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p-adic Heights on Curves

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Dedicated to Professor Kenkichi Iwasawa on the occasion of his 70th birthday

In this paper, we will present a new construction of the p-adic height pairings of Mazur-Tate [MT] and Schneider [S], when the Abelian variety in question is the Jacobian of a curve. Our aim is to describe the local height symbol solely in terms of the curve, using arithmetic intersection theory at the places not dividing p and integrals of normalized differentials of the third kind (Green's functions) at the places dividing p.

It is a pleasure to dedicate this note to Kenkichi Iwasawa, in thanks for the many inspiring things he has taught us.

§ 1. The local pairing

Let p be a rational prime and let $\boldsymbol{Q}_p$ denote the field of p-adic numbers. Let k be a non-archimedean local field of characteristic zero, with valuation ring $\mathcal{O}$, uniformizing parameter π, and residue field $\boldsymbol{F}=\mathcal{O}/\pi\mathcal{O}$ finite of order q. We fix a continuous homomorphism

$$(1.1) \qquad \chi\colon k^* \longrightarrow \boldsymbol{Q}_p.$$

If the residue characteristic of k is not equal to p, then χ is trivial on the subgroup $\mathcal{O}^*$ and is determined by the value $\chi(\pi)$.

Let X be a complete non-singular, geometically connected curve defined over k, and assume for simplicity that X has a k-rational point. Let J denote the Jacobian of X over k. The following statement, as well as its proof, is similar to that of Proposition 2.3 in [G].

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Proposition 1.2. *Assume that the residue characteristic of k is not equal to p. Then there is a unique function $\langle a, b\rangle$, defined on relatively prime divisors a and b of degree zero on X defined over k with values in $\boldsymbol{Q}_p$ which is continuous, symmetric, bi-additive (when all relevant terms are defined) and satisfies*

$$\langle (f), b\rangle = \chi(f(b))$$

for $f \in k(X)^*$.

Proof. The difference of any two such functions gives a continuous homomorphism $J(k)\times J(k)\to\boldsymbol{Q}_p$, which must be trivial for topological resaons. This gives the uniqueness, and the existence is proved using intersection theory. Let $\mathcal{X}$ be a regular model for X over $\mathcal{O}$, and extend a and b to divisors (with rational coefficients) A and B on $\mathcal{X}$ which have zero intersection with each component in the special fiber. Then the formula

$$\langle a, b\rangle = (A\cdot B)\chi(\pi) \tag{1.3}$$

defines a local symbol with all the desired properties. □

An analogue of Proposition 1.2 also holds when k is archimedean: in this case χ must be trivial and we define $\langle a, b\rangle = 0$ for all a and b. The situation is more complicated when k has residue characteristic p, for in this case conditions like those in 1.2 do not determine $\langle a, b\rangle$ uniquely. Indeed, the difference of two such functions would describe a continuous pairing $J(k)\times J(k)\to\boldsymbol{Q}_p$, and many such pairings exist! In the next four sections we will assume k has residue characteristic p and give an analytic treatment of the theory of local heights.

§ 2. Differentials and the logarithm

We say a differential on X over k is of the first kind if it is regular everywhere, and of the second kind if it is locally exact. The differentials of the second kind, modulo the exact differentials, form a finite dimensional k-vector space of dimension $2g$, where g is the genus of X. We will denote this quotient space $H^1(X/k)$. It is canonically isomorphic to the first hypercohomology group of the de Rham complex

$$0\longrightarrow\mathcal{O}_X\longrightarrow\Omega^1_X\longrightarrow 0$$

on X/k (cf. [K1] p. 72–73). Therefore we obtain a canonical exact sequence

$$0\longrightarrow H^0(X, \Omega^1_{X/k})\longrightarrow H^1(X/k)\longrightarrow H^1(X, \mathcal{O}_{X/k})\longrightarrow 0. \tag{2.1}$$

We identify $H^0(X, \Omega^1_{X/k})$, the space of differentials of the first kind, with its image. It has dimension g and we will denote it $H^{1,0}(X/k)$. The space $H^1(X, \mathcal{O}_{X/k})$ also has dimension g and may be canonically identified with the tangent space at the origin of $J = \mathrm{Pic}^0(X)$.

The space $H^1(X/k)$ has a canonical non-degenerate alternating form given by the algebraic cup product

$$\{\ ,\ \}: H^1(X/k) \times H^1(X/k) \longrightarrow k. \tag{2.2}$$

This may be calculated (using a well-known formula of Serre) as follows: Let ν_1 and ν_2 be differentials of the second kind, with classes $[\nu_1]$ and $[\nu_2]$ in $H^1(X/k)$. For each point x of X, choose a formal primitive f_x of ν_1. Then

$$\{[\nu_1], [\nu_2]\} = \textstyle\sum_x \mathrm{Res}_x(f_x \nu_2).$$

In particular, it is apparent from this formula that $H^{1,0}(X/k)$ is a maximal isotropic subspace with respect to $\{\ ,\ \}$.

