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Compact Ricci-Flat Kähler Manifolds

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In this part, we survey general results on compact Kähler manifolds M with $c_1(M)_{\mathbb{R}} = 0$. According to the solution of the Calabi conjecture by Yau [Ya], such a compact Kähler manifold M admits a unique Ricci flat Kähler metric with given Kähler class. Our main interests here are applications of the existence of Einstein-Kähler metrics to studies on topological or holomorphic structures of compact Kähler manifolds M with $c_1(M)_{\mathbb{R}} = 0$.

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§1. Bogomolov decomposition

There are three fundamental types of compact Kähler manifolds whose real first Chern classes vanish:

(1) complex tori T;
(2) *symplectic* Kähler manifolds, i.e., compact Kähler manifolds X of even dimension $2m$ which have a holomorphic 2-form φ with φ^m nowhere vanishing on X (such φ is called holomorphic *symplectic* 2-form);
(3) *special unitary* Kähler manifolds, i.e., compact Kähler manifolds Y of dimension $n \geq 3$ such that the canonical bundle of Y is trivial but $H^0(Y, \Omega^p) = 0$ for $0 < p < n$.

Some examples of compact symplectic Kähler manifolds are given in Section 5. These three types are fundamental in the sense that the following holds:

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Theorem 1.1 (Bogomolov decomposition). *Let M be a compact Kähler manifold with $c_1(M)_{\mathbb{R}} = 0$. Then a certain finite unramified covering space M' of M decomposes holomorphically,*

$$M' \cong T \times X_1 \times \cdots \times X_r \times Y_1 \times \cdots \times Y_s,$$

into a direct product of a complex torus T and simply connected manifolds X_i, Y_j, where

- *X_i are symplectic Kähler manifolds with $h^{2,0}(X_i) = 1$;*
- *Y_j are special unitary Kähler manifolds.*

Moreover this decomposition is unique up to order.

Because of the uniqueness of the decomposition, a simply connected symplectic Kähler manifold X with $h^{2,0}(X) = 1$ cannot decompose any more. Such a manifold X is called *irreducible* symplectic Kähler manifold.

Corollary 1.2. a) *The Albanese map of a compact Kähler manifold M with $c_1(M)_{\mathbb{R}} = 0$ is surjective; and*

b) *any surjective holomorphic map $f: M \to N$ between compact Kähler manifolds M, N with $c_1(M)_{\mathbb{R}} = 0$, $c_1(N)_{\mathbb{R}} = 0$ induces a structure of holomorphic fiber bundle with finite structure group.*

Calabi [Ca-1] noted that the existence of Ricci-flat Kähler metric on M would imply Corollary 1.2 for the Albanese map. Bogomolov [Bo-1] proved Theorem 1.1 for simply connected M without using the Ricci-flat Kähler metrics. The following proof using the Ricci-flat metric was found independently by S. Kobayashi [Ko1] and Michelson [Mi-1][Mi-2].

We begin by quoting several facts:

Cheeger-Gromoll's splitting theorem [CG]. *Let M be a compact Riemannian manifold with non-positive Ricci curvature. Then the universal covering space of M splits into Riemannian direct product of a flat Euclidean space and a compact simply connected manifold.*

The second is the classification of holonomy groups. Let (M, g) be a Riemannian manifold of dimension m. Then the restricted holonomy group $\Phi_o(x)$ at $x \in M$ is always isomorphic to a subgroup of $SO(m)$ with the standard action of $SO(m)$ to T_pM. We call M *irreducible* if the action of the restricted holonomy group $\Phi_o(x)$ to T_xM is irreducible. According to Berger [Ber], if M is irreducible but not locally symmetric, then the restricted holonomy group Φ_p is isomorphic to one of the following subgroups of $SO(m)$ (the action to T_xM is the standard one induced by that of $SO(m)$):

$$\begin{array}{lll} SO(m); & U(n) \quad (m=2n); & SU(n) \quad (m=2n); \\ Sp(r) \quad (m=4r); & Sp(1)\cdot Sp(r) \quad (m=4r); & \\ \mathrm{Spin}(9) \quad (m=16); & \mathrm{Spin}(7) \quad (m=8); & G_2. \end{array}$$

Recall that Ricci flat locally symmetric spaces are flat. Moreover, if (M,g) is a Ricci-flat Kähler manifold, then the restricted holonomy group is a subgroup of $SU(n)(\subset SO(m))$, $m=2n$. Thus we have

Theorem 1.3. *Let M be a irreducible Ricci-flat Kähler manifold with $\dim_{\mathbb{C}} M = n$. Then the restricted holonomy group of M is either $SU(n)$ or $Sp(r)$, $n=2r$; the action to the tangent space is standard.*

We need one more. Let $G=\Phi(x)$ be the holonomy group of M at $x\in M$ and $(\bigwedge^p T_x^*M)^G$ the space of $(p,0)$-forms at x invariant under the action of G.

Proposition 1.4. *Let M be a compact Ricci-flat Kähler manifold with the holonomy group G at $x\in M$. Then $H^0(M,\Omega^p)\cong(\bigwedge^p T_x^*M)^G$.*

In fact, let ξ be a holomorphic p-form on M. Let $||\xi||$ be the pointwise norm of ξ. We compute the laplacian of $||\xi||^2$. Then, since the Ricci curvature is zero, by the Bochner formula we have $\triangle||\xi||^2 = ||\nabla\xi||^2$, where ∇ is the covariant derivative. Integrating the both hands sides yields $\nabla\xi=0$. Conversely any parallel $(p,0)$-form is holomorphic since the $(0,1)$-part of the covariant derivative ∇ is $\bar\partial$. Therefore the mapping $\xi\mapsto\xi(x)$ gives an isomorphism: $H^0(M,\Omega^p)\cong(\bigwedge^p T_x^*M)^G$.

Now we can give

Proof of Theorem 1.1. Let M be a compact Kähler manifold with $c_1(M)_{\mathbb{R}}=0$. Then by Yau [Ya] M has a Ricci-flat Kähler metric. Let $\tilde M$ be the universal covering of M and $\tilde M = E\times\prod_i M_i$ the de Rham decomposition of $\tilde M$, where E is the flat-part. Since M is Kähler, this decomposition is holomorphic and the flat part E is the complex Euclidean space. In view of the Cheeger-Gromoll splitting theorem cited above, the remaining part $\prod_i M_i$ is compact. Therefore, by the theorem of Bieberbach (see [KB]), there is a finite unramified covering $M'\cong T\times\prod_i M_i$ of M, where T is a complex torus covered by E. Note that the holonomy group of M_i coincides with the restricted holonomy group since M_i is simply connected. By the classification of restricted holonomy groups by Berger (Theorem 1.3) the holonomy group G_i of

M_i is isomorphic to either $SU(m_i)$ or $Sp(r_i)$, $m_i = 2r_i$, where $m_i = \dim_{\mathbb{C}} M_i$. According to Weyl [Weyl, Chap. VI],

$$\dim_{\mathbb{C}}(\bigwedge^p T_x^* M_i)^{G_i} = \begin{cases} 1, & \text{if } G_i \cong Sp(r_i),\ m_i = 2r_i \text{ and } p \text{ is even;} \\ 0, & \text{if } G_i \cong SU(m_i) \text{ and } 0 < p < m_i. \end{cases}$$

Thus by Proposition 1.4 M_i is either a symplectic Kähler manifold with $h^{2,0}(M_i) = 1$ or a special unitary Kähler manifold according as G_i is isomorphic to $Sp(r_i)$, $m_i = 2r_i$, or $SU(m_i)$. Q.E.D.

§2. Deformation

Let M be a compact complex manifold and $\mathcal{M} \to S$ the Kuranishi family of M. We call S the (local) universal deformation space of M. This space always exits as complex analytic space [Ku] but in general not smooth (even non-reduced).

Theorem 2.1 (Tian [Ti], Todorov [To-2]). *Let S be the universal deformation space of a compact Kähler manifold M with $c_1(M)_{\mathbb{R}} = 0$. Then S is smooth and* $\dim S = H^1(M, \Theta)$.

Bogomolov [Bo-2] proved this theorem for a symplectic compact Kähler manifold M. On a symplectic manifold the sheaf Θ of holomorphic vector fields is isomorphic to the sheaf Ω^1 of holomorphic 1-forms. He showed that any obstruction for deformation, which is an element of $H^2(M, \Theta)$, should vanish, by regarding it as an element of $H^2(M, \Omega^1)$ via the isomorphism above and then calculating integrals over 3-cycles. Fujiki [Fu-3] also gave a proof for symplectic Kähler manifold, using its hyperKähler structure (cf. Section 3). The proof we overview here is due to Tien and Todorov.

Let M be a compact Kähler manifold of dimension n and TM its holomorphic tangent bundle. For a holomorphic vector bundle E over M let $A^{p,q}(E)$ denote the space of E-valued smooth (p, q)-forms, which is also understood to be the space of $E \otimes \bigwedge^p T^*M$-valued $(0, q)$-forms.

According to the deformation theory of Kodaira-Spencer (cf. [Kod]), each small deformation of M corresponds to $\varphi \in A^{0,1}(TM)$ with the integrability condition

$$\bar{\partial}\varphi + \frac{1}{2}[\varphi, \varphi] = 0.$$

What Tien [Ti] and Todorov [To-2] proved in fact is the following

Theorem 2.1′. *Let M be a compact Ricci-flat Kähler manifold. Then for each TM-valued harmonic $(0,1)$-form φ_1 there exists a unique series $\varphi_\mu \in A^{0,1}(TM)$, $\mu \geqq 1$ such that for $\mu \geqq 2$*

a) $\bar{\partial}\varphi_\mu + \frac{1}{2}\sum_{\nu=1}^{\mu}[\varphi_\nu, \varphi_{\mu-\nu}] = 0$,
b) $\bar{\partial}^*\varphi_\mu = 0$.

