Advanced Studies in Pure Mathematics 22, 1993
Progress in Differential Geometry
pp. 309–319

Diameter and Area Estimates for S^2 and P^2 with Nonnegatively Curved Metrics

Takashi Shioya

§0. Introduction

We consider the quantity

$$F(M) := \frac{\mathrm{Vol}(M)}{\mathrm{Diam}(M)^n}$$

for any closed Riemannian n-manifold M, which is a homothety invariant, where Vol and Diam denote the volume and the diameter respectively. If the Ricci curvature of M is nonnegative everywhere, Bishop's volume comparison theorem implies that $F(M) < \pi$. A.D. Alexandrov conjectured in [A, p.417] (see also [BZ, p.42]) that for any nonnegatively curved metric g on the 2-sphere S^2,

$$F(S^2, g) \leq \frac{\pi}{2},$$

and the equality holds only if g is homothetic to the metric of the double of the Euclidean unit disk $\bar{B}(1) := \{\, x \in \mathbf{R}^2 \mid d(x, o) \leq 1 \,\}$, which is a singular metric of nonnegative Toponogov curvature. Note that Alexandrov deals a class of surfaces containing such a singular space, namely surfaces of bounded curvature in the sense of [AZ]. The volume and the diameter of any such singular surface can be approximated by those of Riemannian 2-manifolds, and thus it suffices to consider only regular metrics.

Alexandrov's conjecture has not been proved as of now. Concerning this, there are two known results as follows.

Theorem (Sakai, [S]). *For any nonnegatively curved Riemannian metric g on the 2-sphere S^2,*

$$F(S^2, g) < 0.985\pi.$$

Received January 28, 1991.
Revised April 28, 1991.

Theorem (Grove-Petersen, [GP1, Theorem B]). *For any integer $n \geq 2$ there exists an $\epsilon(n) > 0$ such that any compact Riemannian n-manifold M with nonnegative sectional curvature satisfies*

$$F(M) < V(n) - \epsilon(n),$$

where $V(n)$ is the volume of the n-dimensional Euclidean unit ball.

In the present paper, we try to extend the above estimates in the 2-dimensional case, i.e. the estimates for the 2-sphere S^2 and the real projective 2-space P^2 with nonnegatively curved metrics, and the 2-torus T^2 and the Klein bottle K^2 with flat metrics. We easily observe that, when $M = (T^2, g)$ or (K^2, g) for a flat metric g, then $F(M) \leq 2$, where the equality holds only if g is the canonical flat metric. Sakai's proof cannot be extended to the case of P^2. On the other hand, although Grove-Petersen's theorem is more general, their proof gives no calculable constant. Accordingly, we develop a proving method independent of the topology and have the following finer estimates.

Main Theorem. (1) *For any nonnegatively curved Riemannian metric g on the 2-sphere S^2, we have*

$$F(S^2, g) \leq \left(\frac{5}{2}\sqrt{10} - 7\right)\pi < 0.906\pi.$$

(2) *For any nonnegatively curved Riemannian metric g on the real projective 2-space P^2, we have*

$$F(P^2, g) \leq \frac{7\sqrt{7} - 10}{9}\pi < 0.947\pi.$$

Different from the case of S^2, the maximum of $F(P^2, g)$ for all nonnegatively curved metrics g on P^2 seems to be $F(P^2, g_c) = 8/\pi > 0.810\pi$, where g_c is the canonical metric on P^2, namely the metric of constant curvature 1.

§1. Preliminaries

Let M be a (not necessarily closed) complete Riemannian 2-manifold without boundary and p a fixed point in M. Consider the metric balls $B(p, r) := \{ x \in M \mid d(p, x) < r \}$ and the metric spheres $S(p, r) := \{ x \in M \mid d(p, x) = r \}$ centered at p for radii $r > 0$, where d denotes the distance function of M induced from the metric. Following Hartman [H] we define the notion of an exceptional radius as follows (actually, he called it an exceptional t-value).

