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Fermion Representations for the 2D and 3D Ising Models

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Abstract

Fermion variables are constructed for the 3D Ising Model in which it becomes a theory of free fermion strings defined on lattice.

§ 1. Introduction

In this paper we report our results on the free fermion string representation for the 3D Ising Model (IM) which have been obtained in 1979–80, but has not yet been published. These results have been announced in [1], and have been given in a fully developed form in the thesis [2], in Russian. With this paper we fill the gap and make these results published.

Throughout the paper we use the analogies with the free fermion variables construction for the 2D IM. The lattice theories for these models, which make the subject of this paper, have in fact very close analogies. So, we consider the developments for the 2D and 3D IMs in parallel, throughout the paper.

The construction of the fermion variables for the 2D IM and for the 3D IM are presented respectively in Sections 2 and 3, where the equations of motion are also derived. Particular feature of these equations—the inhomogeneous or contact terms, which require special technique, is treated in Section 4.

In Section 5 the Path Integral representations for 2D and 3D IMs are reviewed.

In Sections 6 and 7 the solutions of the equations of motion for the fermion variables of the 2D and 3D IMs are presented in the form of a sum, respectively, over the paths equipped with products of path link matrices, and in the form of a sum over the surfaces equipped with surface product of special plaquette matrices. It is this last representation, which we think gets most close to the continuum fermion string

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theory, the sum over surfaces to be replaced by an appropriate functional integral.

Continuum limit for the 2D IM fermion representation is reviewed in Section 6.

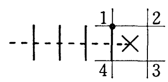
Particular technical details of deriving the inhomogeneous terms of the equations of motion are given in the Appendix.

§ 2. Linear equations for the fermion variables of the 2D Ising Model

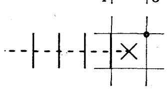
In this section we shall construct the 4-component fermion variables of the 2D IM and show that they satisfy linear equations. For particular representation we shall reproduce the equations for the amplitude of Feynman wanderings over the 2D lattice [3, 4].

Let us consider the following 4-component objects

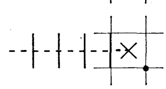
(1)



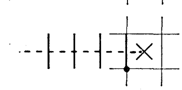
$$=\mu_x\sigma_{x_1}\equiv\chi^1(x)$$



$$=\mu_x\sigma_{x_2}\equiv\chi^2(x)$$



$$=\mu_x\sigma_{x_3}\equiv\chi^3(x)$$



$$=\mu_x\sigma_{x_4}\equiv\chi^4(x)\ .$$

Here μ_x is the so called disorder variable [5], which is given by a product of exponents taken along the dashed line:

$$\mu_x=\prod_{-\infty}^{x} e^{-2\beta\sigma\sigma'}$$

The exponents in the product above correspond to the links which are crossed by the dashed line. This definition implies that the disorder variables are defined on a dual lattice, x being the site of that lattice. σ_{x_1}, σ_{x_2}, σ_{x_3} and σ_{x_4} are order variables placed at four sites of the original lattice, surrounding the dual site x, see (1). The nonlocal variables of this type has been introduced by Kadanoff and Ceva in paper [5].

The essential point is that the $\sigma\mu$ variables are always assumed to be inside some correlation function (under averaging):

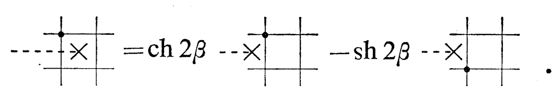
$$\begin{aligned}\chi^\alpha(x)&=\mu_x\sigma_{x_\alpha}\sim\langle\mu_x\sigma_{x_\alpha}\cdots\rangle\\&=\frac{1}{Z(\beta)}\sum_{\{\sigma\}}\exp\{+\beta\sum_{y,\hat{\imath}}\sigma_y\sigma_{y+\hat{\imath}}\}(\mu_x\sigma_{x_\alpha}\cdots)\end{aligned}$$

It is easy to check then that the $\sigma\mu$ variables do not depend on the form of the tail, in the following sence: the form of the tail can be varied, under the average, without changing the value of the correlation function. This is true until the tail crosses the point (lattice site) corresponding to the variable σ. Crossing this point results in changing the sign of the correlation function. This implies that the $\sigma\mu$ variables are doubly defined objects. To make the definition unambiguous, the far end of the tail has to be fixed. We shall choose the definition as in (1): the tail is stretched towards the lefthandside infinity.

All these properties will extensively be used in the following. For more details see [5].

We may notice that the tail of $\chi^{\alpha}(x)$ is analogous to the cut line in the complex plane, required for an unambiguous definition of analytic functions with a square root-like singular points.

We shall start now deriving the equations. We use the following trick. Consider e.g. the object $\chi^1(x)$ and expand the exponent $\exp\{-2\beta\sigma\sigma'\}$, corresponding to the first link of the tail. We find

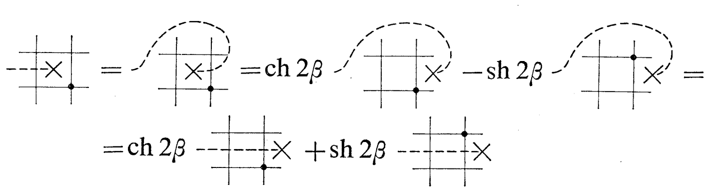


This corresponds to

$$\chi^1(x) = \mathrm{ch}\, 2\beta \cdot \chi^2(x - \hat{1}) - \mathrm{sh}\, 2\beta \cdot \chi^3(x - \hat{1}).$$

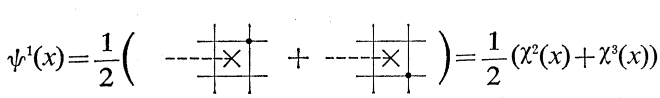
Here $\hat{1}$ is a unit (basic) vector of the lattice.

Additional manipulations are required if the point is placed away from the tail:

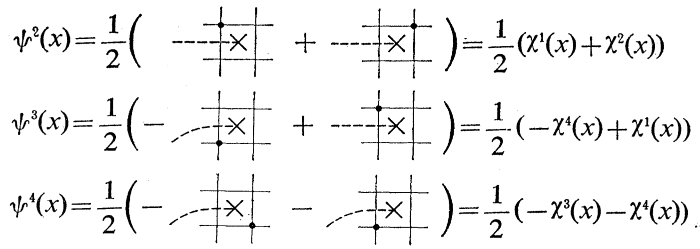


which is $\chi^3(x) = \mathrm{ch}\, 2\beta \cdot \chi^4(x + \hat{1}) + \mathrm{sh}\, 2\beta \cdot \chi^1(x + \hat{1})$. These simple tricks will be sufficient to derive the equations.

Consider the following linear combinations:



(2)



$$\psi^2(x)=\frac{1}{2}(\ldots+\ldots)=\frac{1}{2}(\chi^1(x)+\chi^2(x))$$
$$\psi^3(x)=\frac{1}{2}(-\ldots+\ldots)=\frac{1}{2}(-\chi^4(x)+\chi^1(x))$$
$$\psi^4(x)=\frac{1}{2}(-\ldots-\ldots)=\frac{1}{2}(-\chi^3(x)-\chi^4(x)).$$

It is not difficult to check that the 4-component variables $\psi^\alpha(x)$ satisfy the following linear equations:

$$\begin{pmatrix}\psi^1(x)\\ \psi^2(x)\\ \psi^3(x)\\ \psi^4(x)\end{pmatrix}=e^{-2\beta}\cdot\begin{array}{|c|c|c|c|}\hline 1&1&0&-1\\ \hline 1&1&1&0\\ \hline 0&1&1&1\\ \hline -1&0&1&1\\ \hline\end{array}\begin{pmatrix}\psi^1(x-\hat{1})\\ \psi^2(x-\hat{2})\\ \psi^3(x-\hat{3})\\ \psi^4(x-\hat{4})\end{pmatrix}$$

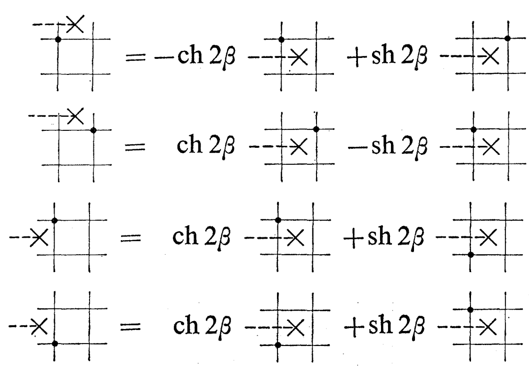
In a compact form

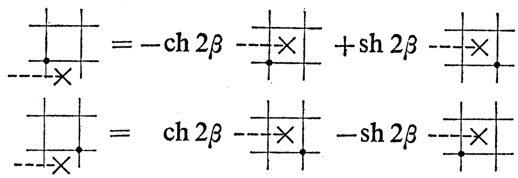
(3) $$\psi^\alpha(x)=e^{-2\beta}\Lambda^{\alpha\gamma}\psi^\gamma(x-\hat{\gamma}).$$

We shall give the derivation of the first equation:

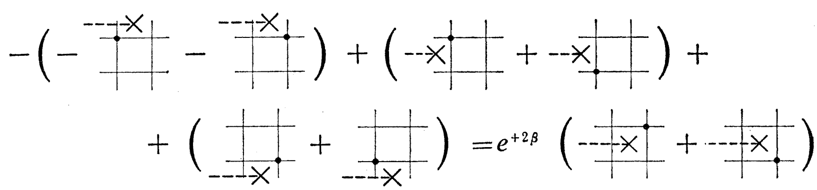
(4) $$\psi^1(x)=e^{-2\beta}(\psi^1(x-\hat{1})+\psi^2(x-\hat{2})-\psi^4(x-\hat{4}))$$

It is found as follows:





Summing up the above equations, we find:



This is precisely the eq. (4), see the definitions (2).

The eq. (3) for $\psi^{\alpha}(x)$ is different from the equation for Feynman wanderings in three respects (comp. [3, 4]).

