Advanced Studies in Pure Mathematics 18-I, 1990
Recent Topics in Differential and Analytic Geometry
pp. 397–416

Compactification of Moduli Spaces of Einstein-Hermitian Connections for Null-Correlation Bundles

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§0. Introduction

In 1970's by an effective use of twistor theory originated from Penrose [P], gauge-theoretic studies of anti-self-dual connections over 4-manifolds were inaugurated by Atiyah, Hitchin and Singer (see for instance [A-H-S], [A-J], [A-W]). Almost at the same time, Hartshorne determined the moduli spaces of anti-self-dual connections for SU(2)-bundles over S^4 through a purely algebraic study of the null-correlation bundles over $\mathbb{P}^3(\mathbb{C})$. A little later, Kobayashi [K] introduced the concept of Einstein-Hermitian vector bundles over Kähler manifolds, which is in some sense a higher dimensional analogue of anti-self-dual connections over 4-manifolds (see for instance Kobayashi [K] for a general theory of Einstein-Hermitian connections).

The purpose of this paper is to construct a compactified family of Einstein-Hermitian connections on null-correlation bundles over odd-dimensional complex projective spaces $\mathbb{P}^{2m+1}(\mathbb{C})$. Let $\mathbb{P}^m(\mathbb{H}) = \mathrm{Sp}(m+1)/\mathrm{Sp}(m) \times \mathrm{Sp}(1)$ be the m-dimensional quaternionic projective space, and $p : \mathbb{P}^{2m+1}(\mathbb{C}) \to \mathbb{P}^m(\mathbb{H})$ the corresponding twistor space. The homogeneous space $\mathrm{Sp}(m+1)/\mathrm{id} \times \mathrm{Sp}(1)$ is a principal fibre bundle over $\mathbb{P}^m(\mathbb{H})$ with typical fibre $\mathrm{Sp}(m)$. Let τ be the standard representation of $\mathrm{Sp}(m)$ in $\mathbb{C}^{2m}$. Then $V := (\mathrm{Sp}(m+1)/\mathrm{id} \times \mathrm{Sp}(1)) \times_\tau \mathbb{C}^{2m}$ is a complex vector bundle over $\mathbb{P}^m(\mathbb{H})$. Since $\mathrm{Sp}(m)$ is contained on $\mathrm{U}(2m)$, the vector bundle V carries a natural Hermitian metric h_V. Salamon introduced in [S] a certain type of connections (which we call B_2-connections) on vector bundles over quaternionic Kähler manifolds, and such connections are later studied by Berard-Bergery and Ochiai [B-O] in a more general setting. We showed that B_2-connections are Yang-Mills connections and studied them in [N1], which is also obtained by Capria and

Received April 21, 1989.

Salamon independently. They constructed an interesting family of Yang-Mills connections for the vector bundle (V, h_V) parametrized roughly by $\mathrm{SL}(m+1, \mathbb{H})/\mathrm{Sp}(m+1)$. By generalizing the Penrose twistor correspondence to higher dimensional quaternionic Kähler manifolds, we obtained the following:

Theorem ([N2]). *The moduli space of B_2-connections on (V, h_V) is imbedded as a totally real submanifold of the moduli space of Einstein-Hermitian connections on (p^*V, p^*h_V).*

This theorem allows us to construct a family of Einstein-Hermitian connections on (p^*V, p^*h_V) parametrized by $\mathrm{PGL}(2m+2, \mathbb{C})/\mathrm{PSp}(m+1, \mathbb{C})$ (cf. Section 1). Thus, we obtained a mapping ψ of $\mathrm{PGL}(2m+2, \mathbb{C})/\mathrm{PSp}(m+1, \mathbb{C})$ to the moduli space of Einstein-Hermitian connectoins for (p^*V, p^*h_V). This mapping ψ is regarded as a complexification of the map constructed by Capria and Salamon, and moreover we obtain (cf. Section 2):

Theorem. *The mapping ψ is injective.*

On the other hand, $\mathrm{PGL}(2m+2, \mathbb{C})/\mathrm{PSp}(m+1, \mathbb{C})$ can be embedded as an open dense subset of $\mathbb{P}^l(\mathbb{C})$ (where $l = m(2m+3)$). Let $\mathcal{L}(p^*V, p^*h_V)$ be the set of Einstein-Hermitian connections for (p^*V, p^*h_V) possibly with singularities, and consider the unitary gauge transformation group $\mathcal{G}(p^*V, p^*h_V)$ consisting of all bundle automorphisms on p^*V preserving p^*h_V. Then we define an equivalence relation on $\mathcal{L}(p^*V, p^*h_V)$ as follows: for $\nabla_1, \nabla_2 \in \mathcal{L}(p^*V, p^*h_V)$, we say that ∇_1 is equivalent to ∇_2 if there is a gauge transformation $s \in \mathcal{G}(p^*V, p^*h_V)$ such that $s^*\nabla_1 = \nabla_2$ off the singular sets. We denote the resulting set of equivalence class by $\mathcal{L}(p^*V, p^*h_V)/\mathcal{G}(p^*V, p^*h_V)$. In Section 4, we extend ψ to a mapping $\tilde{\psi}$ from $\mathbb{P}^l(\mathbb{C})$ to $\mathcal{L}(p^*V, p^*h_V)/\mathcal{G}(p^*V, p^*h_V)$, which gives us a compactification of the image $\psi(\mathrm{PGL}(2m+2, \mathbb{C})/\mathrm{PSp}(m+1, \mathbb{C}))$. Furthermore, we have:

Theorem. *The family of Yang-Mills connections constructed by Capria and Salamon is realized as a connected component of the moduli space of B_2-connections on (V, h_V).*

Finally, the author wishes to thank all those people who encouraged him and gave him suggestions, and in particular Professors H. Ozeki, M. Takeuchi, S. Murakami, M. Itoh, T. Mabuchi, I. Enoki and also Doctors M. Furuta and H. Nakajima. This work was completed when the author stayed at the Max-Plank-Institute für Mathematik at Bonn, to which he also wishes to thank for the support and hospitality.

§1. Notation, conventions and preliminaries

For this section, we refer to [C-S], [N1] and [N2].

(1.1.1) The quaternionic projective space $\mathbb{P}^m(\mathbb{H})$ is the set of all quaternionic lines through 0 sitting in the right $\mathbb{H}$-module $\mathbb{H}^{m+1}$. In this paper we make use of column vectors in order to describe elements in vector space over $\mathbb{C}$ or $\mathbb{H}$. Thus $\mathbb{P}^m(\mathbb{H}) = \{(u) | u = {}^t(u^0, \cdots, u^m) \in \mathbb{H}^{m+1} - \{0\}\}$, where (u) means the quaternionic line including a vector u $(\in \mathbb{H}^{m+1})$. Recall that $\mathbb{P}^m(\mathbb{H})$ has a natural quaternionic Kähler structure. The right $\mathbb{H}$-module $\mathbb{H}^{m+1}$ has a standard quaternionic Hermitian inner product $h_{\mathbb{H}^{m+1}}(u, v) = {}^t\bar{u}v$ $(u, v \in \mathbb{H}^{m+1})$, which induces the quaternionic Hermitian metric h_0 on the trivial vector bundle $F := \mathbb{P}^m(\mathbb{H}) \times \mathbb{H}^{m+1}$. Let V be the quaternionic vector subbundle of $\mathbb{P}^m(\mathbb{H}) \times \mathbb{H}^{m+1}$ such that each fibre $V_{(u)}$ over $(u)(\in \mathbb{P}^m(\mathbb{H}))$ is the orthogonal complement of the quaternionic line (u) with respect to h_0. The restriction of h_0 on V is denoted by h_V.