Definition [H]. A radius $r > 0$ is said to be *exceptional* if and only if there exists a cut point q in $S(p, r)$ from p satisfying one of the following three conditions.

(1) q is a first conjugate point of p along some minimal geodesic segment joining p and q.

(2) There exist more than two distinct minimal geodesic segments joining p and q.

(3) There exist exactly two geodesic segments joining p and q, and moreover the angle between these segments at q is equal to π.

A radius is said to be *nonexceptional* if and only if it is not exceptional.

Note that if M is compact, $S(p, r)$ for any sufficiently large radius $r > 0$ is empty and hence any such r is nonexceptional. Hartman has proved in [H] that the set of all exceptional radii is a closed and measure zero subset of $\mathbf{R}$ and that $S(p, r)$ for each nonexceptional $r > 0$ consists of finitely many simple closed curves of class C^∞ except the cut points in $S(p, r)$ from p, the number of which is finite. For any nonexceptional $r > 0$ we denote by $q_{r,1}, \ldots, q_{r,n(r)}$ $(0 \le n(r) < +\infty)$ the cut points in $S(p, r)$ from p. Then $S(p, r) - \{q_{r,1}, \ldots, q_{r,n(r)}\}$ consists of $n(r)$ disjoint smooth open arcs $\alpha_{r,1}, \ldots, \alpha_{r,n(r)}$. Define a continuous function $\rho\colon M \to \mathbf{R} \cup \{+\infty\}$ by

$$\rho(x) := \sup_{y \in M} d(x, y) \qquad \text{for } x \in M.$$

Clearly, $\rho(x) = +\infty$ if and only if M is open. Denote by $F_{r,i}$ the set of interior points of the minimal segments joining p and all points in $\alpha_{r,i}$ for any nonexceptional $0 < r < \rho(p)$ and any $1 \le i \le n(r)$. Then, $F_{r,i}$ is the open disk bounded by the triangle whose sides are $\alpha_{r,i}$ and two minimal segments joining p and the endpoints of $\alpha_{r,i}$ provided $n(r) \ge 1$. Denote by $\kappa_{r,i}(u)$ the integral of the geodesic curvature of the arc $S(p, u) \cap F_{r,i}$ with respect to $B(p, u)$ for any nonexceptional u and r with $0 < u < r < \rho(p)$ and for any $i = 1, \ldots, n(r)$. Now we will prove

$$(*) \qquad \mathrm{Vol}(F_{r,i}) = \int_0^r \int_0^t \kappa_{r,i}(u)\, du\, dt.$$

Indeed, considering the geodesic polar coordinates (θ, t) on $F_{r,i}$ (θ is the angle at p and t is the distance from p), the volume of $F_{r,i}$ is expressed as

$$\mathrm{Vol}(F_{r,i}) = \int_0^r \int_0^{\Theta_{r,i}} \left\| \frac{\partial}{\partial \theta} \right\| d\theta\, dt,$$

where $\Theta_{r,i}$ is the inner angle of $F_{r,i}$ at p. Moreover, since the geodesic curvature of $S(p,t) \cap F_{r,i}$ with respect to $B(p,t)$ is equal to

$$\left\| \frac{\partial}{\partial \theta} \right\|^{-2} \left\langle \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \theta}, -\frac{\partial}{\partial t} \right\rangle = \frac{\partial}{\partial t} \left\| \frac{\partial}{\partial \theta} \right\|,$$

we have

$$\int_0^{\Theta_{r,i}} \left\| \frac{\partial}{\partial \theta} \right\| d\theta = \int_0^t \kappa_{r,i}(u)\, du.$$

This proves $(*)$.

In particular, if $\bar{B}(p,r) := B(p,r) \cup S(p,r)$ contains no cut points from p, we have

$$\mathrm{Vol}(B(p,r)) = \int_0^r \int_0^t \kappa(B(p,u))\, du\, dt,$$

where $\kappa(D)$ denotes the sum of the integral of the geodesic curvature of the boundary ∂D of D with respect to D and of the exterior angles at all vertices of D (we remark that $B(p,r)$ has no vertices in this case). Fiala [F] and Hartman [H] have extended this to the case where $B(p,r)$ may contain cut points from p as follows.