A differential on X is said to be of the third kind if it is regular, except possibly for simple poles with integral residues. Let $T(k)$ denote the subgroup of differentials of the third kind and $D^0(k)$ the group of divisors of degree zero on X over k. The residual divisor homomorphism gives rise to an exact sequence

$$0 \longrightarrow H^{1,0}(X/k) \longrightarrow T(k) \xrightarrow{\mathrm{Res}} D^0(k) \longrightarrow 0.$$

Let $T_l(k)$ denote the subgroup of $T(k)$ consisting of the logarithmic differentials, i.e., those of the form df/f for $f \in k(X)^*$. Since $T_l(k) \cap H^{1,0}(X/k) = \{0\}$ and $\mathrm{Res}(df/f) = (f)$, we obtain an exact sequence

$$0 \longrightarrow H^{1,0}(X/k) \longrightarrow T(k)/T_l(k) \longrightarrow J(k) \longrightarrow 0. \tag{2.3}$$

It is known that this sequence may be naturally identified with the k-rational points of an exact sequence of commutative algebraic groups over k:

$$0 \longrightarrow H^0(\Omega^1) \longrightarrow E \longrightarrow J \longrightarrow 0 \tag{2.4}$$

Here E is the universal extension of J by a vector group (cf. [MM]) and $H^0(\Omega^1) \cong \boldsymbol{G}_a^g$. The Lie algebra of E is canonically isomorphic to $H^1(X)$, so the exact sequence (2.1) is the resulting exact sequence of Lie algebras over k.

All of the above assertions are true for an arbitrary field k, but we will now exploit the fact that k is p-adic. In this case, there is a logarithmic homomorphism defined on an open subgroup of the points of any commutative p-adic Lie group, G, to the points of its Lie algebra Lie(G) (cf. [Se 5.34]). When $G=E$ or J, the open subgroup on which the logarithm converges has finite index, so the homomorphism can be uniquely extended to the entire group. We denote this extension $\log_E$ or $\log_J$ respectively. Since the logarithm is functorial and equal to the identity on $H^0(\Omega^1)(k)$ we obtain the following.

Proposition 2.5. *There is a canonical homomorphism*

$$\Psi: T(k)/T_l(k) \longrightarrow H^1(X/k)$$

which is the identity on differentials of the first kind and makes the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{1,0}(X/k) & \longrightarrow & E(k) & \longrightarrow & J(k) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \Psi=\log_E} & & \downarrow{\scriptstyle \log_J} & & \\ 0 & \longrightarrow & H^{1,0}(X/k) & \longrightarrow & H^1(X/k) & \longrightarrow & H^1(X, \mathcal{O}_{X/k}) & \longrightarrow & 0. \end{array}$$

The map $\log_J$ is the basis for the study of the group $J(k)$; it has kernel $J(k)_{\mathrm{tor}}$ and its image is an $\mathcal{O}$-lattice of rank g in $H^1(X, \mathcal{O}_{X/k})$. It is the same as the map Λ of § 2 of [C-1]. The map Ψ takes a differential of the third kind on X to a differential of the second kind modulo exact differentials! It can obviously be extended to a linear map from the k-vector space of all differentials on X/k to $H^1(X)$ by writing an arbitrary differential η as a linear combination, $\eta=\sum \alpha_i\omega_i+\nu$, where ω_i is of the third kind, $\alpha_i \in \bar{k}$, and ν is of the second kind on X. We then define $\Psi(\eta)=\sum \alpha_i\Psi(\omega_i)+[\nu]$. In keeping with the aim of this paper, we remark that this homomorphism can be constructed without reference to the Jacobian, using rigid analysis on $X(k)$. The construction is based on the following lemma.

Lemma 2.6. *Let ω be a differential of the third kind and let Y be an affinoid subdomain of X which is conformal to the closed unit ball and contains all the poles of ω. Then on $X-Y$, $\omega=n^{-1}df/f+\nu+dg$, where n is positive integer, $f \in k(X)^*$, ν is a differential of the second kind on X, regular on $X-Y$, and g is a rigid analytic function on $X-Y$.*

Given the lemma (which is proven in [C2]) and the isomorphisms

$$H^1(X/k) \cong H^1_{an}(X/k) \cong H^1_{an}(X-Y/k),$$

which are established in [C2], one can show $\Psi(\omega)=[\nu]$. See [C3] for defails.