(1.1.2) When $\mathbb{H}^{m+1}$ is identified with $\mathbb{C}^{2m+2}$ by the isomorphism which sends each $u_1 + ju_2 \in \mathbb{H}^{m+1}$ to $(u_1, u_2) \in \mathbb{C}^{2m+2}$, we regard V and h_V as a complex vector bundle and a (complex) Hermitian metric respectively. The vector bundle $\wedge^2 \mathrm{T}^*(\mathbb{P}^m(\mathbb{H}))$ of covectors of degree 2 is expressed as a direct sum of three holonomy invariant vector subbundles A_2', A_2'' and B_2 (cf. [N1]). A Hermitian connection ∇ on (V, h_V) is called a B_2-connection, if the curvature R^∇ of ∇ is an End(V)-valued B_2-form. Let ∇ be a B_2-connection on (V, h_V). Then ∇ induces elliptic complexes $C_\nabla = \{(A^i, d_i)\}$ and $\tilde{C}_\nabla = \{(\tilde{A}^i, \tilde{d}_i)\}$ (see [N2;(2.1)] for definition of C_∇ and $\tilde{C}_\nabla$).

(1.1.3) Let $\mathcal{C}_B'(V, h_V)$ be the set of all irreducible B_2-connections ∇ on (V, h_V) where ∇ is said to be irreducible if $\mathrm{H}^0(\mathbb{P}^m(\mathbb{H}), \tilde{C}_\nabla) = \{0\}$. We denote by $\mathcal{B}'(V, h_V)$ the quotient space of $\mathcal{C}_B'(V, h_V)$ by the unitary gauge transformation group $\mathcal{G}(V, h_V)$, and $\mathcal{B}'(V, h_V)$ is often called the moduli space of irreducible Hermitian B_2-connections on (V, h_V). Furthermore, let $\mathcal{C}_B''(V, h_V)$ be the set of all irreducible Hermitian B_2-connections ∇ on (V, h_V) such that $\mathrm{H}^2(\mathbb{P}^m(\mathbb{H}), \tilde{C}_\nabla) = \{0\}$. We then put $\mathcal{B}''(V, h_V) := \mathcal{C}_B''(V, h_V)/\mathcal{G}(V, h_V)$. It is known that $\mathcal{B}''(V, h_V)$ has a natural structure of Riemannian manifold. For examples of Hermitian B_2-connections, see Capria and Salamon [C-S]. Let $M(l, k; \mathbb{H})$ be the set of all quaternionic valued (l, k) matrices. We now set:

$$\mathcal{H} := \{H \in M(l, k; \mathbb{H}) - \{0\} | {}^t\bar{H} = H\},$$

$$\mathcal{H}_0 := \mathcal{H} \cap \mathrm{GL}(m+1, \mathbb{H}).$$

We say that $H_1, H_2 (\in \mathcal{H})$ are equivalent if there exists an element a ($\in \mathbb{R}^*$) such that $H_1 = aH_2$. We write the equivalence class of H ($\in \mathcal{H}$) as $\widetilde{H}$ and the set of all $\widetilde{H}$ ($H \in \mathcal{H}_0$) as $\widetilde{\mathcal{H}}_0$. Now the Lie group $\mathrm{SL}(m+1, \mathbb{H})$ transitively acts on $\widetilde{\mathcal{H}}_0$, which is just $\mathrm{SL}(m+1, \mathbb{H}) / \mathrm{Sp}(m+1)$.

(1.1.4) To each $H \in \mathcal{H}_0$, we associate a quaternionic vector subbundle $W(H)$ of the trivial bundle $F = \mathbb{P}^m(\mathbb{H}) \times \mathbb{H}^{m+1}$ by

$$W(H)_{(u)} = \{v \in \mathbb{H}^{m+1} | {}^t\bar{v}Hu = 0\}, \ (u) \in \mathbb{P}^m(\mathbb{H}),$$

where $W(H)_{(u)}$ denotes the fibre of $W(H)$ over (u). Then given $\widetilde{H} \in \widetilde{\mathcal{H}}_0$, one sees that $W(H)$ is independent of the choice of representations H for $\widetilde{H}$. Let $h(H)$ be the quaternionic Hermitian metric on $W(H)$ induced from the standard quaternionic Hermitian metric on the trivial bundle F. The flat connection d of the vector bundle F over $\mathbb{P}^m(\mathbb{H})$ naturally induces a connection $\nabla(H)$ on $W(H)$

$$\nabla(H) = P(H) \circ d,$$

where $P(H) : F \to W(H)$ denotes the fibrewise orthogonal projection of the vector bundle F onto $W(H)$ over $\mathbb{P}^m(\mathbb{H})$. Then the connection $\nabla(H)$ is compatible with the quaternionic Hermitian metric $h(H)$ on $W(H)$, and the corresponding holonomy group is $\mathrm{Sp}(m)$. Especially, $\nabla(H)$ is irreducible.

(1.1.5) Since $\widetilde{\mathcal{H}}_0$ is connected, the vector bundle $W(H)$ ($H \in \mathcal{H}_0$) is isomorphic to V ($= W(\mathrm{id}_{\mathbb{H}^{m+1}})$) as quaternionic vector bundles. We now note that $\mathrm{Sp}(m)$ is a maximal compact subgroup of $\mathrm{GL}(m, \mathbb{H})$. Hence, for each $H \in \mathcal{H}_0$ there exists a quaternionic isomorphism

$$t_0(H) : (W(\mathrm{id}), h(\mathrm{id})) \xrightarrow{\sim} (W(H), h(H))$$

preserving the Hermitian structure. The resulting pull-back connection

$$D(H) = t_0(H)^* \nabla(H) := t_0(H) \circ \nabla(H) \circ t_0(H)^{-1}$$

is a quaternionic connection on (V, h_V). By identifying $\mathbb{H}^{m+1}$ with $\mathbb{C}^{2m+2}$ we regard $D(H)$ as a Hermitian connection on the complex Hermitian vector bundle (V, h_V). Recall the following result of Capria and Salamon:

Theorem ([C-S]). *For each $H \in \mathcal{H}_0$ the Hermitian connection $D(H)$ is an irreducible B_2-connection on the complex vector bundle (V, h_V).*

(1.1.6) The equivalence class $[D(H)]$ of $D(H)$ modulo the unitary gauge transformation group $\mathcal{G}(V, h_V)$ depends only on $\tilde{H} \in \tilde{\mathcal{H}}_0$ and is independent of the choice of vector bundle isomorphism $t_0(H)$ as above. We then have the mapping

$$\varphi : \widetilde{\mathcal{H}}_0 \ni \widetilde{H} \mapsto [D(H)] \in \mathcal{B}''(V, h_V).$$

(1.2.1) The twistor space corresponding to $\mathbb{P}^m(\mathbb{H})$ is

$$p : \mathbb{P}^{2m+1}(\mathbb{C}) \ni [z] \to (z) \in \mathbb{P}^m(\mathbb{H}),$$

where $[z]$ denotes the complex line including a vector z ($z \in \mathbb{C}^{2m+2} \simeq \mathbb{H}^{m+1}$). The pull-back (p^*V, p^*h_V) over $\mathbb{P}^{2m+1}(\mathbb{C})$ is a Hermitian vector bundle with vanishing first Chern class. A Hermitian connection ∇ on (p^*V, p^*h_V) is an Einstein-Hermitian connection if and only if the corresponding Ricci-curvature is a constant multiple of identity. Since the first Chern class of p^*V is zero, the constant is equal to zero.

