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Projective Degenerations of Surfaces according to S. Tsunoda

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§ 1. Introduction

This is a partial and incomplete account of a work of S. Tsunoda on projective degenerations of algebraic surfaces. We shall begin with some necessary definitions. In the following, we take the complex field $\boldsymbol{C}$ as the ground field.

Degenerations of algebraic surfaces as considered in Persson [9] mean the surfaces $\mathscr{X}_0$ appearing in the following setup:

Let $\pi: \mathscr{X} \to \Delta$ be a proper, flat morphism of a smooth threefold X defined over $\boldsymbol{C}$ onto a small open disc Δ such that π is smooth over $\Delta^* = \Delta - \{0\}$. Then $\mathscr{X}_0 = \pi^{-1}(0)$. Namely, $\mathscr{X}_0$ is thought of as a degeneration of a general smooth fiber $\mathscr{X}_t$ $(t \neq 0)$.

In considering degenerations of algebraic surfaces, it is standard, in view of Mumford's theorem on semistable reduction, to treat the case where the fiber $\mathscr{X}_0$ is a reduced divisor with simple normal crossings, i.e., $\pi: \mathscr{X} \to \Delta$ is a semistable degeneration.

The first important contribution was made by Kulikov [5], where he considered the degenerations of $K3$ surfaces and he employed essentially an operation of non-algebraic nature, called "a generic contraction". The same subject was taken up later by Persson-Pinkham [10]. Their result says that:

If $\pi: \mathscr{X} \to \Delta$ is a semistable degeneration of algebraic surfaces such that a general fiber $\mathscr{X}_t$, $t \neq 0$ has trivial canonical divisor and that all components of $\mathscr{X}_0$ are algebraic, then there exists a semistable modification $\pi': \mathscr{X}' \to \Delta$ of π such that the canonical divisor of the total space $\mathscr{X}'$ is trivial.

D. Morrison [8] and Tsuchihashi [11] considered the degenerations of Enriques' and hyperelliptic surfaces.

On the other hand, concerning the construction of a minimal model in dimension three, one can consider the following problem (cf. [4]).

Given a nonsingular projective threefold X of non-negative Kodaira

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dimension defined over $\boldsymbol{C}$, find a normal projective threefold Y such that

(i) Y is birationally equivalent to X,

(ii) Y has at worst terminal (or canonical) singularities,

(iii) the canonical divisor of Y is numerically effective, *nef* for short.

Moreover, if such a threefold Y exists, describe explicitly a way of finding Y from X.

Tsunoda's theorem which is to be stated below can be thought of as a positive answer to this problem in the case of a degeneration of surfaces. Now we shall turn to Tsunoda's work.

An $\mathcal{S}$-degeneration of algebraic surfaces is a surjective morphism $\pi: \mathcal{X} \to \Delta$ from a nonsingular projective threefold $\mathcal{X}$ onto a nonsingular complete curve Δ such that:

(1) For any point P of Δ, the fiber $\pi^*(P)$ is a connected reduced divisor whose irreducible components are smooth and whose singularities are at worst those of normal crossings; we simply say that $\pi^*(P)$ is a divisor with simple normal crossings.

(2) If $\pi^{-1}(P)$ is nonsingular then the canonical divisor $K(\pi^{-1}(P))$ is nef, whence $\pi^{-1}(P)$ is a minimal surface.

On the other hand, a normal, $\boldsymbol{Q}$-factorial, projective threefold $\mathcal{Y}$ together with a surjective morphism $\pi_0: \mathcal{Y} \to \Delta$ is called *$\mathcal{S}$-regular* if there exist an $\mathcal{S}$-degeneration of surfaces $\pi: \mathcal{X} \to \Delta$ and a sequence of birational morphisms

$$\mathcal{X} = \mathcal{Y}_n \xrightarrow{f_n} \mathcal{Y}_{n-1} \longrightarrow \cdots \longrightarrow \mathcal{Y}_2 \xrightarrow{f_2} \mathcal{Y}_1 \xrightarrow{f_1} \mathcal{Y}_0 := \mathcal{Y}$$

such that the following conditions are satisfied:

(1) $\mathcal{Y}_i$ is $\boldsymbol{Q}$-factorial for each i;

(2) The induced rational mapping $\pi_i := \pi \cdot (f_{i+1} \cdots f_n)^{-1}: \mathcal{Y}_i \to \Delta$ is a morphism;

(3) There exists an irreducible Weil divisor D_i on $\mathcal{Y}_i$ such that f_i induces an isomorphism $\mathcal{Y}_i - D_i \rightleftarrows \mathcal{Y}_{i-1} - f_i(D_i)$ and $f_i(D_i)$ is a point;

(4) $-K(\mathcal{Y}_i)|_{D_i}$ is ample on D_i and the image of the canonical homomorphism $\mathrm{Pic}(\mathcal{Y}_i) \otimes \boldsymbol{Q} \to \mathrm{Pic}(D_i) \otimes \boldsymbol{Q}$ has rank 1;

(5) For any i and any irreducible component X of a fiber of $\pi_i: \mathcal{Y}_i \to \Delta$, the proper transform X' of X on $\mathcal{X}$ is the minimal resolution of singularities of X.

With the above notations, the composite $f := (f_1 \cdot f_2 \cdots f_n): \mathcal{X} \to \mathcal{Y}$ is called *a good resolution of* (singularities of) $\mathcal{Y}$; we also say that $f: \mathcal{X} \to \mathcal{Y}$ is *a good contraction of D_i*'s.

A main result of Tsunoda's is the following:

Main Theorem. *Let $\pi: \mathcal{X} \to \Delta$ be an $\mathcal{S}$-degeneration of surfaces, where Δ has positive genus. Then there exists an $\mathcal{S}$-regular* 3*-fold $\mathcal{Y}$ over Δ such* *that $\mathscr{Y}$ is birational to $\mathscr{X}$ and K_Y is nef, where $\mathscr{Y}$ might have a good resolution other than $\mathscr{X}$.*

Our notations and terminology conform to those in [2], [6] and [7]. We shall add some more. The canonical divisor of a normal projective variety V is denoted by K_V or $K(V)$. We call a nonsingular rational curve C on nonsingular projective surface a $(-n)$ curve if $(C^2)=-n$. Hence a (-1) curve stands for an exceptional curve of the first kind. A reduced effective divisor D is often confused with its support $\mathrm{Supp}(D)$. For a birational morphism $f\colon V\to W$ and a subvariety Z on W, the proper transform of Z on V under f is denoted by $f'(Z)$ if it is well-defined.

In writing this article, the author had long discussions with S. Tsunoda on the subject treated here. Moreover, the notes taken by D. Morrison on the lectures given by Tsunoda at Kyoto University were very helpful. The author wishes to express his gratitude to both of them. The incompleteness of this article is partly due to the lack of a complete paper by Tsunoda himself but mostly due to the lack of the author's competence. After this paper was completed, Tsunoda himself wrote up his paper [13] contributed to this volume, where he treats the case (3) in our Main Lemma. Therefore, this paper plays only a role of introducing the readers to Tsunoda's work.