Then by the argument in [KNS] ([Kod], Chap. 5) the power series $\varphi(t) = \sum \varphi_\mu t^\mu$ converges for sufficiently small $|t|$, satisfying the integrability condition above. This shows that M can be deformed in the direction of any element of $H^1(M, \Theta)$ and hence the Kuranishi space of M is an open subset of $H^1(M, \Theta)$.

Since $K_M^* \cong \bigwedge^n TM$, the contraction induces a holomorphic isomorphism

$$\alpha_r\colon K_M^* \otimes \textstyle\bigwedge^{n-r} T^*M \to \bigwedge^r TM,$$

where $\bigwedge^0 TM$ is understood to be a trivial line bundle. This extends to

$$\alpha_r\colon A^{n-r,q}(K_M^*) \to A^{0,q}(\textstyle\bigwedge^r TM),$$

commuting with $\bar{\partial}$. Via α_1, the Lie bracket on $\bigoplus_{q\geq 0} A^{0,q}(TM)$ induces a Lie bracket [,] on $\bigoplus_{q\geq 0} A^{n-1,q}(K_M^*)$:

$$[\alpha_1(\xi), \alpha_1(\eta)] := \alpha_1([\xi, \eta]) \qquad \text{for} \quad \xi,\, \eta \in \bigoplus_{q\geqq 0} A^{0,q}(TM).$$

On the holomorphic tensor bundle $E = \bigwedge^r TM \otimes \bigwedge^s T^*M$, the Levi-Civita connection ∇ of M defines the hermitian connection relative to the induced metric and the exterior covariant derivative

$$d^\nabla\colon \bigoplus_{p+q=k} A^{p,q}(E) \to \bigoplus_{p+q=k+1} A^{p,q}(E).$$

Let ∂ be the $(1,0)$-component of d^∇.

Lemma 2.2. *For $\xi,\, \eta \in A^{n-1,1}(K_M^*)$*

$$[\xi,\, \eta] = \partial\big(\beta(\alpha_1(\xi) \wedge \eta)\big) - \alpha_0(\partial\xi) \wedge \eta + \xi \wedge \alpha_0(\partial\eta),$$

where β is induced from an interior product

$$TM \otimes \textstyle\bigwedge^{n-1} T^*M \to \bigwedge^{n-2} T^*M.$$

Proof. Recall that K_M and K_M^* are flat line bundles with the induced metrics. Relative to a parallel local trivialization ω of K_M, define $\operatorname{div}_\omega X$ for a local holomorphic vector field X by

$$(\operatorname{div}_\omega X)\omega = \mathcal{L}_X \omega,$$

where $\mathcal{L}_X$ is the Lie derivative with respect to X. Let $Z \lrcorner \omega$ denote the interior product of ω with a vector field Z. Then $\partial \circ \beta$ corresponds to div_ω and the lemma reduces to the fundamental properties of Lie derivative: for holomorphic vector fields X, Y we have

$$\begin{aligned}\mathcal{L}_X(Y \lrcorner \omega) &= (\mathrm{div}_\omega X) Y \lrcorner \omega + (\mathcal{L}_X Y) \lrcorner \omega && \text{(the derivation rule)}\\ &= \partial(X \lrcorner Y \lrcorner \omega) + X \lrcorner \partial(Y \lrcorner \omega) && \text{(H. Cartan's formula)}\end{aligned}$$

and $\mathcal{L}_X Y = [X, Y]$. Q.E.D.

Since K_M^* is a flat line bundle and M is Kähler, the Hodge theory holds also for K_M^*-valued (p,q)-forms; in particular every K_M^*-valued $\bar\partial$-harmonic form is ∂-closed and the $\partial\bar\partial$-lemma holds:

Lemma 2.3 (cf. [GH] p. 149). *If K_M^*-valued $\bar\partial$-closed (p,q)-form η is ∂-exact, then there exists a $(p-1, q-1)$-form γ such that $\eta = \bar\partial\partial\gamma$ and $\bar\partial^*\bar\partial\gamma = 0$.*

The flatness of K_M also implies the following

Lemma 2.4. $\bar\partial^* \alpha_1(\xi) = \alpha_1 \bar\partial^* \xi$ *for* $\xi \in A^{n-1,1}(K_M^*)$.

In fact, the inverse of $a^{(1)}$ can be obtained by contracting with ω and then tensoring ω^*, where ω is a parallel local trivialization of K_M and ω^* its dual. Express $\bar\partial^*$ using ∇. Then, the lemma follows immediately since contractions and the covariant derivative commute each other.

Now the power series $\sum_{\mu \geqq 1} \varphi_\mu t^\mu$ in the theorem can be constructed inductively in terms of $\xi_\mu \in A^{n-1,1}(K_M^*)$ with $\varphi_m = \alpha_1(\xi_\mu)$. The conditions a), b) correspond to

a)′ $\bar\partial\xi_\mu + \frac{1}{2}\sum_{\nu=1}^{\mu}[\xi_\nu, \xi_{\mu-\nu}] = 0$,
b)′ $\bar\partial^*\xi_\mu = 0$.

We pose moreover

c)′ $\partial\xi_\mu = 0$.

Since φ_1 and hence ξ_1 are harmonic, ξ_1 satisfies the conditions above by Lemma 2.4. Suppose there are determined $\xi_1, \ldots, \xi_\mu$ satisfying a)′, b)′ and c)′. Then each $[\xi_\nu, \xi_{\mu-\nu}]$ is ∂-exact by c)′ and Lemma 2.2; the sum is $\bar\partial$-closed by condition a)′. Hence, by the $\partial\bar\partial$-lemma, Lemma 2.4, there exists $\xi_{\mu+1}$ satisfying a)′, b)′ and c)′. By condition b)′ this series is unique.

§3. Symplectic manifolds

In this section we discuss the structure of de Rham cohomology ring of a symplectic Kähler manifold. We begin by recalling Kähler case briefly.

Kähler Case. Let M be a compact Kähler manifold of dimension n. We first note that any parallel endmorphism θ of the tangent bundle TM of M induces a derivation of the de Rham cohomology ring $H^*(M, \mathbb{R})$ of M. In fact we have ([Li-1] or see [Li-2])

Proposition 3.1. *Let M be a Riemannian manifold and $\triangle$ the Laplacian on p-forms. If h is a parallel endmorphism of $\bigwedge^p T^*M$, then $\triangle(h\alpha) = h\triangle\alpha$ for any p-form α.*

The complex structure J on a Kähler manifold M is a parallel endmorphism of TM. Let $v(J)$ denote the derivation (over $\mathbb{R}$) on $\bigwedge^* T^*M$ induced by J, namely, $v(J)\varphi = \sqrt{-1}(p-q)\varphi$ for (p,q)-form φ. Recall that TM admits another parallel endmorphism, that is, the identity id; and it induces a derivation $v(\mathrm{id})$ given by $v(\mathrm{id})\varphi = (p+q)\varphi$ for (p,q)-form φ. These two derivations generate a Lie algebra corresponding to a Lie group $U(1) \times \mathbb{R}^* \cong \mathbb{C}^*$. Thus we have

Proposition 3.2 (see [We]). *Let M be a compact Kähler manifold. Then there is a real representation ρ of $\mathbb{C}^* \cong U(1) \times \mathbb{R}^*$ to the algebra automorphism group of the cohomology ring $H^*(M, \mathbb{R})$.*

Let $H^{p,q}(M) \subset H^{p+q}(M, \mathbb{C})$ be a subspace spanned by classes of d-closed (p,q)-forms in the de Rham cohomology group of M. Then the Hodge decomposition,

$$H^k(M, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M), \quad \bar{H}^{p,q}(M) = H^{q,p}(M),$$

is nothing but the decomposition of $H^*(M, \mathbb{C}) \cong H^*(M, \mathbb{R}) \otimes \mathbb{C}$ into the isotypical components of the real representation ρ in Proposition 3.2 above (an *isotypical* component is, by definition, a direct sum of one and the same irreducible representation). Moreover ρ does not depend on a Kähler metric of M. In fact, for $z = \exp(s + \sqrt{-1}\theta)$, $s, \theta \in \mathbb{R}$, we have

$$\rho(z) = \sum_{p,q} \exp((p+q)s + \sqrt{-1}(p-q)\theta)\pi^{p,q},$$

where $\pi^{p,q} \colon H^r(M, \mathbb{C}) \to H^{p,q}(M)$ denote the projection.