(1.2.2) Take an Einstein-Hermitian connection ∇ on (p^*V, p^*h_V). Then ∇ induces elliptic complexes A_∇ and $\tilde{B}_\nabla$ defined by Itoh and Kim (see [N2;(2.1)] for definition of A_∇ and $\tilde{B}_\nabla$). Let $\mathcal{C}_E(p^*V, p^*h_V)$ be the set of all Einstein-Hermitian connections on (p^*V, p^*h_V). Moreover, let $\mathcal{C}'_E(p^*V, p^*h_V)$ be the set of all irreducible Einstein-Hermitian connections ∇ on (p^*V, p^*h_V) where ∇ is said to be irreducible if $\mathrm{H}^0(\mathbb{P}^{2m+1}(\mathbb{C}), \tilde{A}_\nabla) = \{0\}$. We denote by $\mathcal{E}(p^*V, p^*h_V)$ and $\mathcal{E}'(p^*V, p^*h_V)$ the quotient space of $\mathcal{C}_E(p^*V, p^*h_V)$ and $\mathcal{C}'_E(p^*V, p^*h_V)$ by the unitary gauge transformation group $\mathcal{G}(p^*V, p^*h_V)$. The quotient space $\mathcal{E}'(p^*V, p^*h_V)$ is often called the moduli space of irreducible Einstein-Hermitian connections on (p^*V, p^*h_V). Furthermore, let $\mathcal{C}''_E(p^*V, p^*h_V)$ be the set of irreducible Einstein-Hermitian connections ∇ on (p^*V, p^*h_V) such that $\mathrm{H}^2(\mathbb{P}^{2m+1}(\mathbb{C}), \tilde{B}_\nabla) = \{0\}$. We then put

$$\mathcal{E}''(p^*V, p^*h_V) := \mathcal{C}''_B(p^*V, p^*h_V)/\mathcal{G}(p^*V, p^*h_V).$$

It is known that $\mathcal{E}''(V, h_V)$ has a natural structure of Kähler manifold (cf. [I], [K]).

(1.3) The pull-back $\nabla \mapsto p^*\nabla$ of connections induces an imbedding $p^* : \mathcal{B}'(V, h_V) \to \mathcal{E}'(p^*V, p^*h_V)$ ($p^* : \mathcal{B}''(V, h_V) \to \mathcal{E}''(p^*V, p^*h_V)$). Furthermore we obtained:

Theorem ([N2]). *The embedding $p^*: \mathcal{B}''(V, h_V) \hookrightarrow \mathcal{E}''(p^*V, p^*h_V)$ is totally real, (i.e., $\mathcal{B}''(V, h_V)$ is embedded in $\mathcal{E}''(p^*V, p^*h_V)$ by p^* as a totally real submanifold).*

§2. Construction of Einstein-Hermitian connections

In this section, we construct a family of Einstein-Hermitian connections on the Hermitian vector bundle (p^*V, p^*h_V) over $\mathbb{P}^{2m+1}(\mathbb{C})$. It will be shown that connections constructed here are parametrized by symplectic structures on $\mathbb{C}^{2m+2}$ i.e., we shall obtain a mapping of the set of all symplectic structures of $\mathbb{C}^{2m+2}$ onto a family of Einstein-Hermitian connections on (p^*V, p^*h_V).

(2.1.1) Let $M(k; \mathbb{C})$ be the set of complex-valued square matrices of degree k. A complex-valued skew-symmetric matrix $S \in M(2m+2; \mathbb{C})$ induces a skew-symmetric bilinear form on $\mathbb{C}^{2m+2}$ by

$$S(\xi, \eta) = {}^t\xi S \eta, \quad (\xi, \eta \in \mathbb{C}^{2m+2}).$$

Then this bilinear form is non-degenerate if and only if the matrix S is of full rank. We identify each S with the corresponding bilinear form defined as above, when no confusion is likely to occur.

(2.1.2) We put

$$\mathfrak{S} := \{0 \neq S \in M(2m+2; \mathbb{C}) \mid S \text{ is skew-symmetric}\},$$
$$\mathcal{S} := \{S \in \mathfrak{S} \mid S \text{ is non-degenerate}\}.$$

Then $\mathbb{C}^*$ naturally acts on $\mathfrak{S}$ by

$$\mathbb{C}^* \times \mathfrak{S} \ni (c, S) \mapsto cS \in \mathfrak{S}.$$

Note that this $\mathbb{C}^*$-action preserves the subset $\mathcal{S}$ of $\mathfrak{S}$. We now define:

$$\widetilde{\mathfrak{S}} := \mathfrak{S}/\mathbb{C}^*,$$
$$\widetilde{\mathcal{S}} := \mathcal{S}/\mathbb{C}^*.$$

For each $S \in \mathfrak{S}$, we denote by $\widetilde{S}$ the corresponding element of $\widetilde{\mathfrak{S}}$. Then it is easily seen that $\widetilde{\mathcal{S}}$ is nothing but $\mathrm{PGL}(2m+2, \mathbb{C}) / \mathrm{PSp}(m+1, \mathbb{C})$.

(2.2.1) Recall that the vector bundle p^*F is the trivial bundle $\mathbb{P}^{2m+1}(\mathbb{C}) \times \mathbb{C}^{2m+2}$ over $\mathbb{P}^{2m+1}(\mathbb{C})$. For $\widetilde{S} \in \mathcal{S}$, we define a complex subbundle $V(S)$ of p^*F such that the fibre $V(S)_{[z]}$ over $[z] \in \mathbb{P}^{2m+1}(\mathbb{C})$ is the vector subspace $\{y \in \mathbb{C}^{2m+2} \mid {}^t y S z = 0, \ {}^t\overline{y}({}^t\overline{S}S)^{1/2} z = 0\}$ of

$\mathbb{C}^{2m+2}$. Since the two vectors $\overline{Sz}$ and $({}^t\overline{S}S)^{1/2}z$ are orthogonal, $V(S)$ is a complex vector bundle of rank 2m. Note that $V(S) = V(S')$ whenever $\widetilde{S} = \widetilde{S}'$.

(2.2.2) Let $k(S)$ be the Hermitian metric on $V(S)$ induced from the standard Hermitian metric on p^*F. Then the flat connection d on the trivial bundle p^*F induces a Hermitian connection $\nabla(S)$ on $V(S)$ by

$$\nabla(S) = Q(S) \circ d,$$

where $Q(S)$ denotes the orthogonal projection of p^*F onto $V(S)$. We then obtain:

Theorem 2.2.3. *For each S, the Hermitian connection $\nabla(S) = \nabla$ is an Einstein-Hermitian connection on $(V(S), k(S))$.*

Proof. Let $N(S)$ be the vector subbundle of p^*F obtained as the orthogonal complement of $V(S)$ in p^*F. We denote by $\widetilde{Q} = \widetilde{Q(S)}$ the orthogonal projection of p^*F onto $N(S)$. Put $H = ({}^t\overline{S}S)^{1/2}$. For $z \in \mathbb{C}^{2m+2}$, let A be the $(2m+2, 2)$-matrix consisting of two column vectors Hz and $\overline{Sz}$. Then the projection $\widetilde{Q}$ is written as follows

$$\widetilde{Q} = A({}^t\overline{A}A)^{-1}\ {}^t\overline{A}, \tag{1}$$

at $[z] \in \mathbb{P}^{2m+1}(\mathbb{C})$. For a section $f \in \Gamma(\mathbb{P}^{2m+1}(\mathbb{C}), \mathrm{C}^\infty(V(S)))$,

$$\begin{aligned} \nabla f &= (\mathrm{id} - \widetilde{Q})(df) \\ &= df + d(\widetilde{Q})f, \end{aligned}$$

since $\widetilde{Q}f = 0$. The curvature $R = R(S)$ for ∇ is given by

$$\begin{aligned} R &= (d + d\widetilde{Q}) \circ (d + d\widetilde{Q}) \\ &= d\widetilde{Q} \wedge d\widetilde{Q}. \end{aligned}$$

More precisely, $R = Q(d\widetilde{Q} \wedge d\widetilde{Q})Q$, where we denote $Q(S)$ by Q for simplicity. Since

$$Q(Hz, \overline{Sz}) = 0 \text{ and } {}^t(\overline{Hz}, Sz)Q = 0,$$

we obtain from (1) the expression:

$$R = QdA({}^t\overline{A}A)^{-1}\ {}^t\overline{(dA)}Q, \tag{2}$$

where $dA = (Hdz, \overline{Sdz})$. Moreover,

$$\text{(3)} \qquad {}^t\overline{A}A = \begin{pmatrix} |Hz|^2 & 0 \\ 0 & |Sz|^2 \end{pmatrix}.$$

By (2) and (3),

$$\text{(4)} \qquad \begin{aligned} R &= (\det({}^t\overline{A}A))^{-1} Q dA \begin{pmatrix} |Sz|^2 & 0 \\ 0 & |Hz|^2 \end{pmatrix} {}^t\overline{(dA)}Q \\ &= \frac{Q\{|Sz|^2 Hdz \wedge {}^t\overline{dz}{}^t\overline{H} + |Hz|^2 \overline{Sdz} \wedge {}^t dz {}^t S\} Q}{\det({}^t\overline{A}A)}. \end{aligned}$$