§ 2. Surface pairs with log-terminal singularities

2.1. Let V be a normal, $\boldsymbol{Q}$-factorial, projective variety and let D be a reduced effective Weil divisor on V. We say that a pair (V, D) has only *log-terminal singularities* if there exists a resolution of singularities $f\colon W\to V$ satisfying the following conditions:

(1) Let D' be the union of the proper transform $f'(D)$ of D and all exceptional varieties of f. Then D' as a reduced effective divisor on W is a divisor with simple normal crossings;

(2) $\Gamma := K_W+D'-f^*(K_V+D)$ is effective, where $f^*(K_V+D)$ is the pull-back of a $\boldsymbol{Q}$-Cartier divisor K_V+D, i.e., $f^*(K_V+D)=(1/N)f^*(N(K_V+D))$ provided $N(K_V+D)$ is a Cartier divisor for an integer $N>0$;

(3) $\mathrm{Supp}(\Gamma)$ is the union of *all* exceptional varieties of f.

2.2. In case $\dim V=2$, log-terminal singularities can be characterized as follows:

Lemma ([12; Theorem 1.2]). *Let (S, D) be a pair of a normal, $\boldsymbol{Q}$-factorial, projective surface S and a reduced effective Weil divisor D with simple normal crossings*(*). *Then (S, D) has only log-terminal singularities*

(*) This implies that each irreducible component of D is smooth and if two irreducible components of D meet at a point, say P, V is smooth at P and the two components meet transversally at P.

iff the following condition is satisfied: *Let* $\mathrm{Sing}(S)=\{P_1, \cdots, P_r, P_{r+1}, \cdots, P_s\}$, *which is indexed in such a way that* $P_1, \cdots, P_r$ *are on* D *and* $P_{r+1}, \cdots, P_s$ *are not on* D. *Then, for* $1 \leqq i \leqq r$, P_i *is a cyclic quotient singular point of* S, *whereas, for* $r+1 \leqq j \leqq s$, P_j *is a (not necessarily cyclic) quotient singular point.*

We simply say that a pair (S, D) as above is *log-terminal.* Let (S, D) be a surface pair which is log-terminal. A curve l on S is called a generalized (-1) curve if $(l^2)<0$ and $(K_S+D, l)<0$. We have the following:

2.2.1. Lemma. *Let* l *be a generalized* (-1) *curve on a log-terminal surface* (S, D). *Then there exist a pair* $(\bar{S}, \bar{D})$, *which is log-terminal, and a birational morphism* $\sigma: S \to \bar{S}$ *such that* $\bar{D}=\sigma_*(D)$, $\sigma(l)$ *is a point and* $S-D\cup l \xrightarrow{\sim} \bar{S}-\sigma(D\cup l)$.

Proof. Let $f: W \to S$ be the minimal resolution of singularities of S. Let $E:=f'(l)$, let F be the sum of all (irreducible) exceptional curves and let $G:=f'(D)$ be the proper transform of D. Since S is log-terminal, each connected component of F is either an admissible rational rod or an admissible rational fork. For the terminology and the relevant result, one is referred to [6]. Let $B:=G+F$ and let B^* be the $\boldsymbol{Q}$-divisor obtained from B by peeling the bark of F. Then we have $B^*+K_W \equiv f^*(D+K_S)$ in $N(W)_{\boldsymbol{Q}}$. Suppose, first of all, that l is not a component of D. Then $(E^2)<0$ and $(B^*+K_W, E)=(D+K_S, l)<0$, whence E is an exceptional curve of the first kind. Since $0 \leqq (B^*, E)<1$, we have $([B^*], E)=0$. Moreover, we can show that E meets at most two connected components, meeting each connected component in a single point transversally, so that, under the contraction of E and all subsequently contractible curves in F, the image of the above (at most two) connected components is an admissible rational rod or fork (cf. [6]). Hence, under the contraction $\rho: W \to \bar{S}$ of $E+F$, the pair $(\bar{S}, \bar{D})$ with $\bar{D}:=\rho_*(B)$ is log-terminal. Suppose that l is a component of D. Then E is a component of B, and we have one of the cases considered in [6; 1.4.2], where D_0 stands for E. The hypothesis that $(l^2)<0$ implies that the intersection matrix of $T+D_0$ in the case (i) (resp., $T+D_0+T'$ in the case (ii), or the fork F in the case (iii)) is negative definite. Let $\rho: W \to \bar{S}$ be the contraction of $T+D_0$ (resp., $T+D_0+T'$, or F). Then the pair $(\bar{S}, \bar{D})$ with $\bar{D}:=\rho_* B$ is obviously log-terminal.

A log-terminal surface pair (S, D) is called *almost minimal* if there are no generalized (-1) curves on S.

2.3. Let (V, D) be now a pair of a nonsingular projective surface V and a reduced effective divisor $D=C_1+\cdots+C_r$ with simple normal crossings. A partial sum $L=C_{i_1}+\cdots+C_{i_s}$ of irreducible components of D is a *linear chain with a tip* C_{i_1} if $(C_{i_j}, C_{i_{j+1}})=1$ for $1\leqq j<s$ and C_{i_j} $(1\leqq j<s)$ meets no other curves of D. The chain L is called *a rod*, *a twig* or *a maximal twig*, respectively, according as $(C_{i_s}, D-C_{i_s})=1$, 2, or L is not extended further as a linear chain. A linear chain L is *rational* (resp. *admissible*) if every irreducible component is rational (resp. has self-intersection number $\leqq -2$). *The determinant of* L is $d(L):=|\det(I)|$, where I is the intersection matrix associated with L. *A rational fork* in D is a partial sum of irreducible components of D which consists of an irreducible rational component D_0, called the *central component* of the fork, and three admissible rational maximal twigs L_1, L_2 and L_3 whose determinants are, up to a permutation, one of the following triplets: $\{2, 2, n\}$ $(n\geqq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$. The dual graph of a linear chain L with determinant $n\leqq 5$ is given in Figure 1.

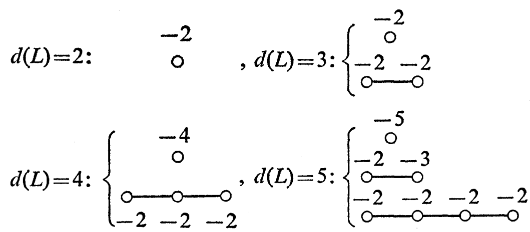

Figure 1

A rational fork F is *admissible* if its intersection matrix is negative definite. This is equivalent to saying $(D_0^2)\leqq -2$. Thus the dual graph of an admissible rational fork or an admissible rational rod is the one of the minimal resolution of a quotient singular point and vice versa.

Given a nonsingular pair (V, D) as above, let $\{T_\lambda\}$, $\{R_\mu\}$ and $\{F_\nu\}$ be respectively the sets of all admissible rational maximal twigs which are not contained in admissible rational forks, all admissible rational rods and all admissible rational forks. Then there exists a unique effective Q-divisor $D^\#$ satisfying the following conditions:

(1) $D\geqq D^\#$, and Supp D and Supp $D^\#$ differ by all admissible rational rods and forks whose irreducible components have self-intersection number -2; such rods and forks are called (-2) rods and (-2) forks, respectively;

(2) Let $Bk(D):=D-D^\#$. Then each connected component of $Bk(D)$ is one of T_λ's, R_μ's and F_ν's;

(3) $(D^\#+K_V, C_i)=0$ for every irreducible component C_i of $Bk(D)$.