First, wanderings are usually described by the function $W_L^{\alpha\alpha_0}(x, x_0)$, which is the amplitude for the propagation of a wandering particle from the point x_0 to the point x, along all possible paths, but with a fixed length L. This function satisfies the equation [3, 4]:

(5) $$W_{L+1}^{\alpha\alpha_0}(x, x_0)=\mathrm{th}\,\beta\cdot \Lambda_F^{\alpha\gamma} W_L^{\gamma\alpha_0}(x-\hat{\gamma}, x_0)$$

Here

$$\Lambda_F=\begin{array}{|c|c|c|c|}\hline 1 & \bar{\varepsilon} & 0 & \varepsilon \\ \hline \varepsilon & 1 & \bar{\varepsilon} & 0 \\ \hline 0 & \varepsilon & 1 & \bar{\varepsilon} \\ \hline \bar{\varepsilon} & 0 & \varepsilon & 1 \\ \hline \end{array}\,, \qquad \varepsilon=e^{i(\pi/4)}\,.$$

The full propagator is found by summing over L:

$$G=\sum_{L=0}^{\infty} W_L$$

From (5) one gets the equation for $G^{\alpha\alpha_0}(x, x_0)$:

(6) $$G^{\alpha\alpha_0}(x, x_0)=\mathrm{th}\,\beta \Lambda_F^{\alpha\beta} G^{\beta\alpha_0}(x-\hat{\beta}, x_0)$$

In this section we ignore the "contact" (inhomogeneous) terms of these equations (terms $\sim\delta_{x,x_0}$ in eq. (6)). They will be derived in Section 4.

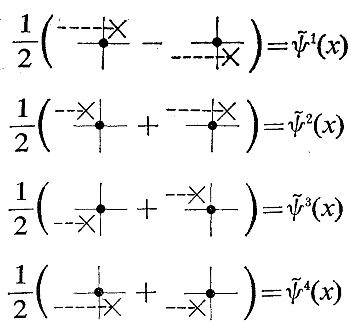
There remain two points of difference between the equations (3) and (6).

First, the matrices Λ_F in (6) and Λ in (3) look rather different, at first sight. In fact, they are equivalent: Λ is transformed into Λ_F by redifining the variables:

$$\begin{aligned}
\psi^1 &\longrightarrow \psi^1\\
\psi^2 &\longrightarrow \varepsilon\psi^2\\
\psi^3 &\longrightarrow \varepsilon^2\psi^3\\
\psi^4 &\longrightarrow \varepsilon^3\psi^4
\end{aligned}$$

Second, eq. (3) has a factor $\exp(-2\beta)$, instead of thβ in (6). This is because our objects $\psi^\alpha(x)$ and $\chi^\alpha(x)$ were defined on a dual lattice. It is well known that dual transformation for the 2D Ising Model relates $\exp(-2\beta)$ and thβ [6] (see also [7]). One can check that for the following objects, defined on the original lattice:

(7)



one would get the eq. (3) with the factor thβ instead of $\exp(-2\beta)$.

So, this is just our choice of variables, defined as in (2). Otherwise, eq. (3) is precisely that for the Feynman wanderings, in our particular representation.

§ 3. Barbed wire variables of the 3D Ising Model

In this section we turn to the 3D Ising Model. We choose to start with the 3D Gauge Ising Model, which is dual, and thus equivalent, to the usual Ising Model [8] (see, also [7]). The partition function has the form:

$$Z(\beta)=\sum_{\{\sigma\}}\exp\{\beta\sum_{x,\mu<\nu}\sigma_{x,\mu}\sigma_{x+\hat{\mu},\nu}\sigma_{x+\hat{\nu},\mu}\sigma_{x,\nu}\} \tag{8}$$
$$=\sum_{\{\sigma\}}\exp\{-\beta S[\sigma]\}$$

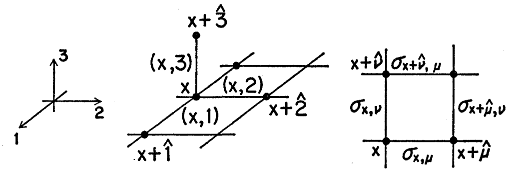


Fig. 1 Definition of the 3D Gauge Ising Model.

Here μ, $\nu=1, 2, 3$; $\hat{\mu}$, $\hat{\nu}$ are unit (basic) vectors of a cubic lattice; $\sigma_{x,\mu}=\pm 1$ are Ising variables defined at links (x, μ) of a cubic lattice (Fig. 1). Characteristic property of this model is its invariance to local (gauge) transformations: the action of the model $S[\sigma]$ is invariant to a simultaneous change of the sign of the six variables σ, placed at lattice links which have one lattice site in common:

$$\begin{aligned}&\sigma_{x,\mu}\longrightarrow-\sigma_{x,\mu}\\&\sigma_{x-\hat{\mu},\mu}\longrightarrow-\sigma_{x-\hat{\mu},\mu}\end{aligned}\qquad \mu=1, 2, 3$$

Such transformation can be represented as follows:

$$\sigma_{x,\mu}\longrightarrow f(x)\sigma_{x,\mu}f(x+\hat{\mu})$$

where $f(x)$ is an arbitrary function, defined at lattice sites, and assuming values ± 1. Because of this gauge invariance, the usual correlation functions, like e.g. the two-point function

$$\langle\sigma_{x,\mu}\sigma_{y,\nu}\rangle=\frac{1}{Z(\beta)}\sum_{\{\sigma\}}\sigma_{x,\mu}\sigma_{y,\nu}e^{-\beta S[\sigma]}$$

– vanish identically. The nonvanishing correlation functions of this model are those corresponding to closed contours (Fig. 2).

$$G(C)=\langle\sigma_1\sigma_2\cdots\sigma_L\rangle\equiv\langle\prod_C\sigma\rangle \tag{9}$$

These functions, similarly to the usual spin correlation functions of the 2D IM, satisfy *nonlinear* equations. We shall demonstrate this fact without further comments:

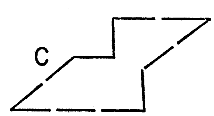
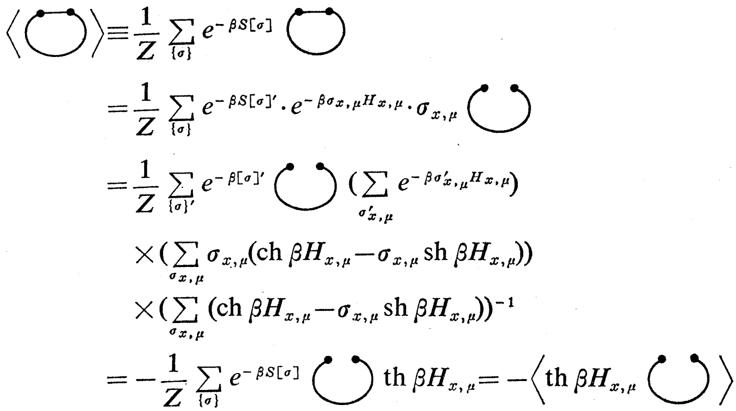


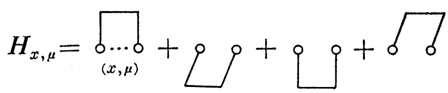
Fig. 2 The contour correlator $\langle\prod_C\sigma\rangle$ of the Gauge Ising **Model.**



$$\times\Big(\sum_{\sigma_{x,\mu}} \sigma_{x,\mu}(\operatorname{ch}\beta H_{x,\mu}-\sigma_{x,\mu}\operatorname{sh}\beta H_{x,\mu})\Big)$$

$$\times\Big(\sum_{\sigma_{x,\mu}} (\operatorname{ch}\beta H_{x,\mu}-\sigma_{x,\mu}\operatorname{sh}\beta H_{x,\mu})\Big)^{-1}$$

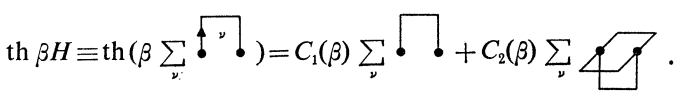
Here



$$=\sum_{\nu}\sigma_{x+\hat{\mu},\nu}\sigma_{x+\hat{\nu},\mu}\sigma_{x,\nu}\,.$$

Other notations are selfexplanatory.

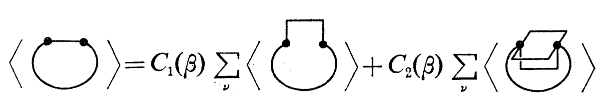
Notice next that

(11) 

Here

$$C_1(\beta)=\frac{1}{2}\left(\frac{\operatorname{th}4\beta}{4}+\frac{\operatorname{th}2\beta}{2}\right),\qquad C_2(\beta)=\left(\frac{\operatorname{th}4\beta}{4}-\frac{\operatorname{th}2\beta}{2}\right).$$

On substituting (11) into (10), we find:

(12) 

This is a nonlinear equation on the contour correlator $G(C)$—(9). Analogous equation can also be derived for the usual correlation functions of the 2D IM.

In the case of 2D IM, the "proper" variables, those which satisfy linear equations, has been described in Section 2. We shall define now "proper" contour variables for the 3D IM, which will be shown to satisfy linear equations.

Consider the following object

$$\chi_{\alpha_1\cdots\alpha_L}(C)=\prod_{\text{along } C}\sigma_{x,\mu}\cdot\mu_{\tilde{x}(\alpha)}=\prod\chi^{\alpha}_{x,\mu}\qquad(\alpha=1,2,3,4). \tag{13}$$

One element of this product is shown in Fig. 3. Here, the disorder variable $\mu_{\tilde{x}}$ of the 3D Gauge IM (order variable of the usual 3D IM) is given by a product, taken along the dashed line, of the plaquette exponents:

$$\mu_{\tilde{x}}=\prod^{\tilde{x}}\exp\{-2\beta(\sigma\sigma\sigma\sigma)\}$$

$\mu_{\tilde{x}}$ is defined at the site $\tilde{x}$ of the dual lattice, see Fig. 3.