Hence, R is an End$(V(S))$-valued (1,1)-form. Hence ∇ is a Hermitian connection of type (1,0) on $(V(S), k(S))$. Secondly, we shall calculate the Ricci curvature $\gamma(S) = \gamma$ for ∇. Let ω be the Fubini-Study form on $\mathbb{P}^{2m+1}(\mathbb{C})$. Recall that the corresponding Kähler operator

$$L: \{p\text{-forms}\} \to \{(p+2)\text{-forms}\} \qquad 0 \leqq p \leqq 2(2m+1)$$

is defined by $L(\eta) := \omega \wedge \eta$ for a p-form η on $\mathbb{P}^{2m+1}(\mathbb{C})$. Let Λ be the formal adjoint of L. Then Λ can be naturally extended to the operator $\mathrm{id} \otimes \Lambda$ (denoted also by Λ for simplicity) on End$(V(S)) \otimes \wedge^*$ $\mathrm{T}^* \mathbb{P}^{2m+1}(\mathbb{C})$. Recall that $\gamma = \sqrt{-1}\Lambda R$. Let $\{(U_j, \varphi_j)\}_{0 \leqq j \leqq 2m+1}$ be the standard affine coordinate system for $\mathbb{P}^{2m+1}(\mathbb{C})$, defined by

$$U_j = \{[z] = [{}^t(z^0, \cdots, z^{2m+1})] \in \mathbb{P}^{2m+1}(\mathbb{C}); z^j \neq 0\}$$

and φ_j is the mapping:

$$U_j \ni [{}^t(x^1, \cdots, 1, \cdots, x^{2m+1})] \mapsto {}^t(x^1, \cdots, x^{2m+1}) \in \mathbb{C}^{2m+1}.$$

Let us calculate $\sqrt{-1}\Lambda R$ on U_0. For $z = {}^t(1, x^1, \cdots, x^{2m+1})$, we have:

$$\text{(5)} \qquad \sqrt{-1}(1 + |x|^2)(dz \wedge {}^t\overline{dz}) = \mathrm{id} + z {}^t\overline{z} - (z, 0) - {}^t(\overline{z}, 0),$$

where $(z, 0)$ denotes the $(2m+2, 2m+2)$-matrix whose first column vector is z and all other entries are 0. Substituting the above expression of R, we now conclude that

$$\gamma = 0.$$

Hence ∇ is an Einstein-Hermitian connection on $(V(S), k(S))$. Q.E.D.

(2.3) Since $\mathcal{S}$ is connected, $(V(S), k(S))$ is isomorphic to (p^*V, p^*h_V) as C^∞-Hermitian vector bundle. We choose such an isomorphism $t(S): (p^*V, p^*h_V) \simeq (V(S), k(S))$. Let $D(S)$ be the pull-back $t(S)^*\nabla(S) := t(S)^{-1} \circ \nabla(S) \circ t(S)$ of $\nabla(S)$. Then the connection $D(S)$ is also an Einstein-Hermitian connection on (p^*V, p^*h_V). Note that the equivalence class $[D(S)]$ modulo $\mathcal{G}(p^*V, p^*h_V)$ is independent of the choice of the isomorphism $t(S)$. We obtain the mapping $\psi: \widetilde{\mathcal{S}} \to \mathcal{E}(p^*V, p^*h_V)$ by

$$\psi(\widetilde{S}) = [D(S)] \qquad S \in \mathcal{S}.$$

Since the holonomy group of $D(S)$ is $\mathrm{Sp}(m)$, the connection $D(S)$ is irreducible (for more details see Section 3). Thus ψ is regarded as a mapping: $\widetilde{\mathcal{S}} \to \mathcal{E}'(p^*V, p^*h_V)$.

(2.4.1) Recall that the element j $(\in \mathbb{H})$ induces a real structure j_0 on $\mathbb{C}^{2m+2}$ $(\simeq \mathbb{H}^{m+1})$:

$$j_0: \mathbb{C}^{2m+2} \ni (a, b) \mapsto (-\bar{b}, \bar{a}) \in \mathbb{C}^{2m+2}.$$

Therefore the subset $\mathcal{S}$ of $M(2m+2; \mathbb{C})$ admits a natural real structure

$$j_{\mathfrak{S}}: \mathfrak{S} \ni S \mapsto j_0^{-1} S j_0 \in \mathfrak{S}.$$

Since $j_{\mathfrak{S}}(cS)$ $(c \in \mathbb{C}^*, S \in \mathfrak{S})$ is $\bar{c} j_{\mathfrak{S}}(S)$, the real structure $j_{\mathfrak{S}}$ on $\mathfrak{S}$ is pushed down on a real structure (denoted by $j_{\widetilde{\mathfrak{S}}}$) on $\widetilde{\mathfrak{S}}$. Furthermore, $j_{\mathfrak{S}}$ and $j_{\widetilde{\mathfrak{S}}}$ restrict to the real structures $j_{\mathcal{S}}$ and $j_{\widetilde{\mathcal{S}}}$ on $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ respectively.

(2.4.2) Recall that the twistor space $\mathbb{P}^{2m+1}(\mathbb{C})$ has the standard real structure

$$\tau: [z^1, z^2] \mapsto [-\overline{z^2}, \overline{z^1}] \qquad z^1, z^2 \in \mathbb{C}^{m+1}.$$

Since p^*V is trivial on each fibre of $p : \mathbb{P}^{2m+1}(\mathbb{C}) \to \mathbb{P}^m(\mathbb{H})$, the real structure τ induces a bundle automorphism $\tilde{\tau}$ on p^*V such that the following diagram is commutative:

$$\begin{array}{ccc} p^*V & \xrightarrow{\tilde{\tau}} & p^*V \\ \downarrow & & \downarrow \\ \mathbb{P}^{2m+1}(\mathbb{C}) & \xrightarrow{\tau} & \mathbb{P}^{2m+1}(\mathbb{C}). \end{array}$$

By the bundle automorphism $\tilde{\tau}$, we define a mapping τ' of $\mathcal{E}'(p^*V, p^*h_V)$ onto itself as follows:

$$\tau'([D]) = [\tilde{\tau}^* D], \quad ([D] \in \mathcal{E}'(p^*V, p^*h_V))$$

(cf. [N2;(3.6)]).

(2.4.3) One can easily check that $\psi \circ j_{\widetilde{S}} = \tau' \circ \psi$. Hence ψ induces the mapping

$$(\psi)_{\mathbb{R}}: \mathcal{S}_{\mathbb{R}} \to \mathcal{E}'(p^*V, p^*h_V)_{\mathbb{R}},$$

where $\mathcal{S}_{\mathbb{R}}$ and $\mathcal{E}'(p^*V, p^*h_V)_{\mathbb{R}}$ are the subsets of all elements of $\widetilde{\mathcal{S}}$ and $\mathcal{E}'(p^*V, p^*h_V)$ fixed by the real structures $j_{\widetilde{S}}$ and τ' respectively. Note that $\mathcal{S}_{\mathbb{R}} \simeq \mathcal{H}_0$ and $(\psi)_{\mathbb{R}} = p^* \circ \varphi$. By [N1;(0.2)], $p^*(\mathcal{B}'(V, h_V))$ is contained in $\mathcal{E}'(p^*V, p^*h_V)_{\mathbb{R}}$. Thus,

$$\operatorname{Image}(\psi) \cap p^*(\mathcal{B}'(V, h_V)) = p^*(\operatorname{Image}(\phi)).$$

§3. Injectivity of the mapping ψ

In this section we shall prove that the mapping ψ is injective. This injectivity allows us to show that the image of ψ is $\mathrm{PGL}(2m+2, \mathbb{C})/\mathrm{PSp}(m+1, \mathbb{C})$.