Let ω be a Kähler form on M. Let L be the multiplication operator on $H^*(M, \mathbb{C})$ by the Kähler class of ω and Λ the adjoint operator of L. For $k \leq n$, $n = \dim M$ we set

$$H^k(M, \mathbb{C})_\omega = \{\alpha \in H^k(M, \mathbb{C}) \mid L^{n-k+1}\alpha = 0\},$$
$$H^{p,q}(M)_\omega = H^k(M, \mathbb{C})_\omega \cap H^{p,q}(M).$$

Then the strong Lefschetz theorem says

$$L^{n-k}\colon H^{n-k}(M, \mathbb{R}) \cong H^{n+k}(M, \mathbb{R}),$$
$$H^k(M, \mathbb{R}) = \bigoplus_{r \geq 0} L^r H^{k-2r}(M, \mathbb{R})_\omega.$$

Elements of $H^k(M, \mathbb{C})_\omega$ are called *ω-effective* (or *primitive*).

hyperKähler manifold. Let M be now a symplectic Kähler manifold of complex dimension $2n$. According to the solution of the Calabi conjecture by Yau [Ya] there is a unique Ricci flat Kähler metric g with given Kähler class. Then (M, g) has a structure of *hyperKähler* manifold (for definition, see below). Before stating results on the structure of cohomology ring of M, we introduce first a few notions related to hyperKähler manifolds.

The restricted holonomy group of (M, g) is a subgroup of $Sp(n)$. (It is exactly $Sp(n)$ if M is irreducible.) Hence the ring of parallel endomorphisms of the tangent bundle TM of M contains a subalgebra $\mathbb{H}$ isomorphic to (and identified with) the standard quaternion algebra over $\mathbb{R}$. Let $P = \{\lambda \in \mathbb{H} \mid \lambda^2 = -1\}$. Then each λ defines an integrable complex structure on M so that g is a Kähler metric under this complex structure. Let M_λ denote the manifold M with complex structure defined by λ and ω_λ the Kähler form of g relative to λ. Thus we have a family $\{(M_\lambda, \omega_\lambda)\}_{\lambda \in P}$ of Kähler structures, which is called the *Calabi family*; and the manifold M together with this family is called a *hyperKähler* manifold. Moreover the family $\{\omega_\lambda\}_{\lambda \in P}$ of d-closed 2-forms on M defines a 3-dimensional subspace $F \subset H^2(M, \mathbb{R})$, which we call the *hyperKähler 3-space* associated to (M, g).

Each member $(M_\lambda, \omega_\lambda)$ of the Calabi family is again a symplectic Kähler manifold (as we will see a little later). Conversely a Kähler class $[\omega]$ on M and a 1-dimensional subspace of $H^0(M, \Omega^2)$ spanned by a holomorphic symplectic 2-form φ determine a hyperKähler structure on M. For a symplectic Kähler manifold M with $h^{2,0}(M) = 1$, in particular, a hyperKähler structure on M is equivalent to a *polarization* on M, i.e., fixing a Kähler class on M.

Let $\mathbb{H}^* := \mathbb{H} - \{0\}$. By Proposition 3.1 each parallel endomorphism of TM induces an algebra automorphism of $H^*(M, \mathbb{R})$. Therefore, corresponding to Proposition 3.2 above we have

Proposition 3.3. *Let M be a hyperKähler manifold. Then there is a real representation ρ_{HK} of $\mathbb{H}^* \cong Sp(1) \times \mathbb{R}_{>0}$ to the algebra automorphism group of the cohomology ring $H^*(M, \mathbb{R})$.*

The decomposition of $H^k(M, \mathbb{R})$ into isotypical components of the action of $\rho_{\mathrm{HK}}(\mathbb{H}^*)$ is compatible to the Hodge decomposition relative to any complex structure $\lambda \in P$. Note that, however, this decomposition depends on the hyperKähler structure of M.

Corollary 3.4 ([Wa]). *Every odd dimensional Betti number of a compact hyperKähler manifold is divisible by* 4.

This follows from Proposition 3.3 and results on the representation of $\mathbb{H}^*$ ([Fu-3]). The following argument is taken from Wakakuwa's long forgotten paper [Wa].

Proof. Let ξ, η be harmonic forms of odd degree on a compact hyperKähler manifold with Ricci flat Kähler metric g. Assume that ξ is orthogonal to η with respect to the L^2-inner product. Since g is hermitian relative to the complex structure corresponding to each $\lambda \in P$, we have $g(\rho(\lambda)\xi, \rho(\lambda)\eta) = g(\xi, \eta)$. On the other hand $\rho(\lambda)^2\xi = -\xi$ for $\lambda \in P$ since the degree of ξ is odd. Thus the de Rham classes of ξ and $\rho(\lambda)\xi$, $\lambda \in P$, span a 4-dimensional subspace in the cohomology group and they are all orthogonal to η. Q.E.D.

We examine here the action of $\rho_{\mathrm{HK}}(\mathbb{H}^*)$ more closely. For $\lambda \in P$ let $v(\lambda)$ denote the derivation on $H^*(M, \mathbb{C})$ induced by the complex structure λ. Then $v(\mu)w_\lambda = 2\omega_{\lambda\mu}$. Let (λ, μ, ν) be a standard basis of pure quaternions such that

$$l^2 = \mu^2 = \nu^2 = -1,$$
$$l\mu = -\mu\lambda = \nu,\ \mu\nu = -\nu\mu = \lambda,\ \nu\lambda = -\lambda\nu = \mu.$$

Then $v(\lambda)\alpha = \sqrt{-1}(p-q)\alpha$ for $\alpha \in H^{p,q}(M)$, relative to the complex structure λ, and

$$\begin{aligned} &(v(\mu) + \sqrt{-1}v(\nu))(H^{p,q}(M)) \subset H^{p+1,q-1}(M), \\ &(v(\mu) - \sqrt{-1}v(\nu))(H^{p,q}(M)) \subset H^{p-1,q+1}(M). \end{aligned} \tag{3.5}$$

Moreover

$$\varphi_\lambda := \omega_\mu + \sqrt{-1}\omega_\nu = \sqrt{-1}(v(\mu) + \sqrt{-1}v(\nu))\omega_\lambda$$

is a holomorphic symplectic 2-form under the complex structure λ. The hyperKähler 3-space F, spanned by the de Rham cohomology classes of ω_λ, ω_μ and ω_ν, is identified with the space of pure quaternions, i.e., the Lie algebra $sp(1)$. Thus we have

Proposition 3.6. *For a compact hyperKähler manifold M,*

1) $\rho_{\mathrm{HK}}(\mathbb{H}^*) \cong \mathbb{H}^* \cong Sp(1) \times \mathbb{R}_{>0}$;

2) *the hyperKähler 3-space F is stable under the action of $\rho_{\mathrm{HK}}(\mathbb{H}^*)$ and the action of $Sp(1)$ on $F \cong sp(1)$ is identified with the adjoint action.*

For each $\lambda \in P$ let L_λ be the endomorphism of $H^*(M, \mathbb{R})$ defined by the multiplication with ω_λ. For $k \leq 2n$, define

$$H^k(M, \mathbb{R})_F := \{\alpha \in H^k(M, \mathbb{R}) \mid L_\lambda^{2n-k+1}\alpha = 0 \quad \text{for } \lambda \in P\ \}.$$

An element of $H^k(M, \mathbb{R})_F$ called *universally effective.* Since each element ω_λ of the hyperKähler 3-space F is a Kähler form relative to the complex structure corresponding to λ, the strong Lefschetz theorem holds with respect to each L_λ. Moreover we have

Theorem 3.7 [Fu-3]. *Let M be a compact symplectic manifold with $\dim_{\mathbb{C}} M = 4n$. Let N^* be the subalgebra of $H^*(M, \mathbb{R})$ generated by the hyperKähler 3-space F. Then:*

1) *The submodule $H^*_\epsilon(M, \mathbb{R})$ generates $H^*(M, \mathbb{R})$ as N^*-module and we have a natural direct sum decomposition*

$$H^l(M, \mathbb{R}) = \bigoplus N^{l-k} H^k(M, \mathbb{R})_F.$$

2) *If $l \leq n$, then the natural map*

$$N^{l-k} \otimes_{\mathbb{R}} H^k(M, \mathbb{R})_F \to N^{l-k} H^k(M, \mathbb{R})_F$$

is an isomorphism of $\mathbb{H}^*$-module.

Let φ be the holomorphic symplectic 2-form on M. Let

$$L_\varphi \colon H^q(M, \Omega^p) \to H^q(M, \Omega^{p+2}),$$
$$L_{\bar\varphi} \colon H^q(M, \Omega^p) \to H^{q+2}(M, \Omega^p)$$

be the linear maps defined by the multiplication with φ and $\bar{\varphi}$ respectively. For $\gamma = \varphi$ or $\bar{\varphi}$ the space of γ-effective Dolbeault classes is defined by

$$H^q(M, \Omega^p)_\gamma := \{\alpha \in H^q(M, \Omega^p) \mid L_\gamma^{n-s+1}\alpha = 0\},$$

where $s = p$ or q according to $\gamma = \varphi$ or $\bar{\varphi}$.

Theorem 3.8 [Fu-3]. *Let M be a compact symplectic Kähler manifold with $\dim_{\mathbb{C}} M = 4n$. Then:*

1) *The linear maps*

$$L_\varphi^{n-p}\colon H^q(M, \Omega^p) \to H^q(M, \Omega^{2n-p}) \quad \textit{for } p < n, \textit{ and}$$
$$L_{\bar{\varphi}}^{n-q}\colon H^q(M, \Omega^p) \to H^{2n-q}(M, \Omega^p) \quad \textit{for } q < n,$$

are both isomorphic.