Let $\sigma\colon V\to\bar{V}$ be the contraction of all connected components of $Bk(D)$, and let $\bar{D}:=\sigma_*(D)$. Then $\bar{V}$ is a normal projective surface, the surface pair $(\bar{V}, \bar{D})$ is log-terminal, and $D^{\#}+K_V \equiv \sigma^*(\bar{D}+K_{\bar{V}})$ in $N(V)_Q$.

By [6], it is known that, given a nonsingular pair (V, D) as above, one can find a surface pair $(\bar{V}, \bar{D})$, which is log-terminal, and a birational morphism $\phi: V \to \bar{V}$ such that $\bar{D}=\phi_*(D)$ and that the following condition (AM) is satisfied:

(AM) Let $\sigma: \tilde{V} \to \bar{V}$ be the minimal resolution of singularities on $\bar{V}$ and let $\tilde{D}$ be the union of $\sigma^{-1}(\bar{D})$ and the exceptional curves. Then $\sigma^*(\bar{D}+K_{\bar{V}}) \equiv \tilde{D}^{\#}+K_{\tilde{V}}$, and there are no curves C on $\tilde{V}$ such that $(\tilde{D}^{\#}+K_{\tilde{V}}, C)<0$ and the intersection matrix of $C+Bk(\tilde{D})$ is negative definite.

It is easy to see that the condition (AM) is equivalent to saying that $(\bar{V}, \bar{D})$ is almost minimal in the above sense. Indeed, the birational morphism $\phi: V \to \bar{V}$ factors as $\phi: V \xrightarrow{\tau} \tilde{V} \xrightarrow{\sigma} \bar{V}$ and $\tilde{D}=\tau_* D$. We call the pair $(\tilde{V}, \tilde{D})$ *an almost minimal model of* (V, D). A log-terminal surface pair (V, D) is called *a log-del Pezzo surface* if $-(D+K_S)$ is ample.

2.4. Now, regaining the notations in Section 1, we shall make several observations.

Remarks. (1) The threefold $\mathscr{Y}$, which is $\mathscr{S}$-regular over Δ, has finitely many singular points. Suppose that P is a singular point of $\mathscr{Y}$. Then we say that P is a singular point *of twig type* (resp. *of rod type*) if P lies on a double curve (resp. P does not lie on a double curve).

(2) Let F be a fiber of $\pi_i: \mathscr{Y}_i \to \Delta$ and let X be an irreducible component of F. Then the pair $(X, (F-X)|_X)$ is log-terminal. Suppose $D_i \subseteq F$, $D_i \neq X$ and $D_i \cap X \neq \emptyset$. Let $C:=D_i \cap X$. Then C is irreducible, $(C^2)_X<0$ and $(K_X+(F-X)|_X, C)<0$, i.e, C is a generalized (-1) curve on X. Thus the contraction of C (induced by the contraction of D_i on $\mathscr{Y}_i$) produces a surface pair which is also log-terminal.

Proof. Note that $X|_{D_i}$ is an effective Cartier divisor and that

$$\mathrm{rk}\{\mathrm{Im}(\mathrm{Pic}(Y_i)\otimes Q \longrightarrow \mathrm{Pic}(D_i)\otimes Q)\}=1.$$

Since $-K(\mathscr{Y}_i)|_{D_i}$ is ample, we know that $X|_{D_i}$ is ample. Hence $C:= X \cap D_i$ is connected. If C is reducible, then the component X must have a curve singularity, which is not allowed. (See Figure 2.) Therefore C is irreducible. Since D_i is contractible on $\mathscr{Y}_i$, C is contractible on X, whence $(C^2)_X<0$. Moreover, we have

$$0>(K(\mathscr{Y}_i), C)=(K_X+(F-X)|_X, C).$$

Hence C is a generalized (-1) curve with respect to the pair $(X, (F-X)|_X)$. Let $\bar{X}:=f_i(X)$ and $\bar{F}:=f_i(F)$. Then $f_i|_X: X \to \bar{X}$ is the contraction of C. Hence the pair $(\bar{X}, (\bar{F}-\bar{X})|_{\bar{X}})$ is log-terminal.

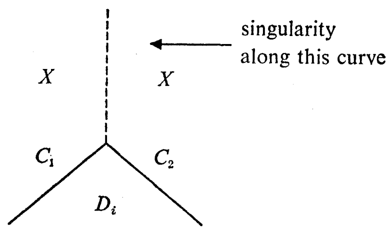

Figure 2

(3) The pair $(D_i, (F-D_i)|_{D_i})$ is a log-del Pezzo surface. In fact, $-K(\mathscr{Y}_i)|_{D_i} \equiv -(K(D_i)+(F-D_i)|_{D_i})$, whence follows the assertion.

§ 3. Extremal rational curves

3.1. Let V be a $\boldsymbol{Q}$-factorial, normal, projective variety. We define $\boldsymbol{R}$-modules $N_1(V)$ and $N^1(V)$ of rank $\rho(V)$ (=the Picard number) by

$$N_1(V) := \{\text{1-cycles}\}/(\equiv) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{R}$$

$$N^1(V) := \{\text{Cartier divisors}\}/(\equiv) \underset{\boldsymbol{Z}}{\otimes} \boldsymbol{R}.$$

In $N_1(V)$ endowed with a Euclidean metric topology, we consider *the closed effective cone* $\overline{NE}(V)$ *of curves on* V. Let D be a reduced effective divisor on V. A curve l on V is called *an extremal curve with respect to* K_V+D if $(K_V+D, l)<0$ and $R=H^{\perp} \cap \overline{NE}(V)$ for a suitable nef $\boldsymbol{Q}$-divisor H, where $R=\boldsymbol{R}_+[l]$ is the ray generated by l. When we say that l is an extremal curve without referring to K_V+D, we mean an extremal curve with respect to K_V.

3.2. Various cone theorems and contraction theorems associated with extremal rational curves are given in Mori [7] and Kawamata [2], [3]. Here we take as V a normal, $\boldsymbol{Q}$-factorial, terminal, projective threefold $\mathscr{Y}$ whose K_y is *not nef*. Let A be an ample Cartier divisor on y and let ε be a small positive number. We set

$$\overline{NE}_\varepsilon(\mathscr{Y}, A) := \{z \in \overline{NE}(\mathscr{Y}) \mid (z, K_y+\varepsilon A) \geqq 0\}.$$

Then we have the following result:

Theorem (cf. [2]). (1) *There exist finitely many* (*possibly singular*) *rational curves* l_i $(1 \leqq i \leqq n)$ *such that*

$$\overline{NE}(\mathscr{Y}) = \sum_{i=1}^{n} \boldsymbol{R}_+[l] + \overline{NE}_\varepsilon(\mathscr{Y}, A).$$

(2) *Let l be an extremal rational curve on $\mathscr{Y}$, and let H be a nef divisor such that $H^{\perp} \cap \overline{NE}(\mathscr{Y}) = \boldsymbol{R}_+[l]$. Then the following assertions hold true*:

(i) *The linear system $|mH|$ is free, i.e.,* $\mathrm{Bs}|mH| = \emptyset$ *for some integer* $m \gg 0$.