(3.1.1) Let $S \in \mathcal{S}$. Then the matrix $H(S)$ in Section 2 induces a Hermitian inner product on $\mathbb{C}^{2m+2}$ by

$$H(S)(\xi, \eta) = {}^t\overline{\xi} H(S)\eta, \quad \xi, \eta \in \mathbb{C}^{2m+2}.$$

This inner product $H(S)(\ ,\ )$ naturally defines a Hermitian metric $k_0(S)$ on the trivial bundle p^*F. Let $k_1(S)$ be the restriction of $k_0(S)$ to the subbundle $V(S)$. The flat connection d on the trivial bundle p^*F induces a Hermitian connection $\nabla_1(S)$ on the Hermitian subbundle $(V(S), k_1(S))$ by

$$\nabla_1(S) := Q_1(S) \circ d,$$

where $Q_1(S)$ denotes the orthogonal projection of p^*F onto $V(S)$. By a calculation similar to Theorem 2.2.3, the Hermitian connection $\nabla_1(S)$ is an Einstein-Hermitian connection on $(V(S), k_1(S))$. By the same argument as in (2.3), there exists an isomorphism $t_1(S) : (p^*V, p^*h_V) \simeq (V(S), k_1(S))$ of C^∞-Hermitian vector bundles. By $D_1(S)$, we denote the pull-back $t_1(S)^*\nabla_1(S)$ of $\nabla_1(S)$ for simplicity. Then $D_1(S)$ is also an Einstein-Hermitian connection on (p^*V, p^*h_V), and its equivalence class $[D_1(S)]$ modulo $\mathcal{G}(p^*V, p^*h_V)$ is independent of the choice of the isomorphism $t_1(S)$. We now define a mapping $\psi_1 : \widetilde{\mathcal{S}} \to \mathcal{E}(p^*V, p^*h_V)$ by

$$\psi_1(\widetilde{S}) = [D_1(S)] \qquad S \in \mathcal{S}.$$

(3.1.2) Let $f_1(S)$ be the automorphism of p^*V defined by

$$f_1(S)(\xi) := (H(S)^{-1})^{1/2}\xi, \quad \xi \in p^*F.$$

Then $f_1(S)$ is an isomorphism between C^∞-Hermitian vector bundles $(V(S'), h(S'))$ and $(V(S), k_1(S))$ where $S' := (\overline{H(S)}^{-1})^{1/2}S)$. Obviously,

$$\nabla_1(S) = f_1(S) \circ \nabla(S') \circ f_1(S)^{-1}.$$

Hence $D(S')$ is equivalent to $D_1(S)$ modulo $\mathcal{G}(p^*V, p^*h_V)$. Note that the mapping:

$$\mathcal{S} \ni S \mapsto S' \in \mathcal{S}$$

is bijective. Thus ψ is injective if and only if so is ψ_1.

(3.2) We prepare the following lemma in linear algebra in order to give an explicit expression of the curvature $R_1(S)$ of $D_1(S)$.

Definition 3.2.1. There exists a $\mathbb{C}$-basis $\{e_1, \cdots, e_{2k}\}$ for $\mathbb{C}^{2k}$ such that the Hermitian inner product $H(S)$ and the symplectic form S are respectively represented by the matrices I and J in terms of the basis, where

$$I := \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & 0 \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \sum_{i=1}^{2k} \overline{e_i}^* \otimes e_i^*,$$

$$J := \begin{pmatrix} 0 & 1 & & 0 & 0 \\ -1 & 0 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & & -1 & 0 \end{pmatrix} = \sum_{j=1}^{k} (e_{2j-1}^* \otimes e_{2j}^* - e_{2j}^* \otimes e_{2j-1}^*).$$

Such a $\mathbb{C}$-basis is called a *symplectic basis* with respect to S.

(3.2.2) Fix an $S \in \tilde{\mathcal{S}}$. Note that S induces a skew symmetric bilinear form fibrewise on the trivial bundle p^*F. Then $k_1(S)$ and the restriction of the symmetric bilinear form to $V(S)$ allow us to regard $V(S)$ as a vector bundle with $\mathrm{Sp}(m)$-structure. Take a point $[z] \in \mathbb{P}^{2m+1}(\mathbb{C})$. Then we choose a $\mathbb{C}$-basis $\{a_1, a_2, \dots, a_{2m+2}\}$ for $\mathbb{C}^{2m+2}$, which is symplectic with respect to the symplectic structure S, such that the fibre $V(S)_{[z]}$ of $V(S)$ at $[z]$ is generated by the flat sections corresponding to

$a_1, a_2, \ldots, a_{2m}$ over $\mathbb{C}$. Obviously, the connection $\nabla_1(S)$ is $\mathrm{Sp}(m)$-invariant. We shall now show that $\nabla_1(S)$ is irreducible. The curvature $R_1(S)$ of $\nabla_1(S)$ is written in the form

$$({}^t\overline{z}H(S)z)^{-1}UB(U^{-1}dz \wedge {}^td\overline{z}{}^t\overline{U}^{-1} + J\overline{U}^{-1}d\overline{z} \wedge {}^tdz{}^tU^{-1}\,{}^tJ)BU^{-1},$$

at $[z] \in \mathbb{P}^{2m+1}(\mathbb{C})$, where $B = \sum_{i=1}^{2m} e_i \otimes e_i^*$ and U denote the square matrix of degree $2m+2$ whose i-th column vector is a_i for each i. Hence the holonomy group of $\nabla_1(S)$ is exactly $\mathrm{Sp}(m)$. Thus $\nabla_1(S)$ is irreducible.

Theorem 3.2.3. *The mapping $\psi_1 : \mathcal{S} \to \mathcal{E}'(p^*V, p^*h_V)$ is injective, i.e., if $[D_1(S_1)] = [D_1(S_2)]$ for $S_1, S_2 \in \tilde{\mathcal{S}}$, then there exists an element $c \in \mathbb{C}^*$ such that $S_1 = cS_2$.*

Proof. Assume $[D_1(S_1)] = [D_1(S_2)]$. We have an isomorphism $g : (V(S_1), k_1(S_1)) \simeq (V(S_2), k_1(S_2))$ such that $g\,\nabla_1(S_1)\,g^{-1} = \nabla_1(S_2)$. The proof is divided into three steps.

Step 1. Let $[z] \in \mathbb{P}^{2m+1}(\mathbb{C})$ be arbitrary. Then there exists a $\mathbb{C}$-basis $\{e_1, \cdots, e_{2m+2}\}$ for $\mathbb{C}^{2m+2}$, which is symplectic with respect to the symplectic structure S_1, such that $V(S_1)_{[z]}$ is generated by the flat sections $a_1, a_2, \ldots, a_{2m}$ over $\mathbb{C}$. Since the normalizer of $\mathrm{Sp}(m)$ in $\mathrm{U}(2m)$ is $\mathrm{U}(1) \cdot \mathrm{Sp}(m)$, we have an element $c \in \mathbb{C}^*$ such that $\{cg(e_1), \cdots, cg(e_{2m})\}$ is a symplectic $\mathbb{C}$-basis for $V(S_2)_{[z]}$ with respect to the symplectic structure induced by S_2. Hence there exist vectors $f_{2m+1}, f_{2m+2} \in \mathbb{C}^{2m+2}$ such that $\{cg(e_1), \cdots, cg(e_{2m}), f_{2m+1}, f_{2m+2}\}$ is a symplectic $\mathbb{C}$-basis for $\mathbb{C}^{2m+2}$ with respect to S_2. Let $H_i := H(S_i)$, $i = 1, 2$ and let $U_1 = (e_1, \cdots, e_{2m+2})$ be the square matrix, of degree 2m+2, whose i-th column vector is e_i. Moreover, put $U_2 = (cg(e_1), \cdots, cg(e_{2m}), f_{2m+1}, f_{2m+2})$. We then obtain:

$$\text{(a)} \qquad \begin{aligned} &({}^t\overline{z}H_1z)^{-1}U_1BK_1BU_1^{-1} \\ &= ({}^t\overline{z}H_2z)^{-1}g^{-1}U_2BK_2BU_2^{-1}g, \end{aligned}$$

on $V(S_1)_{[z]}$, where

$$K_i = B(U_i^{-1}dz \wedge {}^td\overline{z}{}^t\overline{U_i}^{-1} + J\overline{U_i}^{-1}d\overline{z} \wedge {}^tdz{}^tU_i^{-1}\,{}^tJ)B,$$

for $i = 1, 2$. From our definition of U_1 and U_2, we have:

$$g^{-1}U_2B = c^{-1}U_1B.$$

This together with (a) yields

$$({}^t\overline{z}H_2z)({}^t\overline{z}H_1z)^{-1}K_1 = K_2.$$

Therefore, by setting $T := U_1^{-1}dz \wedge {}^t d\bar{z}({}^t\overline{U_1})^{-1}$ and $C := U_2^{-1}U_1$, we obtain:

$$\text{(b)} \qquad \begin{aligned} &({}^t\bar{z}H_2z)({}^t\bar{z}H_1z)^{-1}B(T + J\overline{T}J)B \\ &= B\{CT{}^t\overline{C} + (\overline{JCJ})J\overline{T}\,{}^tJ\,{}^t(JCJ)\}B. \end{aligned}$$