Lemma [F], [H]. *For any* $0 < r \leq \rho(p)$ *we have*

$$(**) \qquad \mathrm{Vol}(B(p,r)) = \int_0^r \int_0^t [\kappa(B(p,u)) - h_p(u)]\, du\, dt$$

where h_p *is the nonnegative function defined by*

$$h_p(u) := \sum_{i=1}^{n(u)} \left(2 \tan \frac{\varphi_{u,i}}{2} - \varphi_{u,i} \right)$$

and where $\varphi_{u,i}$ *for each nonexceptional* $0 < u < \rho(p)$ *denotes the angle at* $q_{u,i}$ *between the two minimal segments joining* p *and* $q_{u,i}$.

Note that Fiala and Hartman deal only with the case where $M = (\mathbf{R}^2, g)$ (Fiala [F] proved $(**)$ for manifolds with real analytic metrics and Hartman [H] later extended this to the case of manifolds with C^2-metrics). However, we observe that their discussions are independent of the topology of M (see [ST]).

§2. Some partial estimates

Assume that M is a nonnegatively curved Riemannian 2-manifold diffeomorphic to either S^2 or P^2 the diameter of which is normalized as $\mathrm{Diam}(M) = 1$. Every curve in M is assumed to have arclength parameter and is often identified with its image. For a while, let p be any fixed point in M.

First we state a basic topological lemma.

Lemma 1. *Let $0 < r < \rho(p)$ be any nonexceptional radius. Then the Euler characteristic $\chi(B(p,r))$ of $B(p,r)$ satisfies*

$$\chi(B(p,r)) \leq 1,$$

and the equality holds if and only if $B(p,r)$ is a disk.

Note that $B(p,r)$ for a nonexceptional $r > 0$ is a disk if and only if it is contractible.

Proof. Since $B(p,r)$ is not closed, the 2-dimensional homology $H_2(B(p,r), \mathbf{Z})$ vanishes, and the first Betti number $b_1(B(p,r))$ is equal to zero if and only if $B(p,r)$ is contractible, namely a disk. Moreover we have

$$\chi(B(p,r)) = 1 - b_1(B(p,r)).$$

This completes the proof. Q.E.D.

Remark. It follows from Lemma 1 and the Gauss-Bonnet theorem that

$$\kappa(B(p,r)) = 2\pi\chi(B(p,r)) - c(B(p,r)) \leq 2\pi$$

for any nonexceptional $0 \leq r < \rho(p)$, where $c(D)$ denotes the total curvature of D, namely the integral $\int_D K\,dv$ of Gaussian curvature K over D with respect to the volume element dv of M.

Applying $(**)$ to $B(q, \inf\rho)$ for a point q in M with $\rho(q) = \inf\rho$ and using the above remark, the following consequence is directly proved.

Proposition 2. *We have*

$$\mathrm{Vol}(M) \leq \pi \cdot (\inf\rho)^2.$$

Note that this is also obtained from Bishop's volume comparison theorem.

The following two lemmas are needed to prove Propositions 5 and 6.

Lemma 3. *Let $0 < R < \rho(p)$ and $a \geq 0$ be any given constants. If $\kappa(B(p,r)) \leq a$ for every nonexceptional r with $R < r < \rho(p)$, then*

$$\mathrm{Vol}(M) \leq \frac{a}{2} + (2\pi - a)\left(R - \frac{R^2}{2}\right).$$

Proof. By $(**)$ and $\rho(p) \leq 1$ we have

$$\begin{aligned}\mathrm{Vol}(M) &\leq \int_0^{\rho(p)} \int_0^t \kappa(B(p,u))\,du\,dt \\ &\leq \int_0^R \int_0^t 2\pi\,du\,dt + \int_R^{\rho(p)} \left(\int_0^R 2\pi\,du + \int_R^t a\,du\right) dt \\ &\leq \frac{a}{2} + (2\pi - a)\left(R - \frac{R^2}{2}\right).\end{aligned}$$

Q.E.D.