(ii) *Let $\phi\colon \mathscr{Y} \to \mathscr{X}$ be the morphism associated with $|mH|$ as above. Then, for an irreducible curve C, $\phi(C)$ is a point iff $C \in \boldsymbol{R}_+[l]$. If $(H^3) > 0$ then ϕ is birational.*

(iii) *For $D \in \mathrm{Pic}(\mathscr{Y}) \otimes \boldsymbol{Q}$, there exists $\bar{D} \in \mathrm{Pic}(\mathscr{X}) \otimes \boldsymbol{Q}$ with $D \equiv \phi^*(\bar{D})$ iff $(D, l) = 0$.*

Remark. The assertion (1) above holds true if one replaces K_y by $K_y + D$ provided $K_y + D$ is not nef and D is a divisor with normal crossings such that $D \cap \mathrm{Sing}(y) = \emptyset$.

3.3. The above morphism $\phi\colon \mathscr{Y} \to \mathscr{X}$ is called *the contraction of* $\boldsymbol{R}_+[l]$ (or simply, of l). Again, retaining the notations of Section 1, assume that $\mathscr{Y}$ is $\mathscr{S}$-regular over Δ. Let F be a fiber of $\pi_0\colon \mathscr{Y} \to \Delta$ and let $F = \sum_i X_i$ be the decomposition of F into irreducible components. Let $B_i := (F - X_i)|_{X_i}$. Let A be an ample Cartier divisor on $\mathscr{Y}$. Then we have the following results:

(3) *There exist (possibly singular) rational curves $l_{i1}, \cdots, l_{ir(i)}$ on X_i such that*

$$\overline{NE}(X_i) = \sum_{j=1}^{r(i)} \boldsymbol{R}_+[l_{ij}] + \{z \in \overline{NE}(X_i) \mid (K_{X_i} + B_i + \varepsilon A|_{X_i}, z) \geqq 0\}.$$

(4) *With the notations in* (3), *we have*

$$\overline{NE}(\mathscr{Y}) = \sum_i \sum_{j=1}^{r(i)} \boldsymbol{R}_+[l_{ij}] + \{z \in \overline{NE}(\mathscr{Y}) \mid (K_{\mathscr{Y}} + \varepsilon A + a\pi_0^{-1}(P), z) \geqq 0\}$$

for an integer $a \gg 0$.

Remark. With the notations in Section 1, the condition (4) there is equivalent to the condition:

There exists an extremal rational curve l with respect to K_{y_i} such that every irreducible curve C belongs to $\boldsymbol{R}_+[l]$.

§ 4. Flippings of (-1) curves

4.1. Let $\pi\colon \mathscr{X} \to \Delta$ be an $\mathscr{S}$-degeneration of surfaces and let $F := \sum_{i=1}^n V_i$ be its fiber. Let $D_{ij} = V_i \cap V_j$, which is a disjoint union of irreducible components. Let C be a (-1) curve on V_i such that (C, D_{ij})

$=1$. Then $N_{C/\mathscr{X}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, i.e., $N_{C/\mathscr{X}}$ has type $(-1, -1)$. The blowing-up of $\mathscr{X}$ along C produces the exceptional divisor E which is isomorphic to $F_0 := \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and hence ruled in two ways. Blowing down E along the other ruling, the curve C is flipped (analytically) over to a (-1) curve on the component V_j, also meeting D_{ij} transversally. This operation is called *the modification of type* I *along* C. A generalization of this modification is given as follows:

Lemma. *Let $\pi: \mathscr{X} \to \Delta$ and F be as above. Let Y and Z be two irreducible components of F and let D be a partial connected sum of irreducible components of F satisfying the following conditions*:

(1) *$D \cap Y \neq \emptyset$, Y is not a component of D and $D \cap Z = \emptyset$*;

(2) *D is contracted to a singular point of rod type on an $\mathscr{S}$-regular threefold by a good contraction*;

(3) *$D|_Y$ is a linear chain of (-2) curves and is a connected component of $(F-Y)|_Y$.*

We assume that there exists a (-1) curve l in Y which meets an edge component of $D|_Y$ and the double curve $Z|_Y$, each in a single point transversally, and that $(Z|_Y) \cap l$ is not a triple point (see Figure 3).

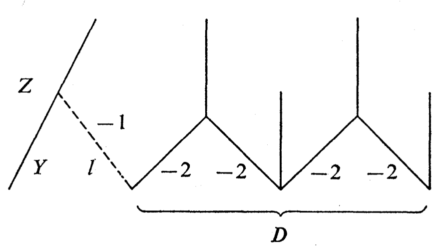

Figure 3

Then there exist a modification $\mathscr{X}'$ of $\mathscr{X}$, which is an analytic space and possibly not projective, and a surjective morphism $\pi': \mathscr{X}' \to \Delta$ satisfying the following conditions:

(1′) *$\pi': \mathscr{X}' \to \Delta$ is an $\mathscr{S}$-degeneration, while the projectivity of $\mathscr{X}'$ is not guaranteed, and $\mathscr{X} - F \rightrightarrows \mathscr{X}' - \pi'^{-1}(\pi(F))$*;

(2′) *Let Y' and Z' be the proper transforms of Y and Z, respectively; the total transform of $l + D$ is a sum $l' + D'$, where l' is a (-1) curve on Z' and D' is a partial connected sum of irreducible components of the fiber $F' := \pi'^{-1}(\pi(F))$; l' and D' together with Y' and Z' satisfy the same conditions as for l and D together with Y and Z*;

(3′) *Y' is obtained from Y by the contraction of $l + D|_Y$* (see Figure 4).

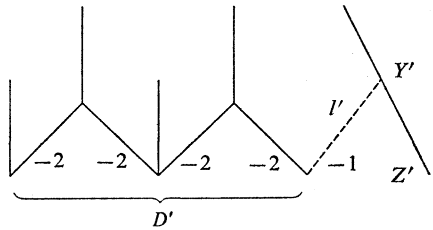

Figure 4

4.2. We say that $\mathscr{X}'$ is obtained from $\mathscr{X}$ by *the (analytic) flipping of* $l+D$ *along the double curve* $Y\cap Z$. The model $\mathscr{X}'$ may not be projective. Indeed, suppose that there exist two (-1) curves l_1 and l_2 on Y such that $(l_1, Z)=(l_2, Z)=1$ and $l_1\equiv l_2$ on $\mathscr{X}$ with the same notations Y and Z as above. Perform the analytic flipping of l_1 along $Y\cap Z$, and let l_i' be the proper transform of l_i on $\mathscr{X}'$, $i=1, 2$. Then $l_1'+l_2'\equiv 0$ on $\mathscr{X}'$, whence $\mathscr{X}'$ is not projective (see Figure 5).

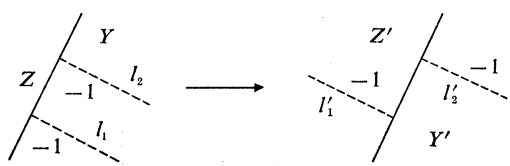

Figure 5

To avoid the inconvenience of this kind, we have to flip l_1 and l_2 simultaneously. A natural question is: How many and which curves should be flipped simultaneously to preserve the projectivity? In a simple case, the answer is given.