Step 2. Put $E_{ij} = e_i \otimes e_j^* - e_j \otimes e_i^* \quad (i \neq j)$ and

$$E_{ii} = e_i \otimes e_i^* \,.$$

Write the matrix C as (c_{ij}). Then there exists a (1,0)-vector $v_1 \in T_{[z]}\mathbb{P}^{2m+1}(\mathbb{C})$ such that

$$T(v_1, \overline{v_1}) = E_{11}.$$

Hence the identity (b) implies

$$|c_{11}|^2 + |c_{21}|^2 = 1, \quad c_{i1} = 0 \quad (3 \leqq i \leqq 2m).$$

Similarly, we have $v_2 \in T_{[z]}\mathbb{P}^{2m+1}(\mathbb{C})$ such that $T(v_2, \overline{v_2}) = E_{22}$. It then follows that

$$|c_{12}|^2 + |c_{22}|^2 = 1, \quad c_{i2} = 0 \quad (3 \leqq i \leqq 2m).$$

Inductively, we obtain

$$\begin{aligned} &|c_{2s-1,2s-1}|^2 + |c_{2s,2s-1}|^2 = 1, \\ &|c_{2s-1,2s}|^2 + |c_{2s,2s}|^2 = 1, \\ &c_{2s-1,j} = c_{2s,j} = 0 \qquad (j \neq 2s-1, 2s)\,, \end{aligned}$$

for all s with $1 \leqq s \leqq m$. For suitable v', $v'' \in \mathrm{T}_{[z]}\mathbb{P}^{2m+1}(\mathbb{C})$ corresponding to the following four values of $T(v', v'')$,

$$T(v', v'') = E_{12}, \sqrt{-1}E_{12}, \sqrt{-1}E_{11}, \sqrt{-1}E_{22}$$

we contract the equality (b) by $v' \wedge v''$. We then have

$$a_{21} = a_{12} = 0$$

and there is a $\theta \in \mathbb{R}$ such that

$$a_{11} = a_{22} = e^{i\theta}.$$

Similarly, taking $T(v', v'')$ to be either $E_{2j-1,2j}, \sqrt{-1}E_{2j-1,2j}, \sqrt{-1}E_{2j,2j}$ or $\sqrt{-1}E_{2j-1,2j-1}$ we have

$$a_{2j-1,2j} = a_{2j,2j-1} = 0 \quad (2 \leqq j \leqq m)$$

and $\theta_j \in \mathbb{R}$ $(2 \leqq j \leqq m)$ such that

$$a_{2j-1,2j-1} = a_{2j,2j} = e^{i\theta_j}.$$

Furthermore, let $T(v', v'')$ be either $E_{2i,2j-1}$ (i $\neq$ j) or $E_{k,2m+1}$ $(1 \leqq k \leqq 2m-1)$. Then the identities

$$\theta_1 = \cdots = \theta_m$$

and

$$a_{i,2m+1} = a_{i,2m+2} = 0 \quad (1 \leqq i \leqq 2m)$$

follow. Hence we obtain:

$$C = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & \vdots & \vdots \\ 0 & 0 & e^{i\theta} & 0 & 0 \\ a_{2m+1,1} & \cdots & a_{2m+1,2m} & a_{2m+1,2m+1} & a_{2m+1,2m+2} \\ a_{2m+2,1} & \cdots & a_{2m+2,2m} & a_{2m+2,2m+1} & a_{2m+2,2m+2} \end{pmatrix}.$$

Step 3. Since ${}^t\overline{U_2}H_2U_2 = I$, the matrix ${}^t\overline{C}$ is just ${}^t\overline{U_1}H_2U_2$. Thus,

(c) $\quad H_2(f_1, \cdots, f_{2m}) = e^{\sqrt{-1}\theta}H_1(e_1, \cdots, e_{2m}) \quad (1 \leqq j \leqq 2m).$

Since $\{e_1, \cdots, e_{2m}\}$ is a unitary basis for $\mathbb{C}^{2m}$ with respect to the Hermitian inner product H_1, the (i, j)-entry $(H_1)_{ij}$ is given by

$$(H_1)_{ij} = ({}^t\overline{H_1^{-1}H_2f_i})H_1(H_1^{-1}H_2f_j) = \delta_{ij},$$

i.e., when restricted to the subspace $\sum_{i=1}^{2m}\mathbb{C}f_i$, the Hermitian inner products associated with $H_2H_1^{-1}H_2$ and H_2 coincide on the space. Changing $[z] \in \mathbb{P}^{2m+1}(\mathbb{C})$ is arbitrarily, we have $H_2H_1^{-1}H_2 = H_2$ on $\mathbb{C}^{2m+2}$. Hence $H_2 = H_1$. Now by (c),

$$f_j = e^{-i\theta}e_j \ (1 \leqq j \leqq 2m),$$

and we have $V(S_2)_{[z]} = V(S_1)_{[z]}$. Now, since $(V(S_2)_{[z]})^\perp = \mathbb{C}z + \mathbb{C}\overline{S_2z}$, and $(V(S_2)_{[z]})^\perp = \mathbb{C}z + \mathbb{C}\overline{S_1z}$, there exists a holomorphic functions $c(z)$ on $\mathbb{C}^{2m+2} \setminus \{0\}$ such that $S_1z = c(z)S_2z$ for all $z \in \mathbb{C}^{2m+2}$. By Hartogs' Theorem, we can extend $c(z)$ to a holomorphic function on $\mathbb{C}^{2m+2}$. Using the Taylor expansion of $c(z)$ at $z = 0$, we see that $c(z)$ is constant on $\mathbb{C}^{2m+2}$. Thus we obtain the composite c such that $S_1 = cS_2$ for constant c, as required. Q.E.D.

(3.3) In (2.4.3), we have $p^* \circ \varphi$ coincides with $(\psi)_{\mathbb{R}}$. Hence in view of (3.2.2), we have

Corollary 3.3.1. *The mapping φ is injective, and so the image of φ is* $\mathrm{SL}(m+1, \mathbb{H})/\mathrm{Sp}(m+1)$.

§4. The moduli space of B_2-connections on (V, h_V)

The moduli space $\mathcal{B}''(V, h_V)$ is written as a union of connected components $\mathcal{B}_i(V, h_V)$:

$$\mathcal{B}''(V, h_V) = \bigcup_{i \in I} \mathcal{B}''_i(V, h_V).$$

By $\mathcal{B}''_1(V, h_V)$, we denote the component containing the image of ϕ. Using the same method as in [A-H-S] and [F], we shall examine $\mathcal{B}''_1(V, h_V)$.

Theorem 4.1.1. *$\mathcal{B}''_1(V, h_V)$ is nothing but the image of φ, i.e., $\mathcal{B}''_1(V, h_V)$ is diffeomorphic to* $\mathrm{SL}(m+1, \mathbb{H})/\mathrm{Sp}(m+1)$.