2) *For any $p, q \geq 0$ we have the direct sum decopositions*

$$H^q(M, \Omega^p) = \bigoplus_{r \geq n-p} L_\varphi^r H^q(M, \Omega^p)_\varphi,$$
$$H^q(M, \Omega^p) = \bigoplus_{r \geq n-q} L_{\bar{\varphi}}^r H^{q-2r}(M, \Omega^p)_{\bar{\varphi}}.$$

Theorems 3.7 and 3.8 are, respectively, hyperKähler and holomorphic symplectic versions of the strong Lefschetz theorem for Kähler manifolds. Recall that the strong Lefschetz theorem is a consequence of the fact that the de Rham cohomology ring of a compact Kähler manifold admits an $sl(2)$-action generated by the operator L and its formal adjoint Λ. One can proof Theorem 3.8 similarly by considering an $sl(2)$-action on $\bigoplus_p H^q(M, \Omega^p)$ (or $\bigoplus_q H^q(M, \Omega^p)$) generated by L_φ (or $L_{\bar{\varphi}}$) and its formal adjoint. We shall give here a proof of Theorem 3.7, which uses the fact that L_λ, their formal adjoint operators, and $v(\lambda)$, $\lambda \in P$, generate an $sp(2)$-action on the de Rham cohomology ring.

Proof of Theorem 3.7. We fix a Ricci-flat Kähler structure on M. Define an operator H by $H\varphi = (p + q - 2n)\varphi$ for (p, q)-form φ. The complex structure corresponding to $\lambda \in P$ induces a derivation $v(\lambda)$ (over $\mathbb{R}$) of the space of forms. Moreover, for $\lambda \in P$, let L_λ denote the multiplication by ω_λ and let Λ_λ be its formal adjoint. Then these operators act on the space of harmonic forms by Proposition 3.1. We shall determine the commutator relations.

First of all, for each $\lambda \in P$ we already know that H, L_λ and Λ_λ generate $sl(2)$:

$$(*) \qquad [H, L_\lambda] = -2L_\lambda,\ [H, \Lambda_\lambda] = 2\Lambda_\lambda,\ [L_\lambda, \Lambda_\lambda] = H,$$

and $v(\lambda)$, $\lambda \in P$, generate $sp(1)$:

$$(**) \qquad [v(\lambda), v(\mu)] = -2v(\lambda\mu) \quad \text{for } \lambda,\ \mu \in P.$$

To derive other relations, take a standard basis of the pure quaternions, say λ, μ and ν, so that λ corresponding to the fixed complex structure. Then $\varphi := (1/2)(\omega_\mu + \sqrt{-1}\omega_\nu)$ is a holomorphic symplectic 2-form. Let L_φ denote the multiplication by φ and Λ_φ its formal adjoint. Moreover let $L_{\bar\varphi}$ and $\Lambda_{\bar\varphi}$ denote the complex conjugate of L_φ and Λ_φ respectively. Since $v(\lambda)\alpha = \sqrt{-1}(p-q)\alpha$ and φ is of type $(2,0)$, we have $[v(\lambda), L_\varphi] = 2\sqrt{-1}L_\varphi$. Taking the real part and its adjoint, we obtain

$$[v(\lambda), L_\mu] = -2L_\nu, \quad [v(\lambda), \Lambda_\mu] = 2\Lambda_\nu.$$

Let α be a (p,q)-form. By the same calculation as in the Kähler case we have

$$[L_\varphi, \Lambda_{\bar\varphi}] = 0, \quad [L_\varphi, \Lambda_\varphi]\alpha = (p-n)\alpha.$$

It follows

$$\begin{aligned}\sqrt{-1}[L_\mu, \Lambda_\nu]\alpha &= [L_\varphi + L_{\bar\varphi}, \Lambda_\varphi - \Lambda_{\bar\varphi}]\alpha \\ &= (p-q)\alpha = \frac{1}{\sqrt{-1}}v(\lambda)\alpha.\end{aligned}$$

Consequently, if $\lambda \neq \mu$, then

$$(***) \qquad \begin{aligned}[v(\lambda), L_\mu] = -2L_{\lambda\mu}, \quad [v(\lambda), \Lambda_\mu] = 2\Lambda_{\lambda\mu},\\ [L_\lambda, \Lambda_\mu] = -v(\lambda\mu).\end{aligned}$$

Any other commutator of H, L_λ, Λ_λ, $v(\lambda)$, $\lambda \in P$, which does not appear in $(*)$, $(**)$ or $(***)$ is zero.

Let $\mathcal{H}^*$ be the space of harmonic forms on M with coefficients in $\mathbb{C}$. By the above commutator relations, the complex Lie algebra generated by H, $v(\lambda)$, L_λ and Λ_λ, $\lambda \in P$, is isomorphic to $sp(2, \mathbb{C})$. Since $sp(2,\mathbb{C})$ is semi-simple, $\mathcal{H}^*$ is decomposed into a direct sum of irreducible $sp(2,\mathbb{C})$-submodules. Let $V \subset \mathcal{H}^*$ be an irreducible subspace. For a subspace $\mathfrak{h} \subset sp(2,\mathbb{C})$ and $U \subset V$ we denote:

$$\mathfrak{h}U := \{X_1X_2\cdots X_m\gamma \mid X_i \in \mathfrak{h},\ \gamma \in U\}.$$

For $\lambda \in P$ let $\mathfrak{g}_\lambda$ be the Lie subalgebra generated by H, L_λ and Λ_λ. Let $V_\lambda \subset V$ be a $\mathfrak{g}_\lambda$-irreducible subspace. As a $\mathfrak{g}_\lambda$-module, V_λ is generated by an ω_λ-effective element $v_\lambda \in V_\lambda$. Choose $\lambda \in P$ so that the degree as a form of v_λ is minimal among those of v_μ, $\mu \in P$. Then, since $\mathfrak{g}_\mu v_\lambda$ is

$\mathfrak{g}_\mu$-irreducible, v_λ is ω_μ-effective for any $\mu \in P$, i.e., universally effective. Let $\mathcal{L}$ be the subspace spanned by L_λ, $\lambda \in P$ and $\mathcal{D}$ be the subspace spanned by $v(\lambda)$, $\lambda \in P$. It follows by $(*)$, $(**)$ and $(***)$ that $\mathcal{L}\mathcal{D}v_\lambda$ is stable under the action of $sp(2,\mathbb{C})$ and hence $V = \mathcal{L}\mathcal{D}v_\lambda$. Moreover by $(***)$ each element of $\mathcal{D}v_\lambda$ is universally effective and of the same degree. This proves 1) of the theorem.

To prove 2) of the theorem, we recall the representations of $sp(2,\mathbb{C})$ (cf. [Weyl]). For any pair (f_1, f_2) of integers with

$$f_1 \geq f_2 \geq 0, \quad f_1 + f_2 = d \geq 1,$$

there is an irreducible representation $V(f_1, f_2)$ of $sp(2,\mathbb{C})$ and any irreducible representation is equivalent to $V(f_1, f_2)$ for some (f_1, f_2). Moreover $V(f_1, f_2)$ and $V(g_1, g_2)$ are equivalent if and only if $(f_1, f_2) = (g_1, g_2)$.

We use the following characterization of $V(f_1, f_2)$. Let V be an irreducible representation of $sp(2,\mathbb{C})$. Consider the set E consisting of all pairs (r, s) of eigen values r of H and eigen values s of $v(\lambda)$ on V. Let (r_0, s_0) be the maximal element of E with respect to the lexicographical order. Then V is equivalent to $V(f_1, f_2)$ with $f_1 + f_2 = r_0$ and $f_1 - f_2 = s_0$.

To observe the action of $sp(2,\mathbb{C})$ on $V(f_1, f_2)$, we shall realize it in an exterior algebra. Let V_0 be a complex vector space with basis $z_1, \ldots, z_n, w_1, \ldots, w_n$. In $V := \bigoplus_{p,q\geq 0} \bigwedge^p V_0 \wedge \bigwedge^q \bar{V}_0$, we set

$$\omega_V = \sum_{i=1}^{n} (z_i \wedge \bar{z}_i + w_i \wedge \bar{w}_i), \quad \varphi_V = \sum_{i=1}^{n} z_i \wedge w_i.$$

Let L_{ω_V} , L_{φ_V} and $L_{\bar{\varphi}_V}$ denote the multiplication in V by ω_V , φ_V and $\bar{\varphi}_V$ respectively. Moreover let Λ_{ω_V}, Λ_{φ_V} and $\Lambda_{\bar{\varphi}_V}$ be the adjoint operators of L_{ω_V} , L_{φ_V} and $L_{\bar{\varphi}_V}$ respectively with respect to the hermitian metric defined by ω_V. Then these generate the action of $sp(2,\mathbb{C})$ on V. Since these operators just correspond to L_ω , L_φ , $L_{\bar{\varphi}}$, Λ_ω, Λ_φ and $\Lambda_{\bar{\varphi}}$ respectively, we will use the same symbols as before. Set

$$X := [\Lambda_\omega, L_{\bar{\varphi}}], \quad Y := [\Lambda_\omega, L_\varphi].$$

Choose integers $0 \leq q \leq p \leq n$ so that $f_1 = n - q$ and $f_2 = n - p$. Let $\xi := z_1 \wedge \cdots \wedge z_p \wedge \bar{w}_1 \wedge \cdots \wedge \bar{w}_q$. Then $sp(2,\mathbb{C})\xi$ is equivalent to $V(f_1, f_2)$. In fact we have

$$\Lambda_\omega \xi = \Lambda_\varphi \xi = \Lambda_{\bar{\varphi}} \xi = Y\xi = 0,$$
$$H\xi = (2n - (p+q))\xi, \quad v(\lambda)\xi = (p-q)\xi.$$

Therefore the eigen space of H with the maximal eigen value, $2n-(p+q)$, in $sp(2,\mathbb{C})\xi$ is spanned by $X^r\xi$, $0 \le r \le p-q$. Moreover $v(\lambda)X^r\xi = (p-q-r)X^r\xi$. Note that all $X^r\xi$ are universally effective.