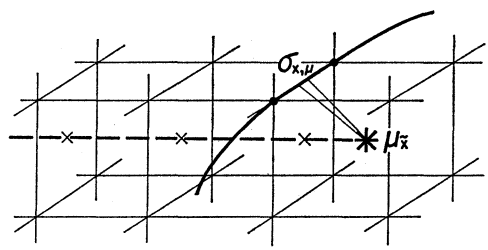


Fig. 3 General view of the link σ_μ variable of the contour object $\chi_{\alpha_1\cdots\alpha_L}(C)$ (13)

Same as in the case of the 2D IM, the objects $\chi_{\alpha_1\cdots\alpha_L}(C)$ are assumed to be inside some correlation function. As a consequence, the tails of the variables μ can be deformed, but so that they do not cross the links of the contour C (corresponding to the variables $\{\sigma_{x,\mu}\}$). Crossing the links results in the change of a sign of the correlator. So, analogously to the corresponding objects of the 2D IM, the contour correlators $\chi_{\alpha_1\cdots\alpha_L}(C)$ are being doubly defined. To make the definition unambiguous, one has to fix the far ends of the tails. We shall use the following representation:

For the variables at links 3 (those which are parallel to the 3-d, or Z-axis of the lattice) the tails will be strectched parallel to the axis 1, towards $-\infty$;

For the variables at links 1-stretched parallel to the axis 2, towards $-\infty$;

For the variables at links 2-stretched parallel to the axis 3, towards $-\infty$.

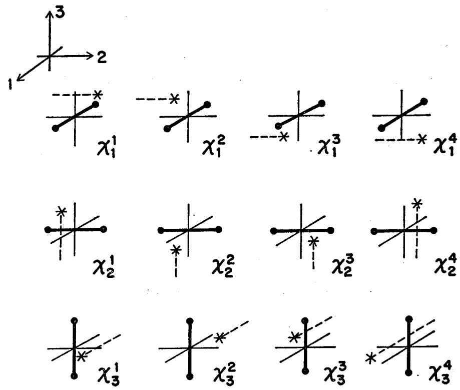


Fig. 4a Definition of link variables χ_μ^α

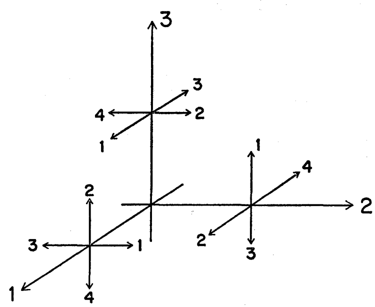


Fig. 4b Definition of link variables $\psi_\mu^\alpha(x)$

The spinor indeces of the link variables are defined as in Fig. 4a. Obviously, we tend to preserve the analogy with the 2D IM by choosing these definitions. The general view of the contour object $\chi_{\alpha_1\cdots\alpha_L}(C)$ for a small loop C is given in Fig. 5.

We remark that it is possible to give a dual definition for the contour object $\chi_{\alpha_1\cdots\alpha_L}(C)$, the one which uses spin variables $\{\sigma_x\}$ of the usual 3D IM with the partition function:

$$Z(\beta)=\sum_{\{\sigma\}} \exp\{\beta \sum_{x,\mu} \sigma_x \sigma_{x+\hat{\mu}}\}$$

In the dual formulation the contour objects would have the following form:

$$\tilde{\chi}_{\alpha_1\cdots\alpha_L}(C)=\oint_C \sigma_{\tilde{x}(\alpha)}\cdot \prod_{\substack{\text{along the}\\ \text{surface bounded}\\ \text{by } C}} e^{-2\beta\sigma\sigma}$$

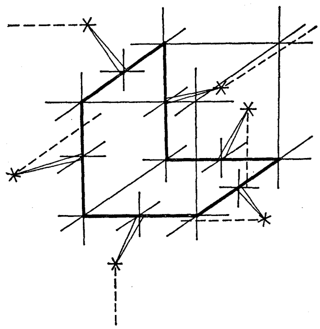


Fig. 5 General view of the contour correlator $\chi_{\alpha_1 \cdots \alpha_L}(C)$ for a small loop C.

Its general view for a small loop C is shown in Fig. 6. Instead of the tails, we would have, in this dual formulation, the surface S_C, bounded by the loop C. Similar to the case with tails, this surface S_C could be deformed, without changing the value of the correlation function, but so that the points around the contour, corresponding to $\{\sigma_{\tilde{x}(\alpha)}\}$, are not crossed.

Thus, we have defined our basic variables $\chi_{\alpha_1 \cdots \alpha_L}(C)$. They have the appearance of a "barbed wire", see Fig. 5. These "barbed" contours of the 3D IM correspond to the variables $\chi^{\alpha}(x)$, in case of 2D IM. Similar to the 2D case, we shall derive equations not for the variables $\chi_{\alpha_1 \cdots \alpha_L}(C)$, but for their linear combinations—the contour product of local (link) linear combinations like (2). The reason is that the coefficients of the equations are simpler in this representation.

Let us consider the following contour object

$$\psi_{\alpha_1 \cdots \alpha_L}(C) = \prod_C \psi^{\alpha}_{\mu}(x). \tag{14}$$

Here $\psi^{\alpha}_{\mu}(x)$ are linear combinations of the form

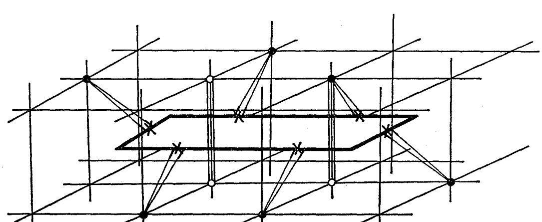
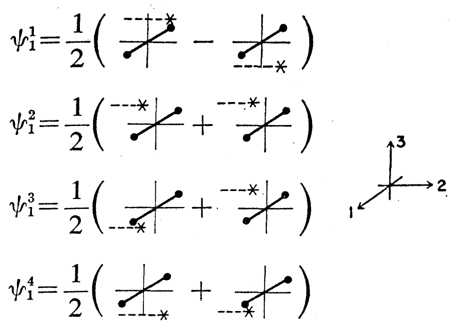


Fig. 6 General view of the contour correlator $\tilde{\chi}_{\alpha_1 \cdots \alpha_L}(C)$ (dual formulation).

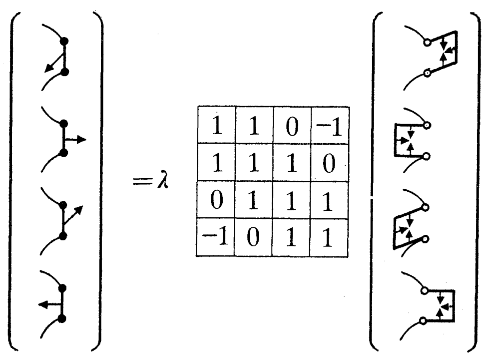
(15)



The variables ψ_2^α, ψ_3^α are obtained by cyclic permutation.

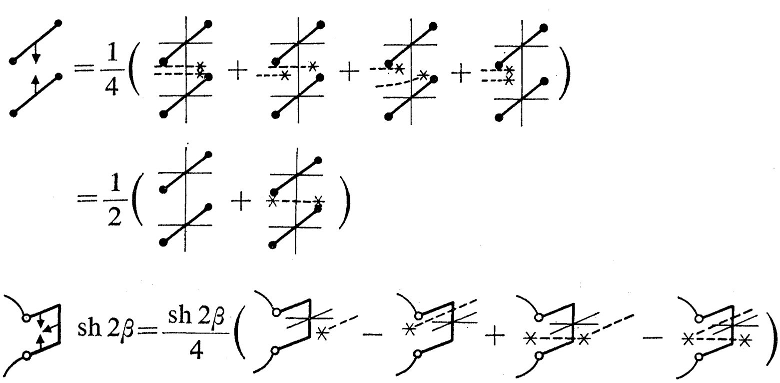
We have found that the contour correlator (14) satisfy the following equations (for each link of the contour):

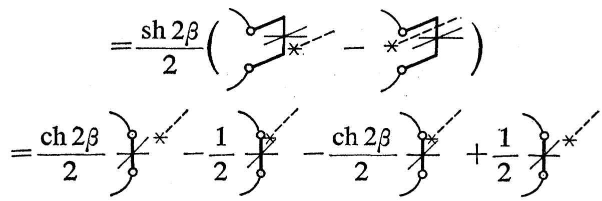
(16)



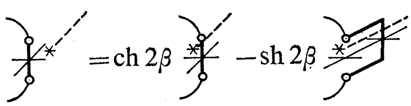
Equations for the links 1 and 2 are found by cyclic permutations. These equations are derived as follows:

(17)





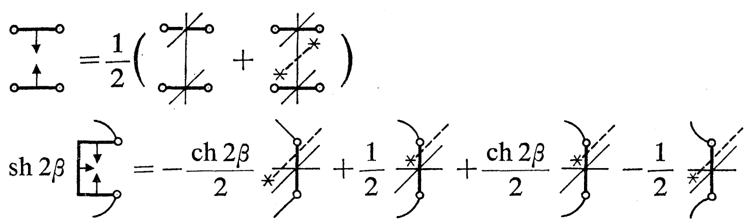
Here we have used the following equation:



which is derived in a way analogous to the 2D case (see Section 2).

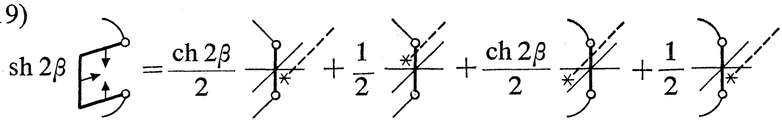
Next:

(18)

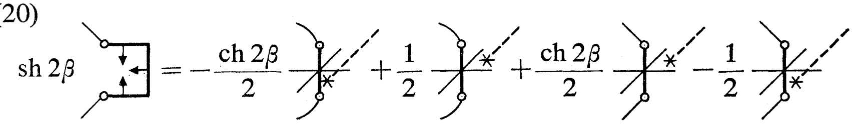


The other two equations are derived in analogous way:

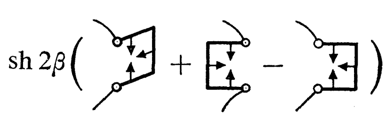
(19)

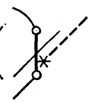
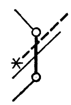
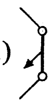


(20)



By summing (17–20) with coefficients provided by the matrix in eq. (16) we can check all the equations. E.g. the first equation is found as follows:



$= \frac{\mathrm{ch}\,2\beta+1}{2}\Big($  $-$  $\Big) = (\mathrm{ch}\,2\beta+1)$ 

This is the first equation in (16), with

$$\lambda = \frac{\mathrm{sh}\,2}{\mathrm{ch}\,2\beta+1} = \mathrm{th}\,\beta.$$

Thus we have shown that our barbed wire string correlators of the 3D IM do satisfy linear contour equations.