Lemma. *With the notations of Lemma* 4.1, *suppose that a given* (-1) *curve* l *on* Y *generates an extremal ray*(*) $\boldsymbol{R}_+[l]$ *in* $\overline{NE}(\mathscr{X})$ *and that* $N_{l/\mathscr{X}}$ *is of type* $(-1, -1)$. *Then the following assertions hold*:

(1) *There are finitely many* (-1) *curves* l_i *on* Y $(1\leqq i\leqq r)$ *such that* $l_i\equiv l_1:=l$ *and that if* C *is an irreducible curve on* $\mathscr{X}$ *with* $[C]\in\boldsymbol{R}_+[l]$ *then* $C=l_i$ *for some* i. *Moreover,* $l_i\cap l_j=\varnothing$ *if* $i\neq j$.

(2) *Let* $\hat{\mathscr{X}}$ *be the bimeromorphic model of* $\mathscr{X}$ *obtained by flipping* $l_1, \cdots, l_r$ *along* $Y\cap Z$ *simultaneously. Then* $\hat{\mathscr{X}}$ *is projective, and the* (-1) *curves* $\hat{l}_i$, *which are the flipped images of* l_i *on the proper transform* $\hat{Z}$ *of* Z, *generate the same extremal ray* $\boldsymbol{R}_+[\hat{l}]$ *of* $\overline{NE}(\hat{\mathscr{X}})$.

(*) We do not require the condition $(K_{\mathscr{X}}, l)<0$ to be satisfied, or we can say that l is an extremal ray with respect to $K_{\mathscr{X}}+Y$.

Proof. (0) Since $\boldsymbol{R}_+[l]$ is an extremal ray, there exists a nef Cartier divisor H on $\mathscr{X}$ such that $\overline{NE}(\mathscr{X})\cap H^{\perp}=\boldsymbol{R}_+[l]$. We shall show $(H^2, Y)>0$. Indeed, since $(Y, l)=-1$, $aH-Y$ is ample for $a\gg 0$. Hence $\kappa(aH)\geqq\kappa(aH-Y)=3$, whence $(H^3)>0$. On the other hand, note that $bH-K_{\mathscr{X}}$ is nef if $b\gg 0$, because $(K_{\mathscr{X}}, l)=(K_Y, l)-(Y, l)=0$, and that $bH-K_{\mathscr{X}}$ is big. Hence, by virtue of [2; Theorem 2.6], $|bH|$ has no base points. Then we obviously have $(H^2, Y)>0$.

(1) Let C be an irreducible curve on $\mathscr{X}$ such that $[C]\in\boldsymbol{R}_+[l]$. So, write $C\equiv\alpha l$ with $\alpha>0$. Since $(Y, C)=\alpha(Y, l)<0$, C lies on Y. Hence $(K_Y, C)=\alpha(K_Y, l)<0$, and $(C^2)_Y<0$ because $(H|_Y, C)=0$ and $(H|_Y)^2>0$. Hence C is a (-1) curve, and $\alpha=1$. Suppose $C\neq l$ and $(C, l)_Y>0$. Then, since $(H, C+l)=0$, we have $(C+l)^2<0$, while $(C+l)^2=-2+2(C, l)\geqq 0$, a contradiction. Hence $C\cap l=\varnothing$ if $C\neq l$. Since $(Z, C)=1$, there are finitely many such (-1) curves.

(2) Let $\sigma\colon\mathscr{X}'\to\mathscr{X}$ be the blowing-up with centers $l_1, \cdots, l_r$ and let $E_i:=\sigma^{-1}(l_i)$, $1\leqq i\leqq r$. For $l:=l_1$, we set $E:=\sigma^{-1}(l)$. Let l' be a fiber of $E\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$ such that $\sigma_*(l')=l$ and let f be a fiber of E with $\sigma_*(f)=0$. Then we have $K_{\mathscr{X}'}=\sigma^*K_{\mathscr{X}}+E_1+\cdots+E_r$, whence $(K_{\mathscr{X}'}, l')=(K_{\mathscr{X}}, l)+(E, l')=-1<0$. We shall show that $\boldsymbol{R}_+[l']$ is an extremal ray of $\overline{NE}(\mathscr{X}')$. Suppose that $l'\equiv z_1+z_2$ with $z_1, z_2\in\overline{NE}(\mathscr{X}')$. Since $l\equiv\sigma_*z_1+\sigma_*z_2$ with $\sigma_*z_1, \sigma_*z_2\in\overline{NE}(\mathscr{X})$ and $(Y', l')=0$ for the proper transform $Y':=\sigma'(Y)$, we may write

$$z_j\equiv\alpha_{j1}l'_1+\cdots+\alpha_{jr}l'_r, \qquad j=1, 2,$$

where l'_i is a fiber of E_i with $\sigma_*(l'_i)=l_i$ and where $\alpha_{ji}\geqq 0$. For $i\neq 1$, we have $(E_i, l')=0$, whence $\alpha_{ji}=0$ for $j=1, 2$. This implies that $\boldsymbol{R}_+[l']$ is an extremal ray of $\overline{NE}(\mathscr{X}')$. Hence l' (and l'_i $(i\geqq 2)$ as well) is an extremal rational curve on $\mathscr{X}'$ with respect to $K_{\mathscr{X}'}$. By 3.2, the contraction τ of $\boldsymbol{R}_+[l']$ is projective, and the image of l'_i $(i\geqq 2)$ under τ is an extremal rational curve. So, we can compose these projective contractions to obtain $\nu\colon\mathscr{X}'\to\hat{\mathscr{X}}$. The rest of the assertion (2) is easily verified.

4.3. This result is generalized as follows:

Lemma. *Let $\pi_0\colon\mathscr{Y}\to\Delta$ be an $\mathscr{S}$-regular 3-fold over Δ and let $f\colon\mathscr{X}\to\mathscr{Y}$ be a good resolution. Let $\bar{F}$ be a fiber of π_0 and let $\bar{Y}$ and $\bar{Z}$ be irreducible components of $\bar{F}$ such that $\bar{Y}\cap\bar{Z}\neq\varnothing$. Let $\bar{l}$ be a generalized (-1) curve on $\bar{Y}$ with $(K_y, \bar{l})=-1$ and $(\bar{Z}, \bar{l})=1$. Assume that $\bar{l}$ generates an extermal ray with respect to $K_y+\bar{Y}$. Let F be the fiber of $\pi:=\pi_0\cdot f\colon\mathscr{X}\to\Delta$ over $\pi_0(\bar{F})$, let Y and Z be the proper transforms of $\bar{Y}$ and $\bar{Z}$ on $\mathscr{X}$, respectively, and let l be the proper transform of $\bar{l}$. Then the following assertions hold:*

(1) *l is a (-1) curve, and the total transform of $\bar{l}$ on Y is written as $l+D|_Y$ as considered in Lemma* 4.1, *where D is a partial connected sum of irreducible components of F which contracts to a singular point of rod type on $\mathscr{Y}$; D might be empty. Moreover, l meets the double curve $Y\cap Z$ transversally in a single point.*

(2) *There are finitely many generalized (-1) curves $\bar{l}_i$ on $\bar{Y}$ $(1\leqq i\leqq r)$ such that $\bar{l}_i\equiv\bar{l}_1:=\bar{l}$ and that if $\bar{C}$ is an irreducible curve on $\mathscr{Y}$ with $[\bar{C}]\in \boldsymbol{R}_+[\bar{l}]$ then $\bar{C}=\bar{l}_i$ for some i. Moreover, $\bar{l}_i\cap\bar{l}_j=\emptyset$ if $i\neq j$. Hence the total transform of $\bar{l}_i$ on Y is of the form $l_i+D_i|_Y$ as $l+D|_Y$ for $\bar{l}$.*