Now we complete the proof of 2). Let $\xi(a,b,c,r) = L_\omega^a L_\varphi^b L_{\bar\varphi}^c X^r\xi$. Then, among these elements, $\xi(a,b,c,r)$ is characterized by the fact that it contains the term $\Xi(k)\wedge\Omega(a)\wedge\Phi(b)\wedge\Psi(c)$, where

$$\Xi(r) = \bigwedge_{i=1}^{p-r} z_i \wedge \bigwedge_{i=1}^{q} \bar{w}_i \wedge \bigwedge_{j=p-r+1}^{p} \bar{w}_j, \quad \Omega(a) = \bigwedge_{j=p+1}^{p+a} (z_j \wedge \bar{z}_j),$$

$$\Phi(b) = \bigwedge_{k=p+a+1}^{p+a+b} (z_k \wedge w_k), \quad \Psi(c) = \bigwedge_{k=p+a+b+1}^{p+a+b+c} (\bar{z}_k \wedge \bar{w}_k).$$

Therefore $\xi(a,b,c,r)$, $2a+2b+2c+p+q \le n$, $k \le p-q$, are linearly independent. Q.E.D.

Theorem 3.9 ([Bea-2], [Fu-3]). *Let M be a compact symplectic Kähler manifold with $h^{2,0}(M) = 1$. Let $v(\alpha) := \int_M \alpha^{2n}$ for $\alpha \in H^2(M,\mathbb{R})$. Then there is a unique quadratic form f on $H^2(M,\mathbb{R})$ such that*

(1) *f is non-degenerate with signature $(3, b_2(M)-3)$; $f(\gamma) > 0$ for any Kähler class γ;*
(2) *$f(\alpha)^n = v(\alpha)$ for $\alpha \in H^2(M,\mathbb{R})$;*
(3) *for α, $\beta \in H^2(M,\mathbb{R})$*

$$\begin{aligned} &v(\alpha)^2 f(\beta) \\ =&f(\alpha)\left[(2n-1)v(\alpha)\int_M \alpha^{2n-2}\beta^2 - (2n-2)\left(\int_M \alpha^{2n-1}\beta\right)^2\right]; \end{aligned}$$

(4) *f is $\mathbb{Q}$-valued on $H^2(M,\mathbb{Q})$.*

We note that (2) of the theorem is due to Fujiki [Fu-3] and (3) is due to Beauville [Bea-2]. Both of their proofs use the Bogomolov unobstructed theorem for deformations; this unobstructed theorem can be proved using the existence of Ricci-flat metrics (see Section 2). The following proof of Theorem 3.8 uses the solution of the Calabi conjecture more directly.

Proof of Theorem 3.9. Let M be a compact symplectic Kähler manifold with $h^{2,0} = 1$. Let φ be a symplectic holomorphic 2-form on M

normalized so that $\int_M (\varphi\bar{\varphi})^n = 1$. Following [Bea-2], for $\alpha \in H^2(M, \mathbb{R})$ let

$$f_o(\alpha) := \frac{n}{2}\int_M (\varphi\bar{\varphi})^{n-1}\alpha^2 + (1-n)\int_M \varphi^{n-1}\bar{\varphi}^n\alpha \cdot \int_M \varphi^n\bar{\varphi}^{n-1}\alpha.$$

We shall show that f_o multiplied by a suitable positive constant has desired properties.

Since f_o is a polynomial on $H^2(M, \mathbb{R})$, it suffices to consider on an open subset of $H^2(M, \mathbb{R})$. Therefore we may assume that the $(1,1)$-component $\alpha^{(1.1)}$ of $\alpha \in H^2(M, \mathbb{R})$ is a Kähler class. According to the solution of the Calabi conjecture by Yau [Ya], there is a Ricci-flat Kähler metric g on M with Kähler class $\alpha^{(1,1)}$. Let ω be the Kähler form of g. We may assume $\int_M \omega^{2n} = 1$. In the following we consider this Riemannian structure. Then the symplectic form φ on T_pM, $p \in M$, is invariant under the action of the holonomy group, $Sp(4n)$ with the standard action. Hence φ can be written as

$$\varphi = \frac{(2n)!}{2^{2n}(n!)^2}\sum_{i=1}^{n} u_i \wedge v_i$$

with a suitable unitary basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ of T_p^*M; note that the Kähler form ω at p is given by

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n}(u_i \wedge \bar{u}_i + v_i \wedge \bar{v}_i).$$

Since φ and ω are parallel, it follows that $(\varphi\bar{\varphi})^{n-1}\omega^2 = c\omega^{2n}$ on the whole M, where c is a positive constant depending only on the dimension of M. Hence $f_o(\alpha) = c' \int_M \alpha^{2n}$ by a direct calculation.

To prove (3) we shall use the $Sp(1)$-action on the cohomology ring $H^*(M, \mathbb{R})$. First we show that f_o *is invariant under this action.* By Theorem 3.7 we have

$$H^2(M, \mathbb{R}) = F \oplus H^2(M, \mathbb{R})_F,$$

as $Sp(1)$-module, where F is the hyperKähler 3-space. Since $h^{2,0}(M) = 1$, any element of $H^2(M, \mathbb{R})_F$ is of type $(1,1)$. Therefore the action of $Sp(1)$ on $H^2(M, \mathbb{R})_F$ is trivial by (3.5). Since ω_λ^{2n}, $\lambda \in P$, are the volume form of the same metric, $f_o(\omega_\lambda)$, $\lambda \in P$, are all equal by (2). By (2) of Proposition 3.6 it follows that f_0 is $Sp(1)$-invariant.

Since the action of $Sp(1)$ on $H^{4n}(M, \mathbb{R})$ is trivial, both hand sides of (3) is $Sp(1)$-invariant. Moreover $Sp(1)$ acts transitively on the hyperKähler 3-space F by Proposition 3.6. Therefore we may assume that

α is of type $(1,1)$ and hence $\alpha = \omega$. Let β_o be the universally effective part of $\beta \in H^2(M, \mathbb{R})$ so that β can be written as $\beta = \omega + c\varphi + \bar{c}\bar{\varphi} + \beta_o$ for some $c \in \mathbb{C}$. Note that

$$\omega_\lambda^a \omega_\mu^b \omega_\nu^c \beta_o = 0 \quad \text{for } a + b + c = 2n - 1,\ a, b, c \geq 0.$$

Therefore $\omega^{2n-2}\varphi\beta_0 = 0$ and $(\varphi \wedge \bar{\varphi})^{n-1}\varphi\beta_o = 0$. Now we have (3) by a direct calculation.

In particular f_o is positive definite on the hyperKähler 3-space F. By Hodge bilinear relation $\int_M \omega^{2n-2}\gamma^2 < 0$ for $\gamma \in H^2(M, \mathbb{R})_F$. Hence by (3) $f_o(\gamma) < 0$ for $\gamma \in H^2(M, \mathbb{R})_F$. Thus the signature of f_o is $(3, b_2(M) - 3)$.

Since $f_o(\varphi + \bar{\varphi}), v(\varphi + \bar{\varphi}) > 0$, we can choose $\gamma \in H^2(M, \mathbb{Q})$ sufficiently near to $\varphi + \bar{\varphi}$ so that $f_o(\gamma) > 0, v(\gamma) > 0$. Note that $v(\gamma)$ is a rational number. Thus by (3) $f := f_o(\gamma)^{-1} f_o$ is $\mathbb{Q}$-valued on $H^2(M, \mathbb{Q})$.

Q.E.D.

Thus $H^2(X, \mathbb{Q})$ of a symplectic Kähler manifold X has the Hodge structure and the quadratic form q_X.

Proposition 3.10. *Let X, Y be compact irreducible symplectic Kähler manifolds. Assume X is bimeromorphic to Y, i.e., there are proper modifications $f\colon Z \to X$ and $g\colon Z \to Y$ of X and Y respectively. Then the bimeromorphic map $h := f \circ g^{-1}\colon Y \cdots\to X$ induces an isomorphism*

$$h^* = g_! \circ f^*\colon H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z}),$$

preserving the Hodge structure and the quadratic forms q_X and q_Y.

Proof. By Hironaka's desingularization theory we may assume Z is smooth. Let E and F be, respectively the exceptional divisors of f and g. Then the canonical bundle K_Z of Z is written as $K_Z = f^*K_X + E$ or $g^*K_Y + F$, where K_X and K_Y are, respectively, the canonical bundles of X and Y, which are both trivial. Thus we have $E = F$ and hence h defines a biholomorphic map of $X - f(E)$ to $Y - g(E)$. Note that both $f(E)$ and $g(E)$ are of codimension ≥ 2. By Lemma 3.11 below it follows that h induces an isomorphism $H^2(X, \mathbb{Q}) \cong H^2(Y, \mathbb{Q})$. Let φ be a holomorphic symplectic form on X. Then $h^*\varphi$ extends holomorphically over the analytic set $g(E)$ of codimension ≥ 2 and defines a symplectic form on Y. Since $h^{2,0}(X) = h^{2,0}(Y) = 1$, this implies that h^* preserves the Hodge structure. Also we have $h^*q_X = q_Y$ since the quadratic forms q_X and q_Y depends only on the symplectic structure ((2) of Theorem 3.9). Q.E.D.