§ 3. Normalized differentials of the third kind

Let a be a divisor of degree zero on X/k. We wish to construct a "normalized" differential ω_a of the third kind on X with $\mathrm{Res}(\omega_a)=a$. In the complex case this is accomplished using Hodge theory (cf. [G] § 3). In the p-adic case, we must first fix a splitting of the exact sequence (2.1). Equivalently, we fix a direct sum decomposition

$$H^1(X/k)=H^{1,0}(X/k)\oplus W. \tag{3.1}$$

We then define ω_a to be the differential of the third kind with residual divisor a such that $\Psi(\omega_a)$ lies in W. Here Ψ is the map defined in Proposition 2.5. The differential ω_a is uniquely specified by these two conditions, as the differentials of the third kind with residual divisor a form a principal homogeneous space for $H^{1,0}(X/k)$, and Ψ restricted to this space is the identity. Since the homomorphism kills logarithmic differentials, we have the following.

Proposition 3.2. *The choice of W gives a section,*

$$D^0(k)\longrightarrow T(k),$$
$$a\longmapsto\omega_a$$

of the residual divisor homomorphism. Moreover if $a=(f)$ is principal, then $\omega_a=df/f$.

We note that in certain cases there is a reasonable choice of a complement W to $H^{1,0}(X/k)$ in $H^1(X/k)$. Namely, when X has good ordinary reduction, we may take W to be the unit root subspace for the action of Frobenius. The resulting normalized differentials then have the following property: on each residue disk R disjoint from $|a|$,

$$\omega_a=n^{-1}df/f+dg$$

where n is a positive integer, f is a rigid analytic unit on R and g is a bounded rigid analytic function on R. This follows from [K2].

§ 4. Integration and the reciprocity law

Integrals of Abelian differentials were defined in [C-1] and [C-dS]. We will sketch here a brief and simplified discussion of integrals of the third kind. We will suppose that X has good reduction (modulo π), and denote its reduction by $\tilde{X}$.

Let ω be a differential of the third kind on X and let $a=\mathrm{Res}(\omega)$. Let

Y be an affinoid obtained from X by removing finitely many residue disks whose union contains $|a|$. Let $A(Y)=(\varprojlim \mathcal{O}_{Y\otimes\mathcal{O}/\pi^n\mathcal{O}})\otimes \mathbf{Q}_p$ be the ring of rigid analytic functions on Y. Finally, let ϕ be an analytic lifting to Y of the Frobenius endomorphism, $\tilde{\phi}$, of $\tilde{Y}$ over the finite field $\mathcal{O}/\pi\mathcal{O}$, and let $P(T)=\sum a_n T^n$ be the characteristic polynomial of the endomorphism induced by $\tilde{\phi}$ on the first l-adic cohomology group of $\tilde{Y}$ for any prime l distinct from p.

Proposition 4.1. *There is a locally analytic function F: $Y(\mathbf{C}_p)\to\mathbf{C}_p$, unique up to an additive constant in k, which satisfies*

(i) $dF=\omega$

(ii) $\sum a_n(F\circ\phi^n)\in A(Y)$

(iii) $F(y^\sigma)=F(y)^\sigma$ *for* $y\in Y(\mathbf{C}_p)$ *and* $\sigma\in \mathrm{Aut}_{\mathrm{cont}}(\mathbf{C}_p/k)$.

The key fact used to prove the existence of F is the result in the theory of Washnitzer and Monsky [MW] which asserts that $\sum a_n(\phi^n)^*\omega$ lies in $dA(Y)$. If b is a divisor of degree zero, on Y, we define

$$\int_b \omega=\sum(\mathrm{ord}_y b)F(y). \tag{4.2}$$

This integral is independent of the ambiguity in F and lies in k. A simple computation on $\mathbf{P}^1$ shows that for f in $k(X)\cap A(Y)^*$,

$$\int_b df/f=\log f(b). \tag{4.3}$$

Here $a=(f)$ and b have disjoint reductions, so that $f(b)$ is a unit and $\log$: $\mathcal{O}^*\to k$ is the unique homomorphism extending the convergent series for $\log(1+T)$ on $1+\pi\mathcal{O}$.

If we wish to define the integral of ω over divisors b which are relatively prime to a but may not necessarily be supported on any Y as above, we must first choose a branch of the p-adic logarithm Log: $\mathbf{C}_p^*\to\mathbf{C}_p$, i.e., a locally analytic homomorphism which extends log on $\mathcal{O}^*$. We may then proceed in one of two ways to define the integral of ω over b. We may choose an appropriate semi-stable cover of $(X, |a|)$ and define the integrals as in [C-dS] or we may use Theorem 4.1 of [C-2] which implies that we may write

$$\omega=\omega'+n^{-1}df/f$$

where ω' is a differential of third kind whose polar locus has disjoint reduction from that of the support of b, n is a positive integer and $f\in$

$k(X)^*$. We then define

$$\int_b \omega = \int_b \omega' + n^{-1}\operatorname{Log}(f(b)).$$

The first integral on the right hand side is defined as above. This depends on the choice of Log but not on the choices of ω', n and f.