(3) *Let $\hat{\mathscr{X}}$ be the bimeromorphic model of $\mathscr{X}$ obtained by flipping $l_i+D_i|_Y$ $(1\leqq i\leqq r)$ along $Y\cap Z$ simultaneously. Then $\hat{\mathscr{X}}$ is projective. If one writes the flipped image of $l_i+D_i|_Y$ by $\hat{l}_i+\hat{D}_i|_{\hat{Y}}$ then one can contract $\hat{D}_i$'s $(1\leqq i\leqq r)$ to singular points of rod type by a good contraction $\hat{f}\colon \hat{\mathscr{X}}\to\hat{\mathscr{Y}}$. Then $\hat{\mathscr{Y}}$ is projective as well.*

One can say that the model $\hat{\mathscr{Y}}$ is obtained from $\mathscr{Y}$ by flipping $\bar{l}_1, \cdots, \bar{l}_r$ along the double curve $\bar{Y}\cap\bar{Z}$.

4.4. There are two more modifications which are used occasionally (see [1]). Regaining the notations at the beginning of this section, suppose that the double curve $D=D_{ij}:=V_i\cap V_j$ is a (-1) curve on V_i and V_j; hence D has two triple points. The blowing-up of $\mathscr{X}$ with center D produces the exceptional divisor E isomorphic to $F_0\cong\boldsymbol{P}^1\times\boldsymbol{P}^1$. Blowing down E along the other ruling, the proper transforms of V_r and V_s (see the picture below) meet along the double curve D'_{rs} which is a (-1) curve and the components V_i and V_j are detached (see Figure 6).

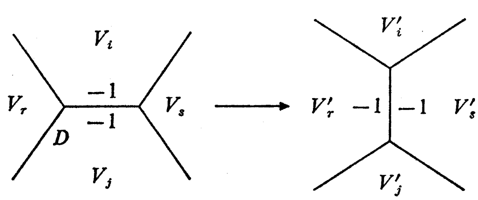

Figure 6

This operation is called *the modification of type* II *along D*. One can show that *if D is an extremal curve with respect to $K_{\mathscr{X}}+V_i$ then the modification of type* II *preserves the projectivity.*

Next, let l be a (-1) curve on an irreducible component X of the fiber F such that $(l, Y)=(l, Z)=1$ for irreducible components Y, Z $(\neq X)$ of F. We suppose that l itself is not a double curve. Let E be the exceptional divisor for the blowing up of $\mathscr{X}$ with center l. Since $N_{l/\mathscr{X}}$ is of type

$(-1, -2)$, E is isomorphic to F_1 and the double curve $E \cap X'$ is of type $(-1, -1)$, where X' is the proper transform of X. Then apply the modification of type II along $E \cap X'$. Thus we obtain a model $\mathscr{X}'$ of $\mathscr{X}$ (see Figure 7).

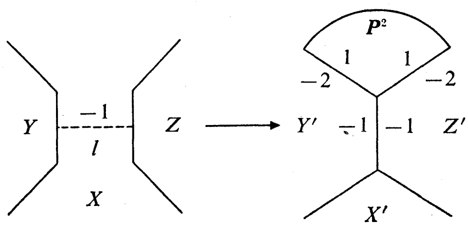

Figure 7

This operation is called *the inverse of the modification of type* III *along* l. One can show that *if* l *is an extremal curve with respect to* $K_{\mathscr{X}} + X$ *then the inverse of the modification of type* III *preserves the projectivity.*

§ 5. Main Lemma

5.1. A crucial result to prove Main Theorem is the following:

Main Lemma. *Let* $\pi_0 \colon \mathscr{Y} \to \Delta$ *be an* $\mathscr{S}$*-regular threefold over* Δ *and let* $f \colon \mathscr{X} \to \mathscr{Y}$ *be a good resolution. Assume that* Δ *has positive genus and* $K_{\mathscr{Y}}$ *is not nef. Let* l *be an extremal rational curve on* $\mathscr{Y}$ *with respect to* $K_{\mathscr{Y}}$*, and let* $g \colon \mathscr{Y} \to \mathscr{Z}$ *be the contraction of* l*. Then there are three possibilities for the structure of* g:

(1) *There exists an irreducible divisor* D *on* $\mathscr{Y}$ *such that* $g(D)$ *is a point and* g *induces an isomorphism* $\mathscr{Y} - D \xrightarrow{\sim} \mathscr{Z} - g(D)$;

(2) *There exists an irreducible divisor* D *on* $\mathscr{Y}$ *such that* $g(D)$ *is a curve and* g *induces an isomorphism* $\mathscr{Y} - D \xrightarrow{\sim} \mathscr{Z} - g(D)$;

(3) *There are finitely many irreducible curves* $l_1, \cdots, l_r$ *on* $\mathscr{Y}$ *such that* $g(l_i)$ *is a point and* g *induces an isomorphism* $\mathscr{Y} - \bigcup l_i \xrightarrow{\sim} \mathscr{Z} - g(l_i)$.

The third case is further divided:

(3.1) *There exists some* l_i *which is a double curve on* $\mathscr{Y}$;

(3.2) *No* l_i*'s are double curves on* $\mathscr{Y}$.

The case (3.2) *is divided as follows*:

(3.2.1) *There exists some* l_i *which meets a double curve*;

(3.2.2) *No* l_i*'s meet the double curves.*

Furthermore, in the cases (1) *and* (2) *above, assume that either* $\mathscr{Z}$ *is not* $\mathscr{S}$*-regular or* $h := g \cdot f \colon \mathscr{X} \to \mathscr{Y} \to \mathscr{Z}$ *is not a good resolution. Then the following assertions hold*:

(I) *In the cases* (1), (2), (3.1) *and* (3.2.1), *there exists an* $\mathscr{S}$*-degeneration* $\mathscr{X}_1$ *over* Δ *such that* $\mathscr{X}_1$ *is birational to* $\mathscr{X}$, $\rho(\mathscr{X}_1)<\rho(\mathscr{X})$ *and* $h\cdot\phi^{-1}\colon \mathscr{X}_1 \to \mathscr{Z}$ *is a morphism, where* $h:=g\cdot f$ *and* $\phi\colon \mathscr{X}\cdots\to\mathscr{X}_1$ *is the birational mapping. Moreover,* ϕ *is isomorphic outside*

$h^{-1}(g(D))$	*in Case* (1)
$h^{-1}(g(D))\cup \mathrm{Sing}(\mathscr{Z}))$	*in Case* (2)
$h^{-1}(\cup g(l_i))$	*in Cases* (3.1) *and* (3.2.1).