§ 4. Contact terms

In this section we shall finish deriving equations for the 2D and 3D Ising Models by supplying them with inhomogeneous, or contact terms. These terms arise whenever points of the 2D correlators, or the contour links of the 3D correlators, get in touch with one another. We have ignored such configurations in Sections 2 and 3.

2D IM, *contact terms.* Now that we are going to study the contact configurations, we have to introduce certain ordering for the tailed objects inside the correlation functions.

In Section 2 we have chosen a representation, in which the ends of the tails are stretched parallel to the axis 1 towards $-\infty$, see (1). Now we have to introduce an ordering of the tails at the lefthand side infinity. We shall define this as it is shown in Fig. 7. One can check that changing the ordering of two tails will result in a change of a sign of the correlator (crossing of tails is not allowed, by definition).

We may remark in passing that our objects $\psi^{\alpha}(x)$, built in a nonlocal way out of Ising variables, have certain analogy with the functional integral formulation of the Mandelstam transformation, connecting the Sin-Gordon and Thirring models. In that case, the tailed objects (2D fermion fields) are build as nonlocal functionals of the scalar field, subject to the Sin-Gordon Lagrangian dynamics [9, 10]. One particular point:

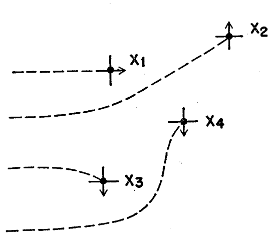


Fig. 7 By definition, this figure corresponds to the correlator $\langle\psi^{\alpha_1}(x_1)\psi^{\alpha_2}(x_2)\psi^{\alpha_3}(x_3)\psi^{\alpha_4}(x_4)\rangle$

the tails are being ordered in a similar way [10].

To demonstrate the technique of deriving the contact terms, it is sufficient to consider the two-point correlator:

$$G^{\alpha\alpha_0}(x-x_0)=\langle\psi^{\alpha}(x)\psi^{\alpha_0}(x_0)\rangle$$

see Fig. 8. Ordering of tails (Fig. 7) define the correlator uniquely, except for the case $x=x_0$. So, the quantity $G^{\alpha\alpha_0}(0)$ have to be defined separately somehow.

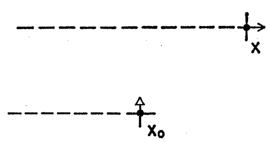


Fig. 8 Two-point correlator $G^{\alpha\alpha_0}(x, x_0)=\langle\psi^{\alpha}(x)\psi^{\alpha_0}(x_0)\rangle$

We notice that $G^{\alpha\alpha_0}(0)$ is encountered in the equation for the contact configuration $(x=x_0)$:

$$G^{\alpha\alpha_0}(x_0, x_0)=\lambda\Lambda^{\alpha\beta}\cdot G^{\beta\alpha_0}(x_0-\hat{\beta}, x_0)+C^{\alpha\alpha_0} \tag{21}$$

and also in the equation for (x, x_0) placed at neighbouring sites of the lattice. We required next that no additional terms appear in the equations for (x, x_0) placed at neighbouring sites. This additional requirement defines uniquely the elements of $G^{\alpha\alpha_0}(0)$: the elements of this matrix are found from the equations for the points (x, x_0) placed at neighbouring sites. After that, the equations for the contact configuration (21) will contain certain additional terms, the elements of the matrix $C^{\alpha\alpha_0}$. They will be our contact terms. We remark, that we could have defined the matrix $G^{\alpha\alpha_0}(0)$ differently. This would have resulted in different contact terms in the equation. Still, the solution $G^{\alpha\alpha_0}(x-x_0)$ (for $x\neq x^0$) is defined uniquely of course.

The approach described above is demonstrated in the Appendix, in which the element C^{21} of the matrix $C^{\alpha\alpha_0}$ is derived. Finally, the following equation is found:

$$\langle\psi^{\alpha}(x)\psi^{\alpha_0}(x_0)\rangle=\mathrm{th}\,\beta\cdot\Lambda^{\alpha\beta}\langle\psi^{\beta}(x-\hat{\beta})\psi^{\alpha_0}(x_0)\rangle+\delta_{x,x_0}C^{\alpha\alpha_0} \tag{22}$$

$$C^{\alpha\alpha_0}=\begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 1 \\ \hline -1 & 0 & 1 & 1 \\ \hline -1 & -1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 \\ \hline\end{array}$$

We may introduce "conjugate" variables

$$\bar{\psi}^{\alpha}(x)=\psi^{\beta}(x)(C^{-1})^{\beta\alpha} \tag{23}$$

with which the equation becomes

$$\langle\psi_{\nu}(x)\bar{\psi}^{\alpha_0}(x_0)\rangle=\text{th}\,\beta\cdot\Lambda^{\alpha\beta}\langle\psi^{\beta}(x-\hat{\beta})\bar{\psi}^{\alpha_0}(x_0)\rangle+\delta_{x,x_0}\delta^{\alpha\alpha_0}. \tag{24}$$

The "conjugate" variables $\bar{\psi}^{\alpha}$ can also be expressed as linear combinations of pairs of the components of the basic variable $\chi^{\alpha}(x)$, similar to those for ψ^{α}, see (2).

3D IM, *contact terms.* Each particular link of the contour correlator $\psi^{\alpha_1\cdots\alpha_L}(C)$ can get in touch with those links of the contour, which happen to be in the same 2D lattice plane, see Fig. 9. The tails of these links are also placed in the same plane (see the definitions in Section 3), and they are stretched in the same direction. Ordering of these tails is essential for the signs of the corresponding contact terms in the equations. We remark that because the contour of the correlator is closed, the ordering of the tails, described above, which uses 2D lattice plane, is defined unambiguously in fact, in a sence that it cannot be changed (without changing the sign of the correlator) by deforming the tails along the 3D lattice.

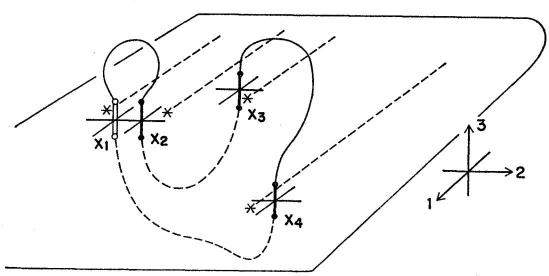


Fig. 9 The links of the contour C are shown which happen to be in the same 2D lattice plane as the particular link chosen—the link $(x_1, 3)$. The 2D plane crosses the link $(x_1, 3)$ and is orthogonal to its direction.

We remark next that, because of contact terms, the contour of the correlator, subject to the linear contour equations, can break, in the course of the dynamics, into two closed contours. Alternatively, it can produced out of two contours in touch (see Fig. 11 below). This means that for the general contour dynamics we have to consider correlation functions $\psi^{\alpha_1\cdots\alpha_L}(C_1\cdots C_K)$ which involve an arbitrary number of closed contours. They satisfy the equations of Section 3 separately for each contour in the correlator. But the contact terms of these equations may

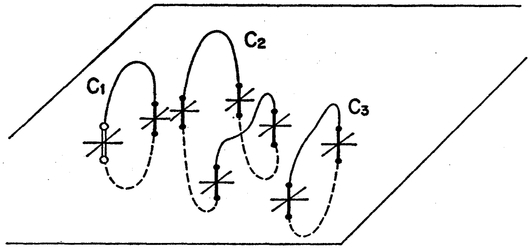
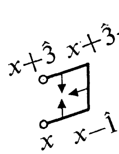


Fig. 10 The particular link chosen may come in contact with any other link of the contour correlator $\psi(C_1, C_2, C_3)$ which happens to be in the same 2D lattice plane.

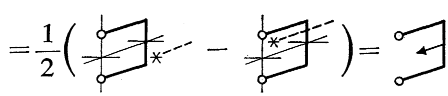
involve, for each particular link, the contact configurations with any other link of the correlator $\psi^{\alpha_1\cdots\alpha_L}(C_1\cdots C_K)$ which happens to be in the same 2D lattice plane as the particular link chosen, see Fig. 10.

We shall assume in the following that the tails of the link variables, in each 2D plane of the 3D lattice, are ordered in the same way as for the tailed objects of the 2D IM (see the beginning of this section, and Fig. 7), with an obvious cyclic permutation of directions (see the definition of link variables of the 3D IM in Section 3 and in Fig. 4).

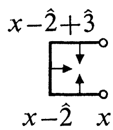
We shall look more closely now at the configurations which appear in the equations for contour correlators (see Section 3). The product of ψ varialbes, corresponding to the three links in this configuration, can be simplified as follows (see (17)):



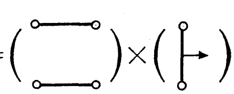
$$\equiv \psi_{\hat1}^{4}(x-\hat{1}+\hat{3})\psi_{\hat1}^{2}(x-\hat{1})\psi_{\hat3}^{1}(x-\hat{1})$$

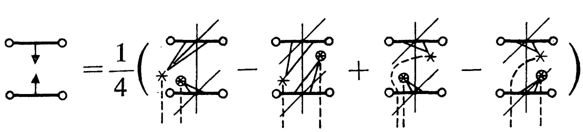


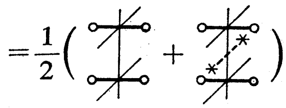
(25)



$$\equiv \psi_{\hat2}^{3}(x-\hat{2}+\hat{3})\psi_{\hat2}^{1}(x-\hat{2})\psi_{\hat3}^{1}(x-\hat{2})=$$

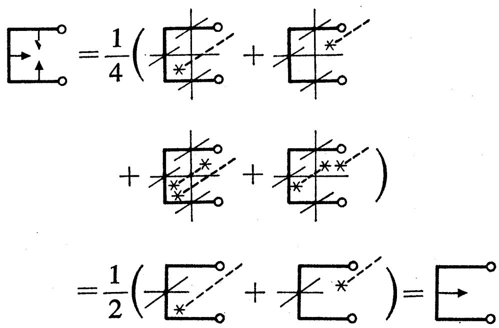






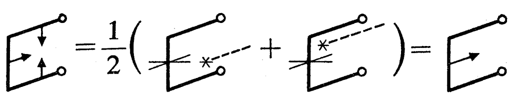
The signs here correspond to the definition of the link variables and to the ordering of the talis described above.