Lemma 4. *If $\bar{B}(p,r)$ for a number $0 < r < \rho(p)$ is not contractible, then there exists a geodesic loop with base point p which is entirely contained in $\bar{B}(p,r)$.*

Proof. Take a continuous loop $\gamma\colon [0,l] \to \bar{B}(p,r)$ with base point p such that

$$L(\gamma) = \inf\{\, L(c) \mid c \text{ is a loop with base point } p \text{ which is not homotopic to the point } p \text{ in } \bar{B}(p,r) \,\}.$$

If γ does not intersect $S(p,r)$, it is a geodesic loop. Thus we consider the case where γ intersect $S(p,r)$. Then $l = L(\gamma) \geq 2r$. Let us first prove the following

Claim. *γ forms a geodesic biangle consisting of two geodesics with length r.*

It suffices to show that $2r = l$. Now suppose that $2r < l$. For a minimal segment σ of M joining p and a point $\gamma(t)$ with $r < t < l - r$, one of the two closed curves $\gamma([0,t]) \cup \sigma$ and $\gamma([t,l]) \cup \sigma$ is not homotopic to the point p in $\bar{B}(p,r)$. Denoting this by γ_1 we have

$$L(\gamma_1) < L(\gamma)$$

because of $L(\sigma) \leq r$. This contradicts the definition of γ and completes the proof of the claim.

We will prove that γ does not break at $\gamma(r)$. Suppose the contrary. For each $0 \le t \le r$ we take a minimal segment σ_t joining $\gamma(r-t)$ and $\gamma(r+t)$ and set $\gamma_t := \gamma([0, r-t]) \cup \sigma_t \cup \gamma([r+t, 2r])$. Since γ breaks we have

$$L(\gamma_t) < L(\gamma) = 2r \qquad \text{and hence} \qquad \gamma_t \subset B(p,r)$$

for any $0 < t \le r$. Moreover, there is a small $\epsilon > 0$ such that $[0,\epsilon] \times [0,1] \ni (t,s) \longmapsto \gamma_t(sL(\gamma_t))$ is a smooth variation entirely contained in $\bar{B}(p,r)$, which is a homotopy with $\gamma_0 = \gamma$ in particular. This contradicts the definition of γ. Q.E.D.

Proposition 5. *Let $0 < R < \rho(p)$. If there exists a number $0 < r_0 \le R$ such that $\bar{B}(p, r_0)$ is not contractible, then*

$$\mathrm{Vol}(M) \le \frac{\pi}{2}(1 + 2R - R^2).$$

Proof. Take any fixed nonexceptional r with $R < r < \rho(p)$. If $B(p,r)$ is not a disk, Lemma 1 implies $\chi(B(p,r)) \le 0$ and hence

$$\kappa(B(p,r)) \le 0$$

by the Gauss-Bonnet theorem. In the case where $B(p,r)$ is a disk, Lemma 4 implies that $\bar{B}(p,r_0)$ contains a geodesic loop, which bounds a disk in $B(p,r)$ whose total curvature greater than π, because of the Gauss-Bonnet theorem. Therefore we have $c(B(p,r)) > \pi$ and hence by Lemma 1

$$\kappa(B(p,r)) = 2\pi\chi(B(p,r)) - c(B(p,r)) < \pi.$$

As a result, in either case we have $\kappa(B(p,r)) < \pi$ for any nonexceptional r with $R < r < \rho(p)$. Applying Lemma 3 under $a := \pi$, the proof is completed. Q.E.D.

Proposition 6. *Let $0 < R < \rho(p)$. Then we have*

$$\mathrm{Vol}(M) \le \pi - \frac{1}{2}(1-R)^2 \min\{\, c(B(p,R)), 2\pi \,\}.$$

Proof. It follows from the Gauss-Bonnet theorem and Lemma 1 that $\kappa(B(p,r)) \le 2\pi - c(B(p,r))$ for all nonexceptional r with $R < r < \rho(p)$. Since the function $t \mapsto c(B(p,t))$ is monotone nondecreasing, we have

$$\kappa(B(p,r)) \le 2\pi - c(B(p,R))$$

for all nonexceptional r with $R < r < \rho(p)$. Setting

$$a := \max\{\, 2\pi - c(B(p,R)), 0 \,\},$$

Lemma 3 completes the proof. Q.E.D.