Lemma 3.11. *Let M be a compact complex manifold of dimension n and $V \subset M$ an analytic subset of codimension ≥ 2. Then the inclusion map $M - V \hookrightarrow M$ induces an isomorphism $H^2(M, \mathbb{Z}) \cong H^2(M - V, \mathbb{Z})$.*

Proof. Consider the cohomology exact sequence of a pair $(M, M - V)$:

$$H^2(M, M - V) \to H^2(M) \to H^2(M - V) \to H^2(M, M - V),$$

where the coefficient $\mathbb{Z}$ is understood and omitted. $H^i(M, M - V) \cong H_{2n-i}(V)$ by the Alexander duality theorem; and $H_{2n-i}(V) = 0$ for $i = 2$ or 3 since $\dim_{\mathbb{R}} V \leq 2n - 4$. Thus $H^2(M) \cong H^2(M - V)$. Q.E.D.

§4. Period map and Weil-Peterson metric

Let M be a compact n-dimensional Kähler manifold with $c_1(M)_{\mathbb{R}} = 0$. Let $h^{r,0}(M) := \dim H^0(M, \Omega^r)$. In this section we consider *periods* of holomorphic r-forms. According to the Bogomolov decomposition (see Theorem 1.1) the study of periods is reduced to the following three cases:

(1) $h^{n,0}(M) = 1$, $n = \dim M$, i.e., the canonical bundle of M is trivial;
(2) $h^{2,0}(M) = 1$, the case where M is an irreducible symplectic Kähler manifold;
(3) the case where M is a complex torus.

Starting with a general situation, we will later restrict our attention to the cases (1) and (2).

Polarized family. A pair (M, ω) of a compact Kähler manifold and a Kähler class $\omega \in H^2(M, \mathbb{R})$ is called a *polarized* Kähler manifold. Two polarized Kähler manifolds (M, ω) and (M', ω') are isomorphic if there is a biholomorphic map $f \colon M \to M'$ with $f^*\omega' = \omega$. A *polarized deformation* family $(\mathcal{M} \to T, \sigma)$ of a polarized Kähler manifold (M, ω) is a deformation family $\pi \colon \mathcal{M} \to T$ of M with a section $\sigma \in \Gamma(S, R^2\pi_*\mathbb{R})$ such that 1) $M \cong M_o$, $\omega = \sigma(o)$ for some $o \in T$; and 2) $\omega_t := \sigma(t)$ is a Kähler class on M_t for each $t \in T$. The universal polarized deformation family is defined analogously to the usual deformation. Existence of an universal polarized deformation family of any polarized Kähler manifold (M, ω) follows from the existence of the usual universal deformation family of M.

Theorem 4.1. *The universal polarized deformation space of a polarized Kähler manifold (M, ω) with $c_1(M)_{\mathbb{R}} = 0$ is smooth.*

Let (M, ω) be a polarized Kähler manifold with $c_1(M)_{\mathbb{R}} = 0$. Let $(\pi\colon \mathcal{M} \to T, \sigma)$ be its universal polarized deformation family with $M \cong \pi^{-1}(o)$, $o \in T$. Then T is, as a germ at o, an analytic subset of the universal deformation space S of M. The tangent space of S at o is identified with $H^1(M, \Theta)$; the tangent space of T at o is the linear subspace

$$(4.2) \qquad H^1(M, \Theta)_\omega := \{\theta \in H^1(M, \Theta) \mid \theta \lrcorner \omega = 0\},$$

where the symbol $\lrcorner$ means a product

$$H^1(M, \Theta) \times H^1(M, \Omega^1) \to H^2(M, \mathcal{O})$$

induced by the contraction $\Theta \times \Omega^1 \to \mathcal{O}$.

We note here

Proposition 4.3. *Let $(\mathcal{M} \to T, \sigma)$ be a universal polarized deformation family of a polarized Kähler manifold (M, ω) with $c_1(M)_{\mathbb{R}} = 0$. Let g_t be the (unique) Ricci-flat Kähler metric on M_t whose cohomology class is $\sigma(t)$. Then there is a C^∞ d-closed form Φ on $\mathcal{M}$ such that the restriction of Φ to M_t is the Kähler form of g_t for each $t \in T$. In particular g_t is C^∞ in t.*

Weil-Peterson metric. Let $\mathcal{M} \to T$ be the universal polarized deformation family of a polarized Kähler manifold (M, ω) with $c_1(M)_{\mathbb{R}} = 0$. Then T has a canonical metric called the Weil-Peterson metric, which is defined as follows. Let g be the Ricci-flat Kähler metric whose Kähler class is ω. The tangent space of T at o, $M \cong M_o$, with $H^1(M, \Theta)_\omega$ defined in (4.2). Moreover we identify each element of $H^1(M, \Theta)_\omega$ with its harmonic representative relative to the metric g. The Weil-Peterson metric g_{WP} on T is defined at o by

$$g_{WP}(\theta, \theta') := \int_M \langle \theta, \theta' \rangle dv_g,$$

where $\langle \theta, \theta' \rangle$ is an inner product of the Θ-valued harmonic $(0,1)$-form θ, $\theta' \in H^1(M, \Theta)_\omega$ relative to g and dv_g is the volume form of g.

Period maps. Let $\pi\colon \mathcal{M} \to S$ be a local universal deformation family of M and $M_s := \pi^{-1}(s)$ with $M_o \cong M$. By the unobstructedness theorem (Theorem 2.1 of Section 2) the base space S is smooth. Let $\Psi\colon M \times S \to \mathcal{M}$ be a C^∞-trivialization. Assume $d := h^{r,0}(M) \neq 0$. Then, since M and hence M_s, $s \in S$, are Kähler, every holomorphic r-forms on M_s is d-closed, i.e., $H^0(M_s, \Omega^r) \subset H^r(M_s, \mathbb{C})$ canonically, and $h^{r,0}(M_s) = d$ are constant. Let $Gr_d(V)$ denote the Grassmannian consisting of d-dimensional subspaces of a vector space V. A *period map* $p_r\colon S \to Gr_d(H^r(M, \mathbb{C}))$ for holomorphic r-forms is defined by

$$p_r(s) := \Psi^* H^0(M_s, \Omega^r) \in Gr_d(H^r(M \times \{s\}, \mathbb{C})) = Gr_d(H^r(M, \mathbb{C})).$$

Then p_r is holomorphic, as proved by Griffiths [Gr] (in a more general setting).

Theorem 4.4 (local Torelli). *The period map p_r for holomorphic r-forms of compact Kähler manifolds with $c_{1,\mathbb{R}} = 0$ is locally injective.*

Proof. We shall show that the differential $(p_r)_*$ of p_r at $o \in S$ is injective. Let $\rho\colon T_oS \to H^1(M, \Theta)$ be the Kodaira-Spencer map. The unobstructed theorem (Theorem 2.1 in Section 2) says that this map is an isomorphism. In view of the Hodge decomposition of $H^r(M, \mathbb{C})$, the tangent space of $Gr_d(H^r(M, \mathbb{C}))$ at $p_r(o)$ is identified with

$$\mathrm{Hom}(H^0(M, \Omega^r), H^1(M, \Omega^{r-1}) \oplus \cdots \oplus H^r(M, \mathcal{O})).$$

Then $(p_r)_*(v)$, $v \in T_oS$, is a map $\rho(v) \lrcorner \bullet$ induced by the contraction $\Theta \otimes \Omega^r \to \Omega^{r-1}$.

Now let M equip with a Ricci-flat Kähler metric. Let θ a Θ-valued harmonic $(0, 1)$-form on M and φ be a holomorphic r-form. Then the $(r-1, 1)$-form $\theta \lrcorner \varphi$ obtained by contraction with Θ-component is also harmonic since φ is parallel by Proposition 1.4 in Section 1. Thus the de Rham cohomology class of $\theta \lrcorner \varphi$ does not vanish whenever $\theta \neq 0$ and $\varphi \neq 0$. It follows that $(p_r)_*$ is injective. Q.E.D.

Let $(\mathcal{M} \to T, \sigma)$ be a deformation family of polarized Kähler manifolds (M_t, w_t), $t \in T$. For $r < n := \dim_{\mathbb{C}} M$ let

$$H^r(M_t, \mathbb{R})_0 := \mathrm{Ker}(L_t^{n-r+1} : H^r(M_t, \mathbb{R}) \to H^{2n-r+2}(M_t, \mathbb{R})),$$

where L_t is the multiplication by the cohomology class of ω_t. Namely $H^r(M_t, \mathbb{R})_0$ is the space of primitive cohomology classes of degree r relative to the Kähler class ω_t. Let $\Psi\colon M \times T \to \mathcal{M}$ be a C^∞-trivialization of the family $\mathcal{M} \to T$. Then, identifying $M \times \{t\}$ with M as usual, we have $\omega = \Psi^*(\omega_t)$ and hence $\Psi^* H^r(M_t, \mathbb{R})_0 = H^r(M, \mathbb{R})_0$ for each $t \in T$. Since holomorphic forms always define primitive classes, the period map p_r for the polarized family takes its value in $Gr_d(H^r(M, \mathbb{C})_0)$, $d = h^{r,0}(M)$.