(26)

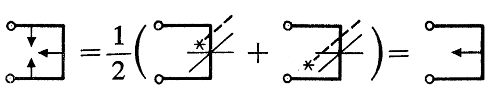


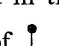
Analogously:

(27)



(28)



Expressions (25–28) reduce the derivation of the equations and their contact terms to the same manipulations which have been used in the case of the 2D IM. This is true, of course, for other orientations of . This means that equations and contact terms for the 3D IM can be written down by using the corresponding expressions for the 2D IM, with obvious replacements-comp. eq. (16) with its 2D analogue, eq. (3).

The contact terms for the contour equations follow from the results of the first part of this section (which deals with 2D IM). An example of the equation for the contact configuration is given in Fig. 11. In a more general case, given in Fig. 9, the equation for the link $(x_1, 3)$ will contain contact terms:

(29) $$\delta_{x_1,x_2}C^{\alpha_1\alpha_2}-\delta_{x_1x_3}C^{\alpha_1\alpha_3}+\delta_{x_1,x_4}C^{\alpha_1\alpha_4}$$

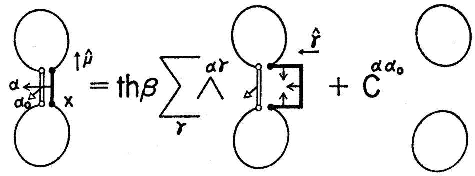


Fig. 11 An equation for the contact configuration. It is assumed that inside the correlator $\psi^{\alpha_1\cdots\alpha_L}(C)$ the link variables are ordered as $\langle\cdots\psi_\mu^\alpha(x)\psi_\mu^{\alpha_0}(x)\cdots\rangle$

It is assumed here that inside the contour correlator $\psi(C)$ the link variables are ordered as:

$$\psi^{\cdots\alpha_1\alpha_2\alpha_3\alpha_4\cdots}(C)=\langle\cdots\psi_3^{\alpha_1}(x_1)\psi_3^{\alpha_2}(x_2)\psi_3^{\alpha_3}(x_3)\psi_3^{\alpha_4}(x_4)\cdots\rangle$$

according to the ordering of the tails in Fig. 9.

We remark, finally, that an alternative ordering is also possible, the one in which the product of link variables is ordered along the contour.

§ 5. Path Integral formalism

Path Integral formalism, which uses Grassmann anticommuting variables (fields on lattice), has been found for the 2D IM a long time ago [11]. In a more developed form it is given e.g. in [12, 13]. Recently the corresponding formalism has also been found for the 3D IM [13, 14]. It happens that technically the derivation of equations in this formalism is simpler (once the Path Integral representation has been proved to give the correct partition function). In this section we shall reproduce the results of Sections 2–4, using this formalism. We shall not go into much detail here, because the Grassmann variables (fields) of this formalism have already appeared in the preceding sections as the tailed objects, built explicitly of the original Ising spin variables.

2D IM

We shall choose the Action for this model in the form:

$$\begin{aligned} A[\psi] &= -\frac{1}{2}\sum_x(\bar\psi\psi)+\frac{\lambda}{2}\sum_{x,\alpha}(\bar\psi_{(x+\hat\alpha)}\hat p_\alpha\psi_{(x)}) \\ &= -\frac{1}{2}\sum_x\psi(x)C^{-1}\psi(x)+\frac{\lambda}{2}\sum_{x,\alpha}\psi(x+\hat\alpha)\tilde p_\alpha\psi(x) \qquad (30)\\ &= \sum_x(\psi^3\psi^1+\psi^4\psi^2+\psi^1\psi^2+\psi^3\psi^4+\psi^2\psi^3+\psi^1\psi^4) \\ &\quad +\lambda\sum_x(\psi^3(x+\hat1)\psi^1(x)+\psi^4(x+\hat2)\psi^2(x) \end{aligned}$$

Here $\psi^{\alpha}(x)$, $\alpha=1, 2, 3, 4$ are four-component anticommuting Grassmann variables, defined at sites of the 2D square lattice. They have one to one correspondence with the tailed objects (2); C is a 4×4 matrix (22):

$$C^{-1}=\begin{array}{|c|c|c|c|}\hline 0 & -1 & 1 & -1 \\ \hline 1 & 0 & -1 & 1 \\ \hline -1 & 1 & 0 & -1 \\ \hline 1 & -1 & 1 & 0 \\ \hline \end{array}$$

$\bar{\psi}=\psi C^{-1}$ are conjugate variables (23). We have also introduced here the matrices $\{\hat{p}_{\alpha}\}$, which shall be used in the following sections:

$$\tilde{p}_{\alpha}=\left\{\begin{array}{|c|c|c|c|}\hline & & 0 & \\ \hline & & & 0 \\ \hline 1 & & & \\ \hline & 0 & & \\ \hline \end{array}\,,\ \begin{array}{|c|c|c|c|}\hline & & 0 & \\ \hline & & & 0 \\ \hline 0 & & & \\ \hline & 1 & & \\ \hline \end{array}\,,\ \begin{array}{|c|c|c|c|}\hline & & -1 & \\ \hline & & & 0 \\ \hline 0 & & & \\ \hline & 0 & & \\ \hline \end{array}\,,\ \begin{array}{|c|c|c|c|}\hline & & 0 & \\ \hline & & & -1 \\ \hline 0 & & & \\ \hline & 0 & & \\ \hline \end{array}\right\}$$

(31)

$$\hat{p}_{\alpha}=C\tilde{p}_{\alpha}=\left\{\begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline 1 & & & \\ \hline 0 & & & \\ \hline -1 & & & \\ \hline \end{array}\,,\ \begin{array}{|c|c|c|c|}\hline & 1 & & \\ \hline & 1 & & \\ \hline & 1 & & \\ \hline & 0 & & \\ \hline \end{array}\,,\ \begin{array}{|c|c|c|c|}\hline & & 0 & \\ \hline & & 1 & \\ \hline & & 1 & \\ \hline & & 1 & \\ \hline \end{array}\,,\ \begin{array}{|c|c|c|c|}\hline & & & -1 \\ \hline & & & 0 \\ \hline & & & 1 \\ \hline & & & 1 \\ \hline \end{array}\right\}$$

(blank cells correspond to zeros).
Notice that $\sum_{\alpha}\hat{p}_{\alpha}=\Lambda$, see eq. (3).

The partition function of the 2D IM is given by the functional Integral over Grassmannian fields on lattice:

$$Z(\lambda)=\int \mathscr{D}\psi \exp\{A[\psi]\} \tag{32}$$

Here

$$\mathscr{D}\psi=\prod_{x}(-d\psi^{1}(x)d\psi^{3}(x)d\psi^{2}(x)d\psi^{4}(x))$$

$$\exp\{A[\psi]\}=\prod_{x}\exp\left\{-\frac{1}{2}\bar{\psi}(x)\psi(x)+\lambda\psi^{3}(x+1)\psi^{1}(x)+\lambda\psi^{4}(x+\hat{2})\psi^{2}(x)\right\}$$

$$=\prod_{x}\left(1-\frac{1}{2}\bar{\psi}(x)\psi(x)-\psi^{3}(x)\psi^{1}(x)\psi^{4}(x)\psi^{2}(x)\right)$$

$$\times \prod_{x} (1+\lambda\psi^3(x+\hat{1})\psi^1(x))(1+\lambda\psi^4(x+\hat{2})\psi^2(x)).$$

It can be proved that (32) provides the correct partition function of the IM. The correspondence is established through the high temperature expansion, with $\lambda = \text{th}\,\beta$.

Let us consider now the correlation function:

$$\begin{aligned}&\langle\psi^{\alpha_2}(x_2)\psi^{\alpha_3}(x_3)\cdots\psi^{\alpha_n}(x_n)\rangle\\ &\quad=\frac{1}{Z}\int \mathscr{D}\psi\, e^{A[\psi]}\psi^{\alpha_2}(x_2)\psi^{\alpha_3}(x_3)\cdots\psi^{\alpha_n}(x_n)\end{aligned} \tag{33}$$

and let us perform a variation, under the functional integral (FI), of the variable $\psi^{\alpha_1}(x_1)$:

$$\psi^{\alpha_1}(x_1) \longrightarrow \psi^{\alpha_1}(x_1) + \varepsilon^\beta C^{\beta\alpha} \tag{34}$$

Here ε^β is an anticommuting parameter. Variation (34) under FI should not change the value of the correlation function. So it produces the equation:

$$\begin{aligned}&C^{\alpha_1\beta}\left\langle \frac{\partial A[\psi]}{\partial\psi^\beta(x_1)}\psi^{\alpha_2}(x_2)\cdots\psi^{\alpha_n}(x_n)\right\rangle\\ &\quad+C^{\alpha_1\beta}\left\langle\frac{\partial}{\partial\psi^\beta(x_1)}(\psi^{\alpha_2}(x_2)\cdots\psi^{\alpha_n}(x_n))\right\rangle=0\end{aligned} \tag{35}$$

Using the following expression for the variation of the Action:

$$\begin{aligned}C^{\alpha_1\beta}\frac{\partial A[\psi]}{\partial\psi^\beta(x_1)}&=-\psi^{\alpha_1}(x_1)+\lambda\sum_\alpha \hat{p}_\alpha\psi(x-\hat{\alpha})\\ &=-\psi^{\alpha_1}(x_1)+\lambda\Lambda^{\alpha_1\beta}\psi^\beta(x-\hat{\beta})\end{aligned}$$

we find:

$$\begin{aligned}&\langle\psi^{\alpha_1}(x_1)\psi^{\alpha_2}(x_2)\cdots\psi^{\alpha_n}(x_n)\rangle\\ &\quad=\lambda\Lambda^{\alpha_1\beta}\langle\psi^\beta(x_1-\hat{\beta})\psi^{\alpha_2}(x_2)\cdots\psi^{\alpha_2}(x_n)\rangle\\ &\quad+\delta_{x_1x_2}C^{\alpha_1\alpha_2}\langle\psi^{\alpha_3}(x_3)\psi^{\alpha_4}(x_4)\cdots\psi^{\alpha_n}(x_n)\rangle\\ &\quad-\delta_{x_1x_3}C^{\alpha_1\alpha_3}\langle\psi^{\alpha_2}(x_2)\psi^{\alpha_4}(x_4)\cdots\psi^{\alpha_n}(x_n)\rangle+\cdots\end{aligned} \tag{36}$$

Thus we have got the equation for the correlators of the 2D IM, together with its contact terms, comp. eq. (22).