§3. Proof of Main Theorem

Lemma 7. *Let $0 < R < \rho(p)$. If $B(p,r)$ for every $0 < r \leq R$ is contractible, then*

$$\mathrm{Vol}(B(p,R)) \geq \frac{1}{2}R^2(2\pi - c(B(p,R))).$$

Proof. In the case where R is exceptional, the above inequality for every nonexceptional R' with $0 < R' < R$ yields the conclusion since the set of nonexceptional radii is dense in $[0,+\infty)$. Thus we may consider only the case where R is nonexceptional. Under the notations as in section 1, it follows from the Gauss-Bonnet theorem that $\kappa_{R,i}(t) = \Theta_{R,i} - c(F_{R,i} \cap B(p,t)) \geq \Theta_{R,i} - c(F_{R,i})$ for all nonexceptional $0 < t \leq R$. This and $(*)$ imply

$$\mathrm{Vol}(F_{R,i}) \geq \int_0^R \int_0^r (\Theta_{R,i} - c(F_{R,i}))\,dt\,dr$$

and hence, by setting $F_R := \bigcup_{i=1}^{n(R)} F_{R,i}$ and $\Theta_R := \sum_{i=1}^{n(R)} \Theta_{R,i}$,

$$\mathrm{Vol}(B(p,R)) \geq \mathrm{Vol}(F_R) \geq \int_0^R \int_0^r (\Theta_R - c(F_R))\,dt\,dr.$$

On the other hand, since $B(p,R) - F_R$ is the union of $n(R)$ disks bounded by geodesic biangles, the Gauss-Bonnet theorem shows that

$$c(B(p,R) - F_R) > 2\pi - \Theta_R.$$

Thus we have

$$\begin{aligned}\mathrm{Vol}(B(p,R)) &\geq \int_0^R \int_0^r (2\pi - c(B(p,R)))\,dt\,dr \\ &= \frac{1}{2}R^2(2\pi - c(B(p,R))).\end{aligned}$$

Q.E.D.

Lemma 8. *For a given constant $R > 0$ we have*

$$\mathrm{Vol}(M) \geq \frac{c(M)\inf_{p\in M}\mathrm{Vol}(B(p,R))}{\sup_{p\in M} c(B(p,R))}.$$

Recall that

$$c(M) = \begin{cases} 4\pi & \text{if } M \cong S^2 \\ 2\pi & \text{if } M \cong P^2. \end{cases}$$

Proof. It suffices to show that

$$\int_M c(B(p,R))\,dp = \int_M K(p)\,\mathrm{Vol}(B(p,R))\,dp,$$

where dp is the volume element with respect to a variable p of M. Define the function $\varphi\colon M \times M \to \mathbf{R}$ by

$$\varphi(p,q) := \begin{cases} 1 & \text{if } d(p,q) < R \\ 0 & \text{if } d(p,q) \geq R \end{cases} \qquad \text{for all } p, q \in M.$$

By Fubini's theorem we have

$$\begin{aligned} \int_M c(B(p,R))\,dp &= \int_M \int_M \varphi(p,q)\,K(q)\,dq\,dp \\ &= \int_M K(q) \int_M \varphi(p,q)\,dp\,dq \\ &= \int_M K(q)\,\mathrm{Vol}(B(q,R))\,dq. \end{aligned}$$

Q.E.D.