Period domains. From now on we assume that $r = 2$ or $n = \dim M$ and $h^{r,0}(M) = 1$. Then the image of the period map p_r is contained in a certain subset of $\mathbb{P}(H^r(M,\mathbb{C})) = Gr_1(H^r(M,\mathbb{C}))$, which is a bounded symmetric domain of type III and written in general as follows: Let V be a vector space over $\mathbb{R}$ and Q a nondegenerate bilinear form on V. Set $V_{\mathbb{C}} = V \otimes \mathbb{C}$ and

$$(4.5)\quad D(V,Q) = \{\ell \in \mathbb{P}(V_{\mathbb{C}}) \mid Q(\varphi,\bar{\varphi}) > 0,\ Q(\varphi,\varphi) = 0 \quad \text{for } \varphi \in \ell\}.$$

The automorphism group G of $D(V,Q)$ is induced by the linear transformation group of V which preserves Q. Let L be the tautological line bundle over $\mathbb{P}(V_{\mathbb{C}})$. Then Q induces a G-invariant hermitian metric h_Q on $L|_{D(V,Q)}$. The curvature $\mathrm{Ric}(h_Q)$ of h_Q defines the G-invariant Kähler form $\sqrt{-1}\,\mathrm{Ric}(h_Q)$ on $D(V,Q)$.

Suppose M is a symplectic manifold with $h^{2,0}(M) = 1$. Let q be the quadratic form on $H^2(M,\mathbb{R})$ introduced by Beauville [Bea-2] and Fujiki [Fu-3] (see §3) and set $D_2(M) := D(H^2(M,\mathbb{R}),q)$. Then any small deformation M_s of M is also a symplectic Kähler manifold with $h^{2,0}(M_s) = 1$; the period map p_2 takes its image in $D_2(M)$. Moreover we have

Theorem 4.6 [Bea-2]. *Let M be a symplectic Kähler manifold with $h^2(M) = 1$. Let $\mathcal{M} \to S$ be the local universal deformation of M. Then the period map $p_2\colon S \to D_2(M)$ is locally isomorphic.*

Proof. The differential of p_2 is injective by Theorem 4.4. Therefore it suffices to show $\dim S = \dim D_2(M)$. By the unobstructedness theorem (Theorem 2.1), we have $\dim S = \dim H^1(M,\Theta)$. The interior product with the holomorphic symplectic form yields an isomorphism $\Theta \cong \Omega^1$ and hence $H^1(M,\Theta) \cong H^1(M,\Omega^1)$. Since $h^{2,0}(M) = 1$, we have $h^{1,1}(M) = b_2(M) - 2$. On the other hand $D_2(M)$ is an open subset of a hypersurface in $\mathbb{P}(H^2(M,\mathbb{C}))$ and hence $\dim D_2(M) = b_2(M) - 2$. Consequently $\dim S = \dim D_2(M)$. Q.E.D.

For a polarized symplectic Kähler manifold M with $h^{2,0}(M) = 1$, we set $D_2(M)_0 := D(H^2(M,\mathbb{R})_0, q)$. Note that $D_2(M)_0$ with the Kähler form $\sqrt{-1}\,\mathrm{Ric}(h_q)$ is isometric to $SO_0(2, b_2(M)-3)/SO(b_2(M)-3)$ with the invariant metric.

Theorem 4.7 (Schumacher [Sc]). *Let $\mathcal{M} \to T$ be a local universal polarized deformation family of a symplectic Kähler manifolds M. Then $\omega_{WP} = p_2^*\sqrt{-1}\,\mathrm{Ric}(h_q)$, i.e., the period map $p_2\colon T \to D_2(M)_0$ is a local isometry, between the Weil-Peterson metric g_{WP} on T and the invariant metric on $D_2(M)_0$.*

For periods of holomorphic n-forms, we take Q in (4.5) to be the intersection form I on $H^n(M,\mathbb{R})$ and set $D_n(M)_0 := D(H^n(M,\mathbb{R})_0, I)$.

Theorem 4.8 (Tian [Ti]). *Let $\mathcal{M} \to T$ be a local universal polarized deformation family a compact n-dimensional Kähler manifold M with trivial canonical bundle. Then $\omega_{WP} = p_n^* \sqrt{-1}\,\mathrm{Ric}(h_I)$, i.e., the period map $p_n \colon T \to D_n(M)_0$ is an isometric immersion between the Weil-Peterson metric g_{WP} on T and the invariant metric on $D_n(M)_0$.*

Proofs of Theorems 4.7 and 4.8 by Schumacher and Tian respectively go parallel; we only sketch here the proof of Theorem 4.8. Let M be a compact Kähler manifold with trivial canonical bundle. Let (M_t, ω_t), $t \in T$, be a local universal polarized deformation family of compact Kähler manifolds M_t with trivial canonical bundle. Let ψ_t be a holomorphic n-form on M_t which depends on t holomorphically. Fix $o \in T$ and we write (M,ω) and ψ for (M_o, ω_o) and ψ_o respectively. Let Φ be the Kähler form of the Ricci-flat Kähler metric g on M whose cohomology class is ω. Since the Ricci curvature vanishes identically, we have $\psi \wedge \bar{\psi} = a\Phi^n$ for some constant a. We identify $\theta \in H^1(M,\Theta)_\omega$ with the Θ-valued harmonic $(0,1)$-form. Then, since Φ is parallel, $\theta \lrcorner \Phi = 0$ as form. It follows by a direct calculation

$$(\theta \lrcorner \psi) \wedge (\overline{\theta \lrcorner \psi}) = -c_n \|\theta\|^2 \Phi^n,$$

where c_n is a positive constant depends only on n. Therefore

$$g_{WP}(\theta,\theta) = -\int_M (\theta \lrcorner \psi) \wedge (\overline{\theta \lrcorner \psi}) \Big/ \int_M \psi \wedge \bar{\psi}.$$

We compute next $\sqrt{-1}p_n^* \,\mathrm{Ric}(h_I)$. Regarding t as a local coordinate of the local deformation space at o, we assume θ is the image of $\partial/\partial t$ by the Kodaira-Spencer map. Then

$$\sqrt{-1}p_n^* \,\mathrm{Ric}(h_I) = -\partial_t \partial_{\bar{t}} \log \textstyle\int_M \psi_t \wedge \bar{\psi}_t\Big|_{t=o}.$$

We can take a local holomorphic coordinates $(z_t^1, \cdots, z_t^n)$ on M_t depending on t smoothly so that

$$\theta = \sum_\alpha \bar{\partial}(\frac{\partial z^\alpha}{\partial t}) \frac{\partial}{\partial z_t^\alpha}\Big|_{t=o}.$$

Using these coordinates we can compute

$$\xi := \frac{\partial \psi_t}{\partial t}\Big|_{t=o} = \psi' + \theta \lrcorner \psi_o,$$

where ψ' is a $(n, 0)$-form. Moreover the left hand side of the above is d-closed; $h \lrcorner \psi_o$ is also d-closed since ψ_o is parallel and θ is harmonic. Therefore ψ' is a constant multiple of ψ_o. It follows

$$I(\psi_o, \bar{\psi}_o)I(\xi, \bar{\xi}) - I(v_o, \bar{\xi})I(\xi, \bar{\psi}_o) = I(\psi_o, \bar{\psi}_o)I(\theta \lrcorner, \overline{\theta \lrcorner \psi_o}).$$

Combining these all together we obtain

$$g_{WP}(\theta, \theta) = \sqrt{-1}p_n^*(\mathrm{Ric}(h_I))(\theta, \bar{\theta}).$$

§5. Examples

In this section we discuss constructions of compact irreducible symplectic Kähler manifolds. A 2-dimensional irreducible symplectic Kähler manifold is a K3 surface, which has a long history of study (cf. [BPV]). Recall that symplectic Kähler manifolds have even (complex) dimensions. Four dimensional examples were discovered by A. Fujiki. Generalizing Fujiki's construction, Beauville [Bea-2] gave two series of examples for each even dimension as follows.

The symplectic manifolds $S^{[r]}$. Let S be a compact complex surface (i.e., complex 2-dimensional manifold). Let $\mathrm{Sym}^r S := S^r/\mathfrak{S}_r$ be the r-th symmetric product of S, where the symmetric group $\mathfrak{S}_r$ acts on S^r as permutaion of components. Let $\pi\colon S^r \to \mathrm{Sym}^r S$ be the quotient map. Let Δ be the set of points $(x_1, \cdots, x_r) \in S^r$ such that at least two components are equal; set $D := \pi(\Delta)$. Then $\mathrm{Sym}^r S$, whose singular points set is D, has a desingularization $\epsilon\colon S^{[r]} \to \mathrm{Sym}^r S$ such that $E := \epsilon^{-1}(D)$ is an irreducible divisor. (In fact $S^{[r]}$ is the Douady space which parametrizes all 0-dimensional analytic subspaces $Z \subset S$ with $lg(\mathcal{O}_Z) = r$. See [Fo-1], [Fo-2], [Ia] for details.) If S is Kählerian, then so is $S^{[r]}$. In fact, if S is Kählerian, then $\mathrm{Sym}^r S$ is a Kähler space according to Varachus [Va]. Moreover any monoidal transformation of a Kähler space is again a Kähler space by Campana [Cam].

Proposition 5.1. *$S^{[r]}$ has a symplectic holomorphic 2-form provided that the canonical bundle of S is trivial.*

Proof. Let $\Delta_3 \subset \Delta$ be the set points $(x_1, \cdots, x_r) \in S^r$ such that at least three components are equal. Set $S_*^{(r)} := S^{(r)} - \pi(\Delta_3)$, $D_* := D - \pi(\Delta_3)$ and $S_*^{[r]} := S^{[r]} - \epsilon^{-1}\pi(\Delta_3)$. Then, since $E = \epsilon^{-1}(D)$ is irreducible, $\epsilon^{-1}\pi(\Delta_3)$ is of codimension 2 in $S^{[r]}$. Therefore it suffices to show that $S_*^{(r)}$ has a holomorphic symplectic 2-form.