3D IM

The Action for the model can be given in the following form (comp. [13, 14]):

$$
\begin{aligned}
A[\psi] &= -\frac{1}{2} \sum_{x,\mu=1,2,3} \bar{\psi}_\mu(x)\psi_\mu(x) \\
&\quad + \lambda \sum_{x,\mu<\nu} (\bar{\psi}_\nu(x+\hat{\mu})\hat{p}_{\alpha_\nu(\vec{\mu})}\psi_\nu(x))(\bar{\psi}_\mu(x+\hat{\nu})\hat{p}_{\alpha_\mu(\vec{\nu})}\psi_\mu(x)) \\
&= -\frac{1}{2} \sum_{x,\mu} \psi_\mu(x) C^{-1} \psi_\mu(x) \\
&\quad + \lambda \sum_{x,\mu<\nu} (\psi_\nu(x+\hat{\mu})\tilde{p}_{\alpha_\nu(\vec{\mu})}\psi_\nu(x))(\psi_\mu(x+\hat{\nu})\tilde{p}_{\alpha_\mu(\vec{\nu})}\psi_\mu(x)) \\
&= \sum_x (\psi^3_\mu\psi^1_\mu + \psi^4_\mu\psi^2_\mu + \psi^1_\mu\psi^2_\mu + \psi^3_\mu\mu^4_\mu + \psi^2_\mu\psi^3_\mu + \psi^1_\mu\psi^4_\mu) \\
&\quad + \lambda \sum_x (\psi^3_1(x+\hat{2})\psi^1_1(x)\psi^4_2(x+\hat{1})\psi^2_2(x) + \psi^3_2(x+\hat{3})\psi^1_2(x)\psi^4_3(x+\hat{2})\psi^2_3(x) \\
&\qquad + \psi^3_3(x+\hat{1})\psi^1_3(x)\psi^4_1(x+\hat{3})\psi^2_1(x))
\end{aligned} \tag{37}
$$

The anticommuting link variables in the Action above are in one to one correspondence with the tailed link objects, built in Section 3–see eq. (15) and Fig. 4b.

The partition function is given by the FI

$$
\begin{aligned}
Z(\lambda) &= \int \mathscr{D}\psi \exp\{A[\psi]\} \\
\mathscr{D}\psi &= \prod_{x,\mu} (-d\psi^1_\mu(x) d\psi^3_\mu(x) d\psi^2_\mu(x) d\psi^4_\mu(x)) \\
\exp\{A[\psi]\} &= \prod_{x,\mu} (1 - \bar{\psi}_\mu\psi_\mu - \psi^3_\mu\psi^1_\mu\psi^4_\mu\psi^2_\mu) \\
&\quad \times \prod_x (1 + \lambda\psi^3_1(x+\hat{2})\psi^1_1(x)\psi^4_2(x+\hat{1})\psi^2_2(x)) \\
&\quad \times (1 + \lambda\psi^3_2(x+\hat{2})\psi^1_2(x)\psi^4_3(x+\hat{2})\psi^2_3(x)) \\
&\quad \times (1 + \lambda\psi^3_3(x+\hat{1})\psi^1_3(x)\psi^4_1(x+\hat{3})\psi^2_1(x))
\end{aligned} \tag{38}
$$

After integration we shall get a sum over closed contours, which is the same as that found in the high temperature expansion for the partition function of the 3D Gauge IM, with $\lambda = \operatorname{th}\beta$ (or low temperature expansion of the usual 3D IM, with $\lambda = \exp(-2\beta)$). This can be checked as follows. First we remark that the integration of each term of the expansion of $Z(\lambda)$ in powers of λ factorizes into integrations in 2D planes of the 3D cubic lattice. In fact, the expression (38) has the form which allows grouping together the products of variables belonging to the same lattice plane (these are the variables which have in common one of the coordinate numbers x_1, x_2 or x_3 and also the link index μ), without producing any sign changes. Moreover, the integration in each 2D plane of the 3D lattice is precisely the same as that for the λ-expansion of the 2D theory (32). So, if it is proved already that the integration of (32) provides the correct signs for the expansion of the 2D IM partition function in the sum over closed contours, then it implies automatically that the integration of (38) will provide the correct signs also for the expansion in closed surfaces of the 3D IM partition function.

Next we consider the contour correlation function:

$$\psi^{\alpha_1\cdots\alpha_L}(C)=\langle \prod_C \psi^{\alpha_k}_{\mu_k}(x_k)\rangle$$
$$(39)\qquad =\frac{1}{Z}\int \mathcal{D}\psi\, e^{A[\psi]} \prod_C \psi^{\alpha_k}_{\mu_k}(x_k)$$

We assume here that the product of anticommuting link variables is ordered along the contour. Suppose we want to derive equation for the link (x_1, μ_1) of the contour. In that case, we take the variable $\psi^{\alpha_1}_{\mu_1}(x_1)$ out of the product in (39), and get in the result the correlator for an open contour:

$$(40)\qquad \langle \prod{}' \psi\rangle=\frac{1}{Z}\int \mathcal{D}\psi\, e^{A[\psi]}\psi^{\alpha_2}_{\mu_2}(x_2)\cdots\psi^{\alpha_L}_{\mu_L}(x_L)$$

which vanishes of course, but it makes no difference for our present purpose. We perform a variation of the variable $\psi^{\alpha_1}_{\mu_1}(x_1)$ under the FI in (40). This gives the equation:

$$(41)\qquad C^{\alpha_1\beta}\left\langle \frac{\partial A}{\partial\psi^{\beta_1}_{\mu_1}(x_1)}(\prod{}'\psi)\right\rangle+C^{\alpha_1\beta}\left\langle \frac{\partial}{\partial\psi^{\beta_1}_{\mu_1}(x_1)}(\prod{}'\psi)\right\rangle=0$$

The variation of the Action has the following form (see also Fig. 12):

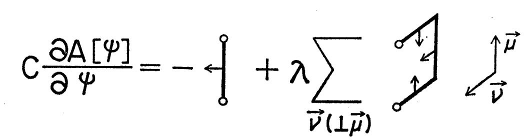


Fig. 12 The graphic form for the variation of the Action (42).

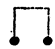
$$(42)\qquad C\frac{\partial A[\psi]}{\partial\psi_\mu(x_1)}=-\psi_\mu(x_1)+\lambda\sum_{\vec\nu(\perp\vec\mu)}\hat p_{\alpha_\mu(\vec\nu)}\psi_\mu(x-\hat\nu)\cdot(\psi_\nu(x+\hat\mu)\hat p_{\alpha_\nu(\vec\mu)}\psi_\nu(x))$$

Here the sum $\vec\nu\ (\perp\vec\mu)$ is taken over all the lattice directions orthogonal to $\vec\mu$. E.g. if $\mu=1$ then $\vec\nu=\vec 2, \vec 3, -\vec 2, -\vec 3$, $\psi_{-2}(x)\equiv\psi_2(x-\hat 2)$, $\psi_{-3}(x)\equiv\psi_3(x-\hat 3)$.

From (41) and (42) we find the equation

$$\langle \prod \psi\rangle=\langle\psi^{\alpha_1}_{\mu_1}(x_1)\psi^{\alpha_2}_{\mu_2}(x_2)\cdots\rangle$$
$$(43)\qquad =\lambda\sum_{\vec\nu(\perp\vec\mu)}\hat p_{\alpha_{\mu_1}(\vec\nu)}\langle\psi_{\mu_1}(x_1-\hat\nu)(\bar\psi_\nu(x+\hat\mu_1)\hat p_{\alpha_\nu(\vec\mu_1)}\psi_\nu(x))\psi_{\mu_2}(x_2)\psi_{\mu_3}(x_3)\cdots\rangle$$

$$+\sum_{k=2}^{L}(-1)^{k-1}\delta_{x_1x_k}\delta_{\mu_1\mu_k}C^{\alpha_1\alpha_k}\langle\widetilde{\prod_{(1,k)}}\psi\rangle$$

which is the same as our equation for the contour correlator (16). We remark only that, because of the ordering along the contour, which is assumed in this section, the sign of the first term in the r.h.s. of eq. (43) may be changed, if we arrange the three variables, making the configuration  (see Fig. 12), according to the adopted ordering.

If we had used an alternative ordering of the contour variables, the one that we used in Section 4 (ordering in the groups of variables, belonging to the same 2D lattice plane):

$$\psi(C)=\langle\prod\psi\rangle=\langle\cdots\psi_3^{\alpha_1}(x_1)\psi_3^{\alpha_2}(x_2)\psi_3^{\alpha_3}(x_3)\psi_3^{\alpha_4}(x_4)\cdots\rangle$$

(shown here is only one group of variables, see Fig. 9) we would have got precisely the contact terms (29).

§ 6. Equipped paths. Continuum limit for the 2D Ising Model. The path amplitude for the spinor particle

It is easy to check that the solution of the equation (24) for the two-point correlator can be represented as a sum over "equipped" paths:

$$G(x, x_0)\equiv\langle\psi(x)\bar{\psi}(x_0)=\sum_{\mathscr{P}_{x,x_0}}\lambda^{L(\mathscr{P})}\prod_{\text{along }\mathscr{P}}\hat{p} \tag{44}$$

The sum here is taken over all the paths on the lattice, which joins the points x, x_0: $L(\mathscr{P})$ is the length of the path $\mathscr{P}$; λ is a scalar weight, and $\hat{p}$ is a matrix corresponding to a particular link of the path $\mathscr{P}$. $\hat{p}$ is one of the four matrices in (31), depending on the direction of the link-see Fig. 13.

In fact, by its definition, the sum over paths (44) satisfies the linear equation

$$G(x, x_0)=\sum_{\alpha=1,2,3,4}\lambda\hat{p}_\alpha G(x-\alpha, x_0)+\delta_{x,x_0} \tag{45}$$

which coincides with (24)-comp. Λ in (3) and $\{\hat{p}_\alpha\}$ in (31).