Proof of Main Theorem. Let us define a constant $0 < R < 1$ by

$$R := \frac{4 - \sqrt{4 + 3c(M)/2\pi}}{4 - c(M)/2\pi} = \begin{cases} 2 - \sqrt{10}/2 & \text{if } M \cong S^2 \\ (4 - \sqrt{7})/3 & \text{if } M \cong P^2. \end{cases}$$

In the case where $\inf \rho \leq R$, Proposition 2 implies

$$\mathrm{Vol}(M) \leq \pi R^2 < \begin{cases} 0.176\pi & \text{if } M \cong S^2 \\ 0.204\pi & \text{if } M \cong P^2, \end{cases}$$

which concludes Main Theorem in particular. Thus assume that $\inf \rho > R$. If there is a point p in M such that $c(B(p,R)) \geq 2\pi$, then by

Proposition 6 we have

$$\mathrm{Vol}(M) \leq \pi \cdot (2R - R^2) < \begin{cases} 0.663\pi & \text{if } M \cong S^2 \\ 0.700\pi & \text{if } M \cong P^2. \end{cases}$$

If there are a point p in M and a radius $0 < r_0 \leq R$ such that $\bar{B}(p, r_0)$ is not contractible, then Proposition 5 implies

$$\mathrm{Vol}(M) \leq \frac{\pi}{2}(1 + 2R - R^2) < \begin{cases} 0.832\pi & \text{if } M \cong S^2 \\ 0.850\pi & \text{if } M \cong P^2. \end{cases}$$

Therefore, it suffices to consider the case where $c(B(p, R)) < 2\pi$ and $\bar{B}(p, r)$ is contractible for all points p in M and all $0 < r \leq R$. Now, setting

$$c := \sup_{p \in M} c(B(p, R)),$$

we have $0 < c \leq 2\pi$. Lemmas 7 and 8 show

$$\mathrm{Vol}(M) \geq \frac{R^2 c(M)(2\pi - c)}{2c}.$$

On the other hand, we have by Proposition 6

$$(\#) \qquad \mathrm{Vol}(M) \leq \pi - \frac{1}{2}(1 - R)^2 c.$$

Combining these two formulas, we have the quadratic inequality:

$$(1 - R)^2 c^2 - (2\pi + R^2 c(M))c + 2\pi R^2 c(M) \leq 0,$$

which gives the estimate of c:

$$c \geq \frac{2\pi + R^2 c(M) - \sqrt{b}}{2(1 - R)^2},$$

where b is the constant defined by

$$b := (2\pi + R^2 c(M))^2 - 8\pi R^2 (1 - R^2) c(M).$$

By this and $(\#)$ we obtain

$$\mathrm{Vol}(M) \leq \frac{\pi}{2} - \frac{1}{4}(R^2 c(M) - \sqrt{b}).$$

This completes the proof of Main Theorem. Q.E.D.

Note that R is determined as the last estimate is finest.

References

[A] A.D. Alexandrov, "Die innere Geometrie der konvexen Flächen", Akad. Verl., Berlin, 1955.

[AZ] A.D. Alexandrov and V.A. Zalgaller, "Intrinsic geometry of surfaces", Amer. Math. Soc., Providence, Rhode Island, 1967.

[BZ] Yu.D. Burago and V.A. Zalgaller, "Geometric inequalities", Springer, Berlin, 1988.

[F] F. Fiala, Le probléme isopèrimétres sur les surface onvretes à courbure positive, Comment. Math. Helv., **13** (1941), 293–346.

[GP1] K. Grove and P. Petersen V, Manifolds near the boundary of existence.

[GP2] ———, Volume comparison à la Aleksandrov.

[H] P. Hartman, Geodesic parallel coordinates in the large, Amer. J. Math., **86** (1964), 705–727.

[S] T. Sakai, On the isodiametric inequality for the 2-sphere, in "Geometry of Manifolds", Perspectives in Math. 8 (K. Shiohama, ed.), Academic Press, 1989, pp. 303–315.

[ST] K. Shiohama and M. Tanaka, An isoperimetric problem for infinitely connected complete open surfaces, in "Geometry of Manifolds", Perspectives in Math. 8 (K. Shiohama, ed.), Academic Press, 1989, pp. 317–343.

Department of Mathematics
Faculty of Science
Kyushu University
Fukuoka 812
Japan