The diagonal set $\Delta - \Delta_3$ is smooth of codimension 2 on S^r_*. Let $\mu\colon Q_\Delta(S^r_*) \to S^r_*$ be the monoidal transformation of S^r_* along $\Delta - \Delta_3$. The action of $\mathfrak{S}_r$ extends to $Q_\Delta(S^r_*)$ and $S^{[r]}_*$ is identified with the quotient of $Q_\Delta(S^r_*)$ by $\mathfrak{S}_r$; let $\varpi\colon Q_\Delta(S^r_*) \to S^{[r]}_*$ be the quotient map. Thus we have a commutative diagram:

$$\begin{array}{ccc} Q_\Delta(S^r_*) & \xrightarrow{\mu} & S^r_* \\ \big\downarrow{\scriptstyle \varpi} & & \big\downarrow{\scriptstyle \pi} \\ S^{[r]}_* & \xrightarrow{\epsilon} & S^{(r)}_* \end{array}$$

Let $\mathrm{pr}_r\colon S^r \to S$ be the projection to the r-th component and let φ be a non-zero holomorphic 2-form on S. Then $\psi_0 := \mathrm{pr}_1^*\varphi + \cdots + \mathrm{pr}_r^*\varphi$ is a symplectic 2-form on S^r; and $\mu^*\psi_0$ is invariant under the action of $\mathfrak{S}_r$ on $Q_\Delta(S^r_*)$. The action of $\mathfrak{S}_r$ on $Q_\Delta(S^r_* - \Delta)$ is free from fixed points. If $g \in \mathfrak{S}_r$ fixes a point $p \in Q_\Delta(S^r_*)$, the tangent space of $Q_\Delta(S^r_*)$ at p decomposes into a direct sum of (± 1)-eigen space; the (-1)-eigen space is one dimensional and the differential ϖ_* of ϖ is injective on the $(+1)$-eigen space. It follows that ψ_0 induces a holomorphic 2-form ψ on $S^{[r]}_*$ such that $\varpi^*\psi = \mu^*\psi_0$. The quotient map ϖ is ramified along E with local ramification index 2. Therefore

$$\mathrm{Zero}(\varpi^*\psi^r) = \varpi^*\,\mathrm{Zero}(\psi^r) + E,$$

where Zero means a zero divisor with multiplicity. On the other hand, since E is an exceptional divisor of μ,

$$\mathrm{Zero}(\varpi^*\psi^r) = \mathrm{Zero}(\mu^*\psi_0^r) + E.$$

Thus ψ^r vanishes nowhere. Q.E.D.

There are two kinds of compact Kähler surfaces with trivial canonical bundle: K3 surfaces and complex 2-dimensional tori. For K3 surfaces we have

Proposition 5.2. *Let S be a K3 surface. Then $S^{[r]}$ is a simply connected irreducible symplectic Kähler manifold. There is an injective homomorphism $i\colon H^2(S,\mathbb{C}) \to H^2(S^{[r]},\mathbb{C})$, compatible to the Hodge structure, such that*

$$H^2(S^{[r]},\mathbb{C}) = i(H^2(S,\mathbb{C})) \oplus \mathbb{C}\cdot[E].$$

For $\alpha \in H^2(S, \mathbb{C})$ we have $i(\alpha) = \mu^*\beta$, where $\beta \in H^2(S^{[r]}, \mathbb{C})$ satisfies $\pi^*\beta = \sum_i \mathrm{pr}_i^* \alpha$ with the notation above.

The manifold $Km^r(T)$. Let T be a 2-dimensional complex torus. Then $T^{[r+1]}$ is a symplectic Kähler manifold but it is not simply connected. In fact $\pi_1(T^{[r+1]}) \cong \pi_1(T)$; and the Albanese torus of $T^{[r+1]}$ is isomorphic to T and the Albanese map $\alpha\colon T^{[r+1]} \to T$ is induced by the map $(x_1, \cdots, x_{r+1}) \in T^{r+1} \mapsto \sum_i x_i \in T$. Moreover, by the Bogomolov decomposition theorem (Theorem 1.1), $\alpha\colon T^{[r+1]} \to T$ is a holomorphic fiber bundle with structure group finite. Let $Km^r(T)$ denote the typical fiber of α.

Proposition 5.3. *$Km^r(T)$ is a simply connected irreducible symplectic Kähler manifold.*

Deformations of $S^{[r]}$ and $Km^r(T)$. By deformation we have irreducible symplectic Kähler manifolds which are neither $S^{[r]}$ nor $Km^r(T)$. In fact Beauville [Bea-2] showed the following:

Theorem 5.4. *Let S be a K3 surface. Then the local universal deformation space V of $S^{[r]}$ is of dimension 21. Each point of V corresponds to an irreducible symplectic Kähler manifold; points corresponding to the manifolds of type $X^{[r]}$ with X a K3 surface form a countable union of smooth hypersurfaces on V.*

Theorem 5.5. *Let T be a complex torus of dimension 2. Then the local universal deformation space V of $Km^r(T)$ is of dimension 5. Each point of V corresponds to an irreducible symplectic Kähler manifold; points corresponding to the manifolds of type $Km^r(T')$ with T' a 2-dimensional complex torus form a countable union of smooth hypersurfaces on V.*

Elementary transformation. Although irreducible symplectic Kähler manifolds enjoy similar properties as K3 surfaces, there are phenomena peculiar to higher dimensional manifolds. For example Mukai [Mu-1] found

Theorem 5.6. *Let X be a symplectic manifold of dimension $2n \geq 4$ which contains a submanifold Y isomorphic to the n-dimensional projective space $\mathbb{P}^n$. Then there are a symplectic manifold $X^\vee$ with a submanifold $Y^\vee \cong \mathbb{P}^n$ and a bimeromorphic map $f\colon X \cdots\to X^\vee$ such that f does not define a holomorphic map on Y but it induces a biholomorphic map $X - Y \to X^\vee - Y^\vee$.*

$X^\vee$ is called an *elementary transformation* of X along Y; and the construction goes as follows.

A n-dimensional complex submanifold Y of a $2n$-dimensional complex symplectic manifold X with symplectic form φ is called *Lagrangean* if $\iota_Y^*\varphi = 0$ on Y, where $\iota_Y : Y \to X$ is the inclusion map. The bundle isomorphism $\varphi \llcorner : TX \cong T^*X$ induces an isomorphism $TY \cong N^*_{X/Y}$, where $N_{X/Y}$ is the normal bundle of Y in X and $N^*_{X/Y}$ is the dual bundle of $N_{X/Y}$. Assume now $Y \cong \mathbb{P}^n$. Let $\mu\colon X^\square \to X$ be the monoidal transformation of X along Y and let $Y^\square := \mu^{-1}(Y)$. Then $Y^\square$ is isomorphic to the projectification $\mathbb{P}(N_{X/Y})$ of $N_{X/Y}$. Since $N_{X/Y} \cong T^*Y$, it follows $Y^\square \cong \mathbb{P}(T^*Y)$. Let $Y^\vee$ denote the projective space dual to $Y = \mathbb{P}^n$, namely Y parametrizes complex lines ℓ on $\mathbb{C}^{n+1}$ while $Y^\vee$ parametrizes hyperplanes. Then we have

$$\mathbb{P}(T^*Y) \cong \{(\ell, H) \in Y \times Y^\vee \mid \ell \subset H\}.$$

This means that $Y^\square$ admits a $\mathbb{P}^{n-1}$-bundle structure $\nu\colon Y^\square \to Y^\vee$. Moreover we can blow down $X^\square$ onto a complex manifold $X^\vee$ along the fibers of ν, namely ν extends to $X^\square \to X^\vee$ so that $\nu(Y^\square) = Y^\vee$. The elementary transformation f is given by $\nu \circ \mu^{-1}$. Since $Y^\vee$ has codimension $n \geq 2$ in $Y^\vee$, the 2-form $f^*\varphi$ on $X^\vee - Y^\vee$ extends to a holomorphic symplectic form on $Y^\vee$.

Counterexample to Torelli. For 2-dimensional compact irreducible symplectic Kähler manifolds, i.e., for K3 surfaces, the Torelli theorem holds:

Theorem 5.7 (Pjateckiĭ-Šapiro and Šafarevič [PS],[BR], [LP]). *Two K3 surfaces S, S' are biholomorphic if and only if there is an isomorphism $h\colon H^2(S', \mathbb{Z}) \to H^2(S, \mathbb{Z})$ preserving the Hodge structure and the quadratic forms $q_{S'}$, q_S.*

For the higher dimensional case, however, this type of a theorem does not hold in a biholomorphic level. In fact Debarre [De] gave an example:

Proposition 5.8. *There is a K3 surface S such that*

1) *$X := S^{[n]}$ admits an elementary transformation $h\colon X \cdots\to Y$;*

2) *Y is an irreducible symplectic Kähler manifold not biholomorphic to X.*

By Proposition 3.10, h induces an isomorphism $H^2(Y, \mathbb{Z}) \cong H^2(X, \mathbb{Z})$ which preserves the Hodge structure and the quadratic forms q_X and q_Y.

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