To prove Theorem 4.1.1, we compute the dimension of $\mathcal{B}''_1(V, h_V)$. By Borel-Weil-Kostant-Bott's theorem (cf. [M]) we shall show the following:

Lemma 4.1.2. *The real dimension of $\mathcal{B}''_1(V, h_V)$ is $m(2m+3)$* ($= \dim_{\mathbb{R}} \mathrm{SL}(m+1, \mathbb{H})/\mathrm{Sp}(m+1)$).

Proof. By [N2], $\mathcal{B}''_1(V, h_V)$ is $\dim_{\mathbb{C}} \mathrm{H}^1(\mathbb{P}^{2m+1}(\mathbb{C}), A_D)$, where D denotes the Einstein-Hermitian connection $\nabla(I)$ on (p^*V, p^*h_V). Since the vector bundle p^*V is homogeneous, and since $\mathbb{P}^{2m+1}(\mathbb{C}) = \mathrm{Sp}(m+1)/\mathrm{Sp}(m) \times \mathrm{U}(1)$, we can write the vector bundle $\mathrm{End}(p^*V)$ as $\mathrm{Sp}(m+1) \times_{(\rho \otimes \rho^*)} \mathfrak{gl}(2m, \mathbb{C})$, where ρ is the unitary representation of $\mathrm{Sp}(m) \times \mathrm{U}(1)$ on $\mathbb{C}^{2m}$ defined by

$$\rho : \mathrm{Sp}(m) \times \mathrm{U}(1) \ni (a, b) \mapsto \rho(a, b) := a \in \mathrm{Sp}(m) \subset \mathrm{U}(2m).$$

The representation $\rho \otimes \rho^*$ is equivalent to $\rho^* \otimes \rho^*$ and is expressible as a direct sum $\mathbb{C}\omega_{\mathbb{C}^{2m}} \oplus \wedge_0^2 \rho^* \oplus \mathrm{S}^2\, \rho^*$ of irreducible representations $\mathbb{C}\omega_{\mathbb{C}^{2m}}$, $\wedge_0^2 \rho^*$ and $\mathrm{S}^2\, \rho^*$, where $\omega_{\mathbb{C}^{2m}}$ is such that

$$\omega_{\mathbb{C}^{2m}}(a, b)(\xi, \zeta) := {}^t\xi J \zeta, \quad \xi, \zeta \in \mathbb{C}^{2m}.$$

Recall that $\wedge_0^2 \rho^* := (\mathbb{C}\omega_{\mathbb{C}^{2m}})^{\perp} \cap \wedge^2 \rho^*$ and that $\mathrm{S}^2\, \rho^*$ is the symmetric part of $\rho^* \otimes \rho^*$. Now, the vector bundle is written as a direct sum $L_1 \oplus L_2 \oplus L_3$ of homogeneous vector bundles L_1, L_2, L_3 corresponding to representations $\mathbb{C}\omega_{\mathbb{C}^{2m}}$, $\wedge_0^2 \rho^*$, $S^2 \rho^*$, respectively. Hence the complex A_D

is decomposed into three components $A(L_1)$, $A(L_2)$, $A(L_3)$. Applying Borel-Weil-Kostant-Bott's theorem to $A(L_i)$ (i=1,2,3), we obtain

$$\dim_{\mathbb{C}} \mathrm{H}^1(A(L_i)) = 0 \qquad (i = 1, 3),$$
$$\dim_{\mathbb{C}} \mathrm{H}^1(A(L_2)) = (2m + 3)m.$$

Summing these up, we have $\dim_{\mathbb{C}} \mathrm{H}^1(A_D) = (2m+3)m$, as required. Q.E.D.

(4.1.3) By using Lemma 4.1.2, we prove Theorem 4.1.1. Consider the frame bundle P of unitary bases. Let $M(2m+2, 2m, \mathbb{C})$ be the set of $(2m+2, 2m)$-matrices. Then P is naturally regarded as a submanifold of $M(2m+2, 2m, \mathbb{C})$ as follows:

Let $(u) \in \mathbb{P}^m(\mathbb{H})$ and let $(f_1, \cdots, f_{2m})$ be a unitary basis for $(V_{(u)}, (h_V)_{(u)})$. Now, the Lie group $\mathrm{SL}(m+1, \mathbb{H})$ acts on P by

$$\nu : \mathrm{SL}(m+1, \mathbb{H}) \times P \ni (g, B) \mapsto gB({}^t\overline{gB}gB)^{-1/2} \in P.$$

Let η be the action of $\mathrm{SL}(m+1, \mathbb{H})$ on $\mathbb{P}^m(\mathbb{H})$ such that

$$\eta : \mathrm{SL}(m+1, \mathbb{H}) \times \mathbb{P}^m(\mathbb{H}) \ni (g, (u)) \mapsto ({}^t\overline{g}^{-1}u) \in \mathbb{P}^m(\mathbb{H}).$$

In terms of these actions, the natural projection of P onto $\mathbb{P}^m(\mathbb{H})$ is equivalent. The vector bundle $\wedge^i \mathrm{T}^* \mathbb{P}^m(\mathbb{H})$ splits into a direct sum $A_i \oplus B_i$ in such a way that A_i and B_i are holonomy invariant vector subbundles (cf. [N1;(3.1)]). Since the decomposition $\wedge^i \mathrm{T}^* \mathbb{P}^m(\mathbb{H}) = A_i \oplus B_i$ $(1 \leqq i \leqq 2m)$ depends only on the $\mathrm{GL}(m, \mathbb{H}) \cdot \mathrm{GL}(1, \mathbb{H})$-structure of the tangent bundle of $\mathbb{P}^m(\mathbb{H})$, the action ν induces the one of $\mathrm{SL}(m+1, \mathbb{H})$ on $\mathcal{B}_1''(V, h_V)$. By an argument similar to [A-H-S;Section 9] and [F; Section 2], the isotropy subgroup of $\mathrm{SL}(m+1, \mathbb{H})$ is compact. Since $\mathrm{Sp}(m+1)$ is a maximal compact subgroup of $\mathrm{SL}(m+1, \mathbb{H})$ and $\dim_{\mathbb{R}}(\mathcal{B}_1''(V, h_V)) =$ (2m+1)m (Lemma (4.1)), the isotropy subgroup is equal to $\mathrm{Sp}(m+1)$. Hence $\mathcal{B}_1''(V, h_V) = \mathrm{SL}(m+1, \mathbb{H})/\mathrm{Sp}(m+1)$ and it coincides with the image of φ, as required.

(4.2) Let N be a holomorphic vector bundle of rank $2m$ over $\mathbb{P}^{2m+1}(\mathbb{C})$. Recall that N is a null-correlation bundle if there exists a following exact sequence:

$$0 \to N \to T \otimes H^{-1} \to H \to 0,$$

where T, H are respectively the holomorphic tangent bundle and the hyperplane bundle over $\mathbb{P}^{2m+1}(\mathbb{C})$. By $\mathcal{N}$ we denote the set of null-correlation bundles over $\mathbb{P}^{2m+1}(\mathbb{C})$. Then we obtain:

Proposition 4.2.1. *We have a natural bijection of $\mathcal{N}$ onto the image of ψ.*

Proof. Given $S \in \mathfrak{S}$, we denote by σ_S the holomorphic section to $H^2 \otimes T^*$ defined by

$$\sigma_S([z]) = {}^t z S z, \quad [z] \in \mathbb{P}^{2m+1}(\mathbb{C}).$$

Then the mapping $\mathfrak{S} \ni S \mapsto \sigma_S \in \mathrm{H}^0(\mathbb{P}^{2m+1}(\mathbb{C}), H^2 \otimes T^*)$ is bijective. Restricting to $\mathcal{S}$, we have the parametrization of $\mathcal{N} = \{N_{\widetilde{S}}; S \in \mathfrak{S}\}$ by $\mathcal{S}$. Endow the tangent bundle T of $\mathbb{P}^{2m+1}(\mathbb{C})$ with the Fubini-Study metric. Since the natural (1,0)-connection on the holomorphic subbundle $N_{\widetilde{S}}$ of $T \otimes H^{-1}$ is obtained from the dual bundle $(V(S), \nabla(S))^*$, we obtain the bijections

$$\mathcal{N} \approx \widetilde{\mathcal{S}} \approx \mathrm{Image}\,\psi, \quad N_{\widetilde{S}} \leftrightarrow \widetilde{S} \leftrightarrow (V(S), \nabla(S)),$$

as required. Q.E.D.