We remark that, in general, if the matrix function $G^{\alpha\alpha_0}(x, x_0)$ satisfies linear equation relating its values in the point x to those in the neighbouring points on the lattice $\{\vec{x}-\vec{t}, \vec{t}$ being the lattice vectors$\}$, then the equation can always be given in the form:

$$G^{\alpha\alpha_0}(x, x_0)=\sum_{\vec{t}}M^{\alpha\beta}(\vec{t})G^{\beta\alpha_0}(x-\vec{t}, x_0)+\delta_{x,x_0}C^{\alpha\alpha_0} \tag{46}$$

It assumes that its solution can be presented in the form of a sum over equipped (with matrices $\hat{M}(\vec{t})$) paths, joining the points x and x_0:

$$G^{\alpha\alpha_0}(x, x_0) = \sum_{\{\mathscr{P}_{x,x_0}\}} \Big(\prod_{\text{along } \mathscr{P}} \hat{M}(\vec{t}) \Big)^{\alpha\beta} \cdot C^{\beta\alpha_0} \tag{47}$$

We shall describe now in short the continuum limit for the sum over paths (44).

Singularities in the thermodynamic functions of the 2D IM arise when one of the eigenvalues of the matrix

$$\lambda\Lambda = \lambda \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

(see eqs. (3) and (24)) becomes equal to 1. The matrix becomes diagonal in the representation of angular momenta:

$$\psi_{\pm(1/2)} = \frac{1}{2} e^{\pm(i\varphi/2)} = \frac{1}{2} \begin{pmatrix} 1 \\ e^{\pm(i\pi/4)} \\ e^{\pm(i\pi/2)} \\ e^{\pm(i3\pi/4)} \end{pmatrix}; \quad \psi_{\pm(3/2)} = \frac{1}{2} e^{\pm(i3\varphi/2)} = \begin{pmatrix} 1 \\ e^{\pm(i3\pi/4)} \\ e^{\pm(i3\pi/2)} \\ e^{\pm(i9\pi/4)} \end{pmatrix} \tag{48}$$

In this basis

$$\lambda\tilde{\Lambda} = \lambda \begin{pmatrix} \sqrt{2}+1 & & & \\ & \sqrt{2}+1 & & \\ & & -\sqrt{2}+1 & \\ & & & -\sqrt{2}+1 \end{pmatrix}$$

From this diagonal form for the matrix Λ we find the phase transition point of the 2D IM:

$$\lambda_C \equiv \text{th}_{\beta_C} = \frac{1}{\sqrt{2}+1}$$

For the continuum limit (which describe critical behaviour of the model in the vicinity of the phase transition point) only the components $\psi_{\pm(1/2)}$ (having eigen values close to 1) will be relevent. This implies that the 4×4 matrices reduce to 2×2 ones. To the first order in $(\lambda-\lambda_C)$ and $\vec{k}$ (the space momentum, after the Fourier transform):

$$\langle \mathscr{S} | \Lambda(\vec{k}) | \mathscr{S}' \rangle = \frac{1}{\lambda_C} - \frac{i}{2\lambda_C} \hat{k} \quad \left(\mathscr{S}, \mathscr{S}' = \pm \frac{1}{2} \right) \tag{49}$$

and the eq. (24), in the momentum representation

$$G(\vec{k}) = \lambda \Lambda(\vec{k}) G(\vec{k}) + 1$$

reduces to the Dirac equation (in 2D Euclidian space-time):

$$(i\hat{k} + m) G(\vec{k}) = 2 \tag{50}$$

Here $\hat{k} = \vec{k}\vec{\gamma}$; $\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$;

$$m = 2(\lambda_C - \lambda)/\lambda_C.$$

It is easy to check that

$$\langle \mathscr{S} | \hat{p}_\alpha | \mathscr{S}' \rangle = \frac{1 + \vec{t}_\alpha \vec{\gamma}}{2} \cdot \frac{1 + \sqrt{2}}{2} \tag{51}$$

here $\vec{t}_\alpha$ is a unit vector in the direction α (see Fig. 13). The sum over paths (44) reduces to

$$G(x, x_0) = \sum_{\mathscr{P} x_1 x_0} (\tilde{\lambda})^{L(\mathscr{P})} \prod_{\mathscr{P} x_1 x_0} \frac{1 + \hat{t}}{2} \tag{52}$$

and the equation (45) takes the form:

$$G(x_1 x_0) = \sum_{\vec{t}} \tilde{\lambda} \, \frac{1 + \hat{t}}{2} G(x - \vec{t}, x_0) \tag{53}$$

It is easy to check that in the straightforward continuum limit this equation reduces to (50) (with 2 in the r.h.s. replaced by 1, which corresponds to a simple change of the normalization).

We shall make now a series of remarks on the Feynman representation for the propagator of a free spinor particle, in the space time of any number of dimensions, not just 2D, as above. We remark that the path amplitude for the spinor particle in the expression for the propagator (52)

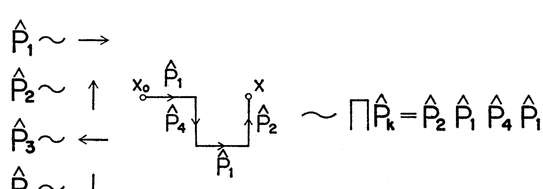


Fig. 13 The definition of the phase factor $\prod_{\mathscr{P}_{x,x_0}} \hat{p}$ in eq. (44).

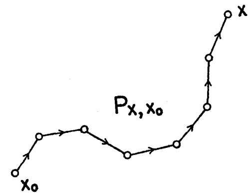


Fig. 14 The path of the propagating particle having the form of a broken line, which is made of straight segments of equal length.

is correct in the space of any number of dimensions. We just have to use the corresponding γ-matrices. Also, the expression (52) applies to any regularization of the sum over paths, not just to the regular lattice case. It is easy to check e.g. that if we use in (52) the summation over paths made by broken lines (Fig. 14), we shall find, in the continuum limit, the same Dirac equation (50).

An analogue of the proper time regularization (often used for the scalar particle propagator) is also possible for the sum over paths (52):

$$G(\vec{x}, \vec{x}_0) = \int_0^\infty d\tau e^{-m\tau} \mathscr{F}_\tau(\vec{x}, \vec{x}_0)$$

$$(54) \qquad \mathscr{F}_\tau(\vec{x}, \vec{x}) = \int_{\substack{\vec{x}(0)=\vec{x}_0 \\ \vec{x}(\tau)=\vec{x}}} \mathscr{D}\vec{x}(\tau') \exp\left\{-\frac{1}{a}\int_0^\tau \frac{\dot{\vec{x}}^2}{2} d\tau'\right\} \cdot P \exp\left\{-\frac{1}{a}\int \vec{\gamma} d\vec{x}\right\}$$

Here P stands for the ordering along the paths; a is a cut-off scale in the coordinate space, which appears explicitly in this regularization in case of a spinor particle.

The function $\mathscr{F}_\tau(\vec{x})$ satisfies the equation:

$$(55) \qquad \frac{\partial}{\partial\tau}\mathscr{F}_\tau(\vec{x}) = -\hat{\partial}\mathscr{F}_\tau(\vec{x}) + \frac{a}{2}\partial^2\mathscr{F}(\vec{x})$$

with the initial condition $\mathscr{F}_\tau(\vec{x})|_{\tau\to 0} \to \delta(\vec{x})$.

The representation (54) is derived as follows:

$$G(\vec{k}) = \frac{1}{i\hat{k}+m} = \int_0^\infty d\tau\, e^{-(m+i\hat{k})\tau}$$

$$G(\vec{x}) = \int_0^\infty d\tau\, e^{-m\tau} \int d^2k\, e^{i\hat{k}\tau - i\vec{k}\vec{x}}$$

We introduce next a cut-off at short distances:

$$(56) \qquad \to G(\vec{x}) = \int_0^\infty d\tau\, e^{-m\tau} \int d^2k \exp\left\{i\hat{k}\tau - i\vec{k}\vec{x} - \frac{ak^2}{2}\right\}$$

$$=\int_0^\infty d\tau\, e^{-m\tau}\int_{\substack{\vec{x}(0)=\vec{x}_0=0\\ x(\tau)=\vec{x}}}\mathscr{D}\vec{x}(\tau')\exp\left\{-\frac{1}{a}\int_0^\tau\frac{\dot{\vec{x}}^2}{2}d\tau'\right\}\cdot P\exp\left\{\frac{1}{a}\int\vec{\gamma}d\vec{x}\right\}$$

We remark also, that for sufficiently smooth paths (which change substantially their direction at distances $\gg a$) the spinor phase factor of the path amplitude in (54) can be transformed as follows:

(57)
$$P\exp\left\{\frac{1}{a}\int\vec{\gamma}\Delta\vec{x}\right\}\approx\prod_{\mathscr{P}}\exp\left\{\vec{\gamma}\frac{\Delta\vec{x}}{a}\right\}=\prod_{\mathscr{P}}\left(\mathrm{ch}\frac{\Delta x}{a}+\vec{\gamma}\frac{\Delta\vec{x}}{\Delta x}\mathrm{sh}\frac{\Delta x}{a}\right)$$
$$=\left(\prod_{\mathscr{P}}\mathrm{ch}\frac{\Delta x}{a}\right)\cdot\prod_{\mathscr{P}}\left(1+\vec{\gamma}\vec{t}\cdot\mathrm{th}\frac{\Delta x}{a}\right)\approx\exp\left\{\frac{L(\mathscr{P})}{a}\right\}\prod_{\mathscr{P}}\frac{1+\vec{\gamma}\vec{t}}{2}$$

We again arrived at the phase factor in (52). Transformations (56), (57) can be regarded as its derivation.