(II) *In the case* (3.2.2), *there exists an* $\mathscr{S}$*-degeneration* $\mathscr{X}_1$ *over* Δ *such that* $\mathscr{X}_1$ *is birational to* $\mathscr{X}$, *the birational mapping* $\phi\colon \mathscr{X}\cdots\to\mathscr{X}_1$ *is isomorphic outside* $h^{-1}(\cup g(l_i))$ *and* $\rho(\mathscr{X}_1)\leqq\rho(\mathscr{X})$. *If* $\rho(\mathscr{X}_1)=\rho(\mathscr{X})$ *there exist an* $\mathscr{S}$*-regular threefold* $\mathscr{Y}_1$ *over* Δ *and a good contraction* $f_1\colon \mathscr{X}_1\to\mathscr{Y}_1$ *satisfying the conditions*:

(i) $\rho(\mathscr{Y}_1)=\rho(\mathscr{Y})$;

(ii) $\phi\colon \mathscr{X}\cdots\to\mathscr{X}_1$ *and* $f_1\cdot\phi\colon \mathscr{X}\cdots\to\mathscr{Y}_1$ *are isomorphic outside* $h^{-1}(\cup g(l_i))$;

(iii) *If* $F_1, \cdots, F_r$ (*resp.* $G_1, \cdots, G_s$) *are the irreducible components of the singular fibers of* $\mathscr{Y}\to\Delta$ (*resp.* $\mathscr{Y}_1\to\Delta$) *then* $\tilde{\rho}(\mathscr{Y}):=\sum\rho(F_i)>\tilde{\rho}(\mathscr{Y}_1):=\sum\rho(G_j)$.

5.2. Concerning the statements of Main Lemma, we note the following:

Remark. (1) $\dim\mathscr{Z}=3$. Indeed, let H be a nef $\boldsymbol{Q}$-divisor such that $H^{\perp}\cap\overline{NE}(\mathscr{Y})=\boldsymbol{R}_+[l]$. Then $bH-K_{\mathscr{Y}}$ is ample for $b\gg0$. Hence $|bH|$ is free, and we can take $\mathscr{Z}=\Phi_{|bH|}(\mathscr{Y})$. If $\dim\mathscr{Z}<3$ then there exists an irreducible curve C contained in a general fiber of $\pi_0\colon\mathscr{Y}\to\Delta$ such that $(H, C)=0$, i.e., $[C]\in\boldsymbol{R}_+[l]$. Then $(K_{\mathscr{Y}}, C)<0$, which is not the case because $K_{\mathscr{Y}}|_{(\text{general fiber})}$ is nef. Thus $\dim\mathscr{Z}=3$, and we can take $g\colon\mathscr{Y}\to\mathscr{Z}$ to be birational.

(2) In the case (3.1), if some l_i is a double curve then every l_j is a double curve. Moreover, if $l_i\subseteq X\cap Y$ then $l_j\subseteq X\cap Y$ for every j. Note that $X\cap Y$ is a disjoint union of irreducible components.

(3) In the case (3.2.1), if $l_1\subset Y$, $l_1\cap X\cap Y\neq\varnothing$ and l_1 is not a double curve, where X and Y are irreducible components of a fiber of π_0, then $l_i\subset Y$ and $l_i\cap X\cap Y\neq\varnothing$ for every i.

(4) $\mathscr{Z}$ has finitely many singular points.

5.3. Suppose that Main Lemma is proved, and prove Main Theorem. Let $\mathscr{X}_1$ be now an $\mathscr{S}$-degeneration birational to $\mathscr{X}$ over Δ such that $\rho(\mathscr{X}_1)$ is minimal among such $\mathscr{S}$-degenerations. Let $\mathscr{Y}_1$ be an $\mathscr{S}$-regular threefold over Δ with a good contraction $\mathscr{X}_1\to\mathscr{Y}_1$ such that the sequence $\mathscr{X}_1\to\mathscr{Y}_1$ cannot be further extended. If $K_{\mathscr{Y}_1}$ were not nef, by Main

Lemma, we have an $\mathscr{S}$-degeneration $\mathscr{X}_2$ over Δ such that $\rho(\mathscr{X}_2)\leqq\rho(\mathscr{X}_1)$. By the minimality of $\rho(\mathscr{X}_1)$, we have $\rho(\mathscr{X}_2)=\rho(\mathscr{X}_1)$. Hence we have the case (3.2.2), and $\tilde{\rho}(\mathscr{Y}_2)<\tilde{\rho}(\mathscr{Y}_1)$ for some good contraction $\mathscr{X}_2\to\mathscr{Y}_2$, where $\mathscr{Y}_2$ is $\mathscr{S}$-regular over Δ. We can then proceed by induction on $\tilde{\rho}(\mathscr{Y}_2)$.

5.4. Remark. Let $\mathscr{X}$ be an $\mathscr{S}$-degeneration over Δ and let l be an extremal rational curve. Let $R:=\boldsymbol{R}_+[l]$ and let $f\colon \mathscr{X}\to\mathscr{Y}$ be the contraction of R. Suppose R is numerically effective. Then there exists an irreducible component X of the fiber F containing l such that $l\subseteq X$, and l does not meet any double curve, i.e., we are in the case (3.2.2). Then $N_{l/\mathscr{X}}$ is of type $(-1, 0)$. Hence $H^0(N_{l/\mathscr{X}})\cong\boldsymbol{C}$ and $H^1(N_{l/\mathscr{X}})=(0)$. Then l is stable by a theorem of Kodaira (cf. Persson [9; Proposition 5]). This is a contradiction because a general fiber of $\pi\colon\mathscr{X}\to\Delta$ is assumed to be minimal. Therefore R is not numerically effective. By Mori [7; Theorem 3.3], either $\mathscr{Y}$ is nonsingular or $\mathscr{Y}$ is $\mathscr{S}$-regular over Δ and $f\colon\mathscr{X}\to\mathscr{Y}$ is its good resolution.

Main Lemma will be proved by induction on $\rho(\mathscr{X})-\rho(\mathscr{Y})$. The case $\rho(\mathscr{X})=\rho(\mathscr{Y})$ is taken care of by the above remark.

§ 6. Singularities appearing on an $\mathscr{S}$-regular 3-fold

Let $\mathscr{Y}$ be an $\mathscr{S}$-regular threefold over Δ and let

$$f\colon \mathscr{X}=\mathscr{Y}_n\xrightarrow{f_n}\mathscr{Y}_{n-1}\xrightarrow{f_{n-1}}\cdots\longrightarrow\mathscr{Y}_1\xrightarrow{f_1}\mathscr{Y}_0=\mathscr{Y}$$

be a sequence of birational morphisms which defines a good resolution $f\colon\mathscr{X}\to\mathscr{Y}$. Let $P\in\mathscr{Y}$ be a singular point. Consider first *the case where P is of twig type.*

6.1. Lemma. *Suppose $P\in X\cap Y$ is a singular point of twig type, where X and Y are irreducible components and P is a simple point of $X\cap Y$. We may assume that $f_1^{-1}(P)$ is an irreducible divisor D on $\mathscr{Y}_1$. Let X_1, Y_1 be the proper transforms of X, Y on $\mathscr{Y}_1$, let $l:=X_1|_D$ and $M:=Y_1|_D$, and let X', Y', D', l', M' be the proper transforms of X_1, Y_1, D, l, M on $\mathscr{X}$, respectively. Then D is a rational surface and there is at most one singular point of D on l (similarly on M), i.e., $l\cap\operatorname{Sing}D=\{P_1\}$ or $\emptyset$, and $M\cap\operatorname{Sing}D=\{Q_1\}$ or $\emptyset$. Let $B_1=f'^{-1}(P_1)|_{D'}$ and $B_2=f'^{-1}(Q_1)|_{D'}$, where $f':=f_2\cdots f_n$. We have one of the following three cases*:

Case $l\cap\operatorname{Sing}D=M\cap\operatorname{Sing}D=\emptyset$. Then $D'\cong\boldsymbol{P}^2$ (see Figure 8).