As in the classical case, we have a reciprocity law for differentials of the third kind. The proof in [C2] is modelled on a combination of the classical proof and the algebraic proof of the Weil reciprocity law for curves.

Proposition 4.5. *Let ω and ω' be two differentials of the third kind on X, whose residual divisors are relatively prime. Then*

$$\int_{\operatorname{Res}(\omega')} \omega - \int_{\operatorname{Res}(\omega)} \omega' = \{\Psi(\omega), \Psi(\omega')\}$$

where Ψ is the map to $H^1(X/k)$ defined in § 2 and $\{\ ,\ \}$ is the cup product.

§ 5. The local pairing at the p-adic completions

Recall that X is a non-singular complete curve over the p-adic field k, and that we have fixed a continuous character $\chi\colon k^* \to \boldsymbol{Q}_p$ in § 1. To apply the results of the previous two sections to construct as local height symbol, we shall assume that X has good reduction (modulo π) and that we have fixed a complement, W, to $H^{1,0}(X/k)$ in $H^1(X/k)$ as in § 3.

Since χ takes values in a torsion-free group, its restriction to $\mathcal{O}^*$ factors through the logarithm

$$\begin{array}{ccc} \chi\colon \mathcal{O}^* & \xrightarrow{\qquad} & \boldsymbol{Q}_p \\ & {\scriptstyle \log}\searrow \quad \nearrow{\scriptstyle t} & \\ & k & \end{array}.$$

The map t is $\boldsymbol{Q}_p$-linear, and uniquely determined by χ. We fix an extension $\operatorname{Log}\colon \boldsymbol{C}_p^* \to \boldsymbol{C}_p$ of log as in § 4 which satisfies $\chi = t \circ \operatorname{Log}$. We use this branch of the logarithm to define the integrals below as in § 4.

Let a and b be relatively prime divisors of degree zero on X and let ω_a be the normalized differentials of the third kind determined by the complement W. We define

$$\langle a, b\rangle = t\left(\int_b \omega_a\right). \tag{5.1}$$

Proposition 5.2. *The symbol* $\langle a, b\rangle$ *is continuous, bi-additive and satisfies*

$$\langle (f), b\rangle = \chi(f(b))$$

for $f \in k(X)^*$. *It is symmetric iff the subspace* W *of* $H^1(X/k)$ *is isotropic with respect to the cup product pairing.*

Proof. The continuity and bi-additivity are clear, as they hold for the normalized differentials and the integrals we have defined. By the reciprocity law (4.5), we have

$$\begin{aligned}\langle a, b\rangle - \langle b, a\rangle &= t\left(\int_b \omega_a - \int_a \omega_b\right)\\ &= t\{\Psi(\omega_a), \Psi(\omega_b)\}.\end{aligned}$$

Since the image of the normalized differentials via Ψ spans the subspace W and $t \neq 0$, the right hand side is identically zero iff W is isotropic. Finally, if $f \in k(X)^*$ and $a = (f)$, then $\omega_a = df/f$ and so by (4.3–4.4)

$$\langle a, b\rangle = t(\operatorname{Log} g(b)) = \chi(f(b)). \qquad \square$$

In particular, if W is the unit root subspace (in the case when X has ordinary reduction), the resulting local pairing is symmetric.

§ 6. Further Remarks

As in the classical case, one can combine the local symbols with the product formula to obtain a global p-adic height pairing on the Jacobian J. The initial data are

1) a curve X defined over a number field k, with good reduction at each place dividing p.

2) a continuous idèle class character $\chi\colon A_k^*/k^* \to \mathbf{Q}_p$.

3) a splitting $H^1(X/k_v) = H^{1,0}(X/k_v) \oplus W_v$ for each place v dividing p.

One can then define the local symbols as in § 1 and § 5, and the global symbol is defined to be their sum (cf. [G] § 4).

In [C-3], it is shown that splittings of the Hodge filtration of the first de Rham cohomology group of an Abelian variety are canonically in one-to-one correspondence with formal splittings of the bi-extension of this Abelian variety. When the Abelian variety has good ordinary reduction the splitting of the Hodge filtration which corresponds to the canonical formal splitting of the bi-extension in [MT] is that given by the unit root subspace. Using this one can show that when our curve X has good ordinary reduction at all places v dividing p, and W_v is the unit root subspace, our local and global pairings correspond to the canonical pairings of Schneider [SC1], [SC2] and Mazur-Tate [MT].

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