Notice, finally, that for the smooth paths, the spinor phase factor

(58)
$$\phi(\mathscr{P})=\prod_{\mathscr{S}(\mathrm{along}\,\mathscr{P})}\frac{1+\vec{t}(\mathscr{S})\vec{\gamma}}{2}$$

can be given a more natural form of a path ordered product of rotational matrices:

(59)
$$\phi(\mathscr{P})=P\exp\left\{\frac{i}{4}\int\sum\nolimits^{\mu\nu}d\omega_{\mu\nu}(\mathscr{S})\right\}\cdot\frac{1+\hat{t}_0}{2}$$

Here

$$\sum\nolimits^{\mu\nu}=-\frac{i}{2}[\gamma^\mu,\gamma^\nu]$$

$$d\omega_{\mu\nu}(\mathscr{S})=t^\mu(\mathscr{S})\wedge dt^\nu(\mathscr{S})\equiv t^\mu(\mathscr{S})dt^\nu(\mathscr{S})-t^\nu(\mathscr{S})dt^\nu(\mathscr{S})$$

and $\mathscr{S}$ is the length parameter along the path. We have used the following transformations in deriving (59):

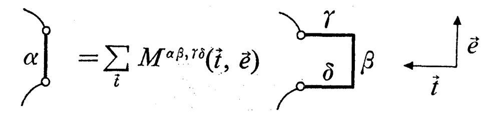
(60)
$$\frac{1+\hat{t}(\mathscr{S}+\Delta\mathscr{S})}{2}\,\frac{1+\hat{t}(\mathscr{S})}{2}\approx\left(\frac{1+\hat{t}(\mathscr{S})}{2}+\frac{dt(\mathscr{S})}{2}\right)\frac{1+\hat{t}(\mathscr{S})}{2}$$
$$=\left(1+\frac{d\hat{t}(\mathscr{S})}{2}\right)\frac{1+\hat{t}(\mathscr{S})}{2}=\left(1+\frac{d\hat{t}(\mathscr{S})}{2}\hat{t}(\mathscr{S})\right)\frac{1+\hat{t}(\mathscr{S})}{2}$$
$$=\left(1+\frac{1}{4}[\gamma^\mu,\gamma^\nu]dt^\mu(\mathscr{S})t^\nu(\mathscr{S})\right)\frac{1+\hat{t}(\mathscr{S})}{2}\to\phi(\mathscr{S}+d\mathscr{S})$$
$$=\left(1+\frac{i}{4}\sum\nolimits^{\mu\nu}d\omega_{\mu\nu}(\mathscr{S})\right)\phi(\mathscr{S})$$

§ 7. Equipped surfaces

As it was discussed in the preceding section, the solution of the linear equation on lattice can be represented by a sum over free (no additional weight on the path cross-sections) but, in general, equipped paths on lattice-see eq. (46) and its solution (47).

Analogously, in case of contour linear equations on lattice, which, by definition, can be presented in the form:

(61)



their solution can be given by a sum over equipped surfaces, bounded by the contour of the correlator. The amplitude for a particular surface in the sum is given by the product of plaquette matrices

$$M^{\alpha\beta,\gamma\delta}(\vec{t}, \vec{e}) \tag{52}$$

taken along the surface:

$$\psi^{\alpha_1 \cdots \alpha_L}(C)=\sum_{\{S_c\}} \prod_{S_c} \hat{M} \tag{63}$$

When taking this product, the internal indeces are contracted and summed over, while the external ones make the indeces of the contour correlator (Fig. 15).

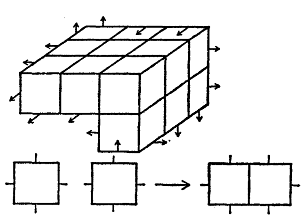


Fig. 15 Product of plaquatte matrices taken along the surface.

In our particular case of the contour correlators of the 3D IM, we have the eq. (16) (or (43)), which has the form of eq. (61) with the plaquette matrices

$$M^{\alpha\beta,\gamma\delta}(\vec{\alpha}_\mu(\nu), \vec{\alpha}_\nu(\mu))=\lambda \hat{p}^{\alpha\beta}_{\alpha_\mu(\nu)} \hat{p}^{\gamma\delta}_{\alpha_\nu(\mu)} \tag{64}$$

In conclusion, the contour correlators of the 3D IM, introduced in Section 3, can be given by the sum over free surfaces, equipped with matrices (64), -see also Fig. 16.

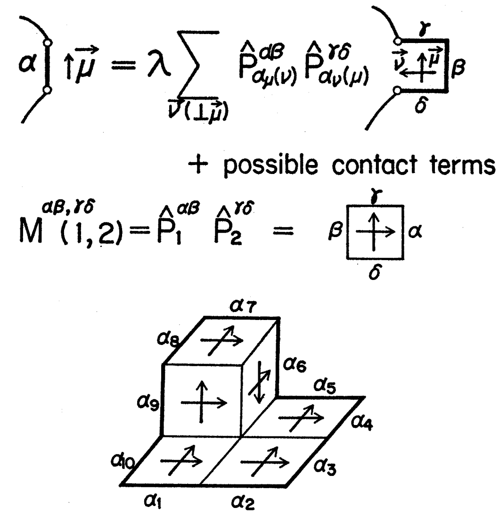
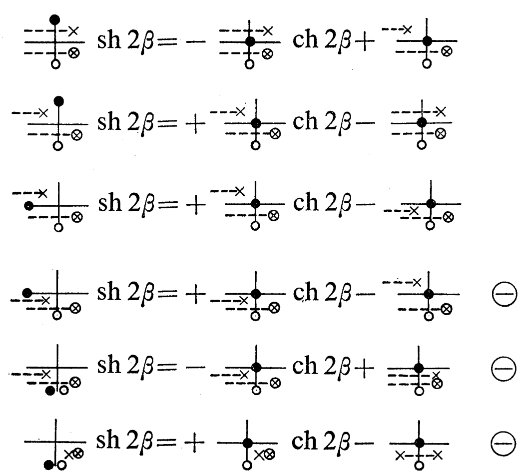


Fig. 16 Contour equation and an example of the equipped surface for the contour correlators of the 3D Ising Model.

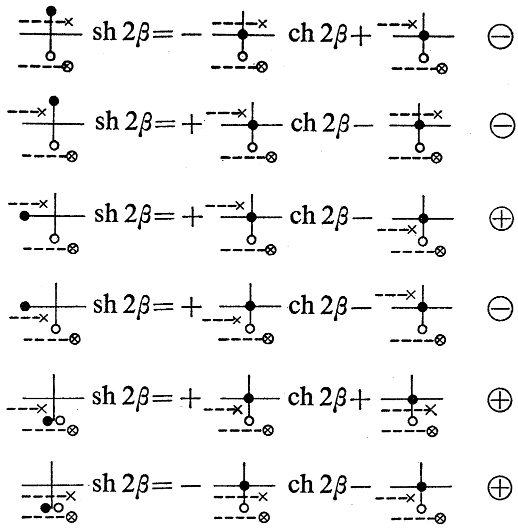
§ 8. Appendix

1. The quantity $G^{21}(0)$ is defined from the equations for the nearest neighbours in the following way (comp. with the derivation of equations in Section 2):



(65)

(66)



(67)

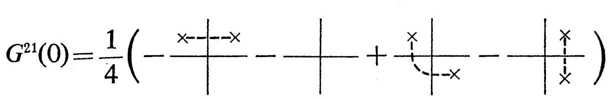
(68)

The signs in the contact configurations (65–68) are chosen so that the correct equation is found:

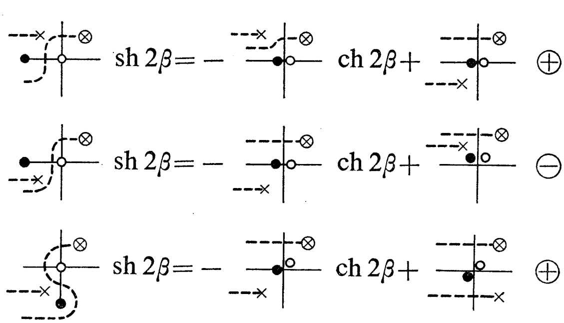
$$G^{\alpha\alpha_0}(x, x-\hat{2}) = \mathrm{th}\,\beta\cdot \Lambda^{\alpha\gamma}\cdot G^{\gamma\alpha_0}(x-\hat{\gamma}, x-\hat{2})$$

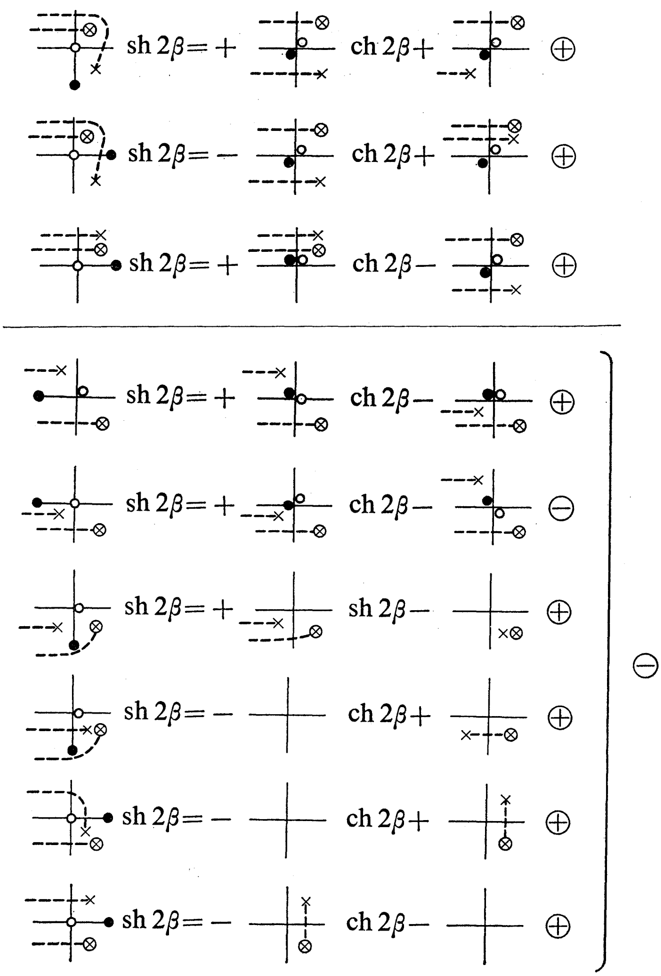
It gives

(69)

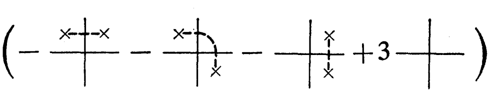


2. We can find now the component C^{21} in the eq. (22) from the equation for the contact configuration:





By summing the above we find:

$$\mathrm{th}\,\beta\cdot\Lambda^{2r}\cdot G^{r1}(x_0\hat{r},\,x_0)=\frac{1}{4}\Big($$


By using the expression (69) for $G^{21}(0)$ we finally get:

$$G^{21}(x_0,\,x_0)=\mathrm{th}\,\beta\cdot\Lambda^{2r}\cdot G^{r1}(x_0-\hat{r},\,x_0)-1$$

It gives $C^{21}=-1$, — comp. the eq. (21).

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