degenerate principal series on tube type domains, Contemp. Math., **138** (1992), 175–187.

[L1] S-T. Lee, On some degenerate principal series representations of $U(n,n)$, J. Fun. Anal., **126** (1994), 305–366.

[L2] ———, Degenerate principal series representations of $Sp(2n,\mathbb{R})$, Comp. Math., **103** (1996), 123–151.

[Sa] S. Sahi, Jordan algebras and degenerate principal series, J. reine angew. Math., **462** (1995), 1–18.

[Sp] B. Speh, The unitary dual of $GL(3,\mathbb{R})$ and $GL(4,\mathbb{R})$, Math. Ann., **258** (1981), 113–133.

[Za] G-K. Zhang, Jordan algebras and generalized principal series representations, Math. Ann., **302** (1995), 773–786.

[Ze] D. Zhelobenko, Compact Lie groups and their representations, Transl. Math. Mono **40**, American Mathematical Society, Providence, 1973.

Mathematics Department
Yale University
10 Hillhouse Avenue
P.O. Box 208283
New Haven, CT 06520-8283, U.S.A.