§5. Compactification of $\psi(\mathcal{S})$

In this section, we give a certain type of compactification of $\tilde{\mathcal{S}}$, by which we study the ends of the family of Einstein-Hermitian connections constructed in Section 2.

(5.1.1) Let $\mathfrak{S}_k$ be the subset of $\mathfrak{S}$ defined by

$$\mathfrak{S}_k := \{S \in \mathfrak{S}; \mathrm{rank}_{\mathbb{C}}\, S = 2k\}.$$

Then $\mathfrak{S}_{m+1}$ is nothing but $\mathcal{S}$ and $\mathfrak{S}$ is represented as a union of $\mathfrak{S}_k$'s, $1 \leqq k \leqq m+1$. Each $\mathfrak{S}_k$ is isomorphic to the complex homogeneous manifold $\mathrm{GL}(2m+2, \mathbb{C}) / \mathrm{G_k}$ where

$$\mathrm{G_k} = \{\begin{pmatrix} C & 0 \\ D & E \end{pmatrix} \in \mathrm{GL}(2(m+1), \mathbb{C}); C \in \mathrm{Sp}(k, \mathbb{C})\}.$$

(5.1.2) Note that $\widetilde{\mathfrak{S}}$ is a complex projective space of complex dimension $m(2m+3)$. Since $\mathfrak{S}$ is a union of $\mathfrak{S}_k$+s,

$$\widetilde{\mathfrak{S}} = \bigcup_{1 \leqq k \leqq m+1} \widetilde{\mathfrak{S}}_k,$$

by setting $\widetilde{\mathfrak{S}}_k = \mathfrak{S}_k/\mathbb{C}^*$. Obviously, we have $\widetilde{\mathfrak{S}}_k \cong \mathrm{PGL}(2m+2;\mathbb{C})/\widetilde{\mathrm{G}}_k$, where

$$\widetilde{\mathrm{G}}_k = \{\begin{pmatrix} \widetilde{C} & 0 \\ \widetilde{D} & \widetilde{E} \end{pmatrix} \in \mathrm{PGL}(2(m+1),\mathbb{C}); \widetilde{C} \in \mathrm{PSp}(k,\mathbb{C})\}.$$

Since $\widetilde{\mathfrak{S}}_{m+1}$ is just $\widetilde{\mathfrak{S}}$, the boundary of $\widetilde{\mathcal{S}}$ in $\widetilde{\mathfrak{S}}$ is a union $\bigcup_{1 \leqq k \leqq m} \widetilde{\mathfrak{S}}_k$.

(5.1.3) Let $\mathcal{L}(p^*V, p^*h_V)$ be the set of all Einstein-Hermitian connections on (p^*V, p^*h_V) possibly with singularities. Then we have an equivalence relation on $\mathcal{L}(p^*V, p^*h_V)$ as follows. For ∇_1, $\nabla_2 \in \mathcal{L}(p^*V, p^*h_V)$, we say that ∇_1 is equivalent to ∇_2 if (1) the singular sets for ∇_1 and ∇_2 coincide, and (2) there exists a unitary gauge transformation $t \in \mathcal{G}(p^*V, p^*h_V)$ such that $t\ \nabla_1\ t^{-1} = \nabla_2$ outside the singularities. We denote the equivalence class of ∇ by $[\nabla]$ and the set of all equivalence classes

$$\{[\nabla] : \nabla \in \mathcal{L}(p^*V, p^*h_V)\} = \mathcal{L}(p^*V, p^*h_V)/\mathcal{G}(p^*V, p^*h_V)$$

by $\widetilde{\mathcal{E}}(p^*V, p^*h_V)$. We shall now study the Einstein-Hermitian connections corresponding to the boundary of $\widetilde{\mathcal{S}}$ in $\widetilde{\mathfrak{S}}$. Let $\widetilde{S} \in \widetilde{\mathfrak{S}} \setminus \widetilde{\mathcal{S}}$. Then, we can define $V(S)$, $h(S)$ and $\nabla(S)$ for $\widetilde{S} \in \widetilde{\mathfrak{S}}$ by the method similar to (2.2.2). Moreover, we put

$$F(S) = \{[z] \in P^{2m+1}(\mathbb{C}); Sz = 0\}.$$

Then, outside $F(S)$, the vector bundle $V(S)$ has a natural holomorphic structure such that $\nabla(S)$ is an Einstein-Hermitian connection on $(V(S), h(S))$. Since $\widetilde{\mathcal{S}}$ is open-dense in $\widetilde{\mathfrak{S}}$, there exists a sequence $\{\widetilde{S}_i\}$ in $\widetilde{\mathcal{S}}$ converging to $\widetilde{S}$. For the corresponding sequence $\{D(S_i)\}$, we have unitary gauge transformations g_i such that $\{g_i D(S_i) g_i^{-1}\}$ converges to $D(S) \in \mathcal{L}(p^*V, p^*h_V)$ with respect to C^∞-topology on every compact subset of $P^{2m+1}(\mathbb{C}) \setminus F(S)$.

(5.1.4) We now have C^∞-bundle isomorphism $t : (p^*V, p^*h_V) \to (V(S), h(S))$ outside $F(S)$, such that

$$tD(S)t^{-1} = \nabla(S).$$

The gauge equivalence class $[D(S)]$ depends only on $\widetilde{S}$. Furthermore, there is an element $K \in \mathrm{PGL}(2m+2,\mathbb{C})$ such that $\widetilde{S}$ is written as ${}^tK\widetilde{J}_jK$ where $J_j = \sum_{i=1}^{j}(e^*_{2i-1} \otimes e^*_{2i} - e^*_{2i} \otimes e^*_{2i-1})$. Hence the set $F(S)$

is $K^{-1}F(J_j)$, which is a space of complex dimension $2m+1-2j$. Hence we obtain the mapping

$$\tilde{\psi} : \tilde{\mathfrak{S}} \ni \tilde{S} \to [D(S)] \in \tilde{\mathcal{E}}(p^*V, p^*h_V).$$

Obviously, $\tilde{\mathfrak{S}}$ is compact and the image of $\tilde{\psi}$ is a compactification of $\psi(\mathcal{S}) \approx \mathcal{N}$ with respect to C^∞-topology on every compact set without singular sets.

(5.2) The space $\tilde{\mathcal{E}}(p^*V, p^*h_V)$ carries the real structure

$$\tilde{\tau} : \tilde{\mathcal{E}}(p^*V, p^*h_V) \ni [D] \mapsto \tilde{\tau}([D]) := [\tau^\sharp \circ D \circ \tau^\sharp] \in \tilde{\mathcal{E}}(p^*V, p^*h_V),$$

which is a natural extension of the real structure τ' on $\mathcal{E}'(p^*V, p^*h_V)$. By calculation, $\tilde{\psi}$ is compatible with the real structures $j_{\mathfrak{S}}$ (cf. (2.4.1)) and $\tilde{\tau}$. Hence $\tilde{\psi}$ restricts to the real points

$$(\tilde{\psi})_{\mathbb{R}} : \tilde{\mathfrak{S}}_{\mathbb{R}} \to \tilde{\mathcal{E}}(p^*V, p^*h_V)_{\mathbb{R}}.$$

Since we have a natural identification of $\tilde{\mathfrak{S}}_{\mathbb{R}}$ with

$$\{\text{positive semi-definite quaternionic Hermitian matrices}\}/\mathbb{R}^*,$$

the image of $(\tilde{\psi})_{\mathbb{R}}$ gives us a compactification of φ.

Added in Proof. After the completion of this paper, the author received a preprint by H.Doi and T.Okai entitled "1-instantons on $\mathbb{H}P^n$", which gives a result slightly stronger than Theorem 4.1.1 .

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