Case $M\cap\operatorname{Sing}D=\emptyset$ and $l\cap\operatorname{Sing}D\neq\emptyset$. Then D' is a minimal ruled surface (see Figure 9).

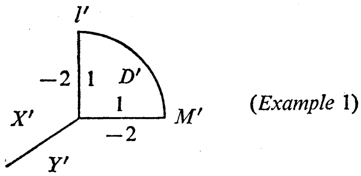

Figure 8

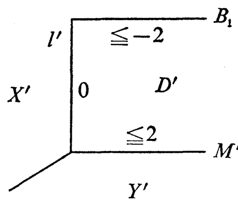

Figure 9

Case $l \cap \operatorname{Sing} D \neq \emptyset$ *and* $M \cap \operatorname{Sing} D \neq \emptyset$. *Then one of* $|l'|$ *and* $|M'|$, *say* $|l'|$, *defines a* $\boldsymbol{P}^1$*-ruling* (see Figure 10).

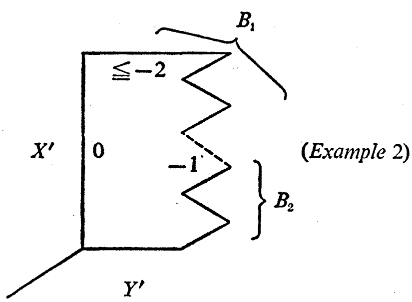

Figure 10

where $|l'|$ *has a unique singular member whose configuration is a linear chain as shown.*

Proof. The proof consists of several steps.

(I) Let F be the fiber of $\mathscr{Y}_1 \to \varDelta$ which contains D, X_1, Y_1, etc. Let $X_1, \cdots, X_r$ be all irreducible components of F, except Y_1 and D, which meet M. Let $\operatorname{Sing}(D) \cap M = \{P_1, \cdots, P_s\}$. Since the pair $(D, (F-D)|_D)$ is a log-del Pezzo surface of rank 1, Lemma 2.2 implies that $\{P_1, \cdots, P_s\} \cap (\bigcup_{i=1}^r X_i) = \emptyset$ and that each P_i is a cyclic quotient singular point. Let a_i $(\geqq 2)$ be the order of the corresponding cyclic group. Since D contracts to a log-terminal singular point on $\mathscr{Y}$ and $M \subset D$, we have $(D, M) < 0$ and

$$0>(K(\mathscr{Y}_1), M)=(K_D+(F-D)|_D, M)=2g(M)-2+r+\sum_{i=1}^{s}\left(1-\frac{1}{a_i}\right),$$

where $g(M)$ is the genus of the (smooth) curve M. Hence we conclude $g(M)=0$, $r\leqq 1$ (hence $r=1$) and $s\leqq 1$. This argument applies to l as well. Therefore each of l and M has at most one singular point of D.

(II) Let $a:=(l'^2)_{D'}$ and $b:=(M'^2)_{D'}$. We claim that $a\geqq 0$ and $b\geqq 0$. Suppose $b<0$. If $\mathrm{Sing}(D)\cap M=\emptyset$, then $b=(M^2)_D>0$ because M is ample on D. Suppose $\mathrm{Sing}(D)\cap M=\{Q_1\}\neq\emptyset$. Since Q_1 is a cyclic quotient singularity and since each irreducible component C of $B_2:=f'^{-1}(Q_1)|_{D'}$ has $(C^2)\leqq -2$, the intersection matrix of $M'+B_2$ is negative definite. However, $f'^{-1}(M)|_{D'}$ is supported by M' and B_2 and M is ample on D. This is a contradiction. By a similar argument, we have $a\geqq 0$.

By the Hodge index theorem, we easily show that $ab\leqq 1$. By symmetry, we may assume that either $a=b=1$ or $a=0$.

(III) Suppose $a=b=1$. Then $B_1=B_2=0$. Indeed, if $B_2\neq 0$, then there exists an effective $\boldsymbol{Q}$-divisor B_2^* such that $\mathrm{Supp}(B_2^*)=\mathrm{Supp}(B_2)$ and $(M'+B_2^*, C)=0$ for each irreducible component C of D_2. Hence $(M'+B_2^*)^2>(M'^2)_{D'}=1$. Consider a divisor $M'+B_2^*-l'$. Since $(M'+B_2^*-l', l')=0$ and $(l'^2)_{D'}=1$, we have $(M'+B_2^*-l')^2=(M'+B_2^*)^2-2+1\leqq 0$, which is a contradiction. Thus $B_2=0$. Similarly, $B_1=0$. We claim that $D'\cong\boldsymbol{P}^2$. Suppose that D' contains a (-1) curve E'. Let E be the image of E' on D. Then E is a curve because D' is the minimal resolution of singularities on D. Since l and M are ample on D, we have $(E, l)>0$ and $(E, M)>0$. Since $(D, (F-D)|_D)$ is a log-del Pezzo surface,

$$0>(K(\mathscr{Y}_1), E)=(K_D+l+M+\Gamma, E),$$

where $\Gamma=(F-D-X_1-Y_1)|_D$. If $E\subset\Gamma$, then E is a double curve and hence does not meet M (cf. the step (I)). This is a contradiction because $(E, M)>0$. Therefore $(E, \Gamma)\geqq 0$. Thus we have

$$0>(K_D+l+M+\Gamma, E)\geqq -1+1+1,$$

a contradiction. Therefore D' is a minimal rational surface which has two nonsingular rational curves l' and M' with $(l'^2)=(M'^2)=1$ and $(l', M')=1$. Then $D'\cong\boldsymbol{P}^2$. In particular, $D=D'$.

(IV) Suppose $a=0$. It is easy to see that D' is a rational surface and the linear system $|l'|$ defines a $\boldsymbol{P}^1$-fibration $\phi\colon D'\to\boldsymbol{P}^1$ for which M' is a cross-section. Let E' be a (-1) curve on D' and let E be the image of E' on D; E is a curve as before. We have

$$0>(K(\mathscr{Y}_1), E)=(K_D+l+M+\Gamma, E),$$

where $\Gamma=(F-D-X_1-Y_1)|_D$ and l and M are ample on D. Any double curve G on D meets both l and M, on the one hand, and there are no double curves (other than l and M) meeting l or M. Hence, either $G=l$ or $G=M$. Therefore $\Gamma=0$. Since

$$g^*(K_D+l+M)\geqq K_{D'}+l'+M'$$

with $g=f'|_{D'}$: $D'\to D$, we have

$$0>(K_{D'}+l'+M', E')=-1+(l', E')+(M', E').$$

Therefore $(l', E)=(M', E)=0$. In particular, E' is contained in a fiber of ϕ. Since $(l, E)>0$ and $(M, E)>0$, E must meet both B_1 and B_2. Moreover, if E' exists, then $B_1\neq 0$ and $B_2\neq 0$. In other words, if $B_1=0$ or $B_2=0$ then D' is a minimal ruled surface. In the present case if $B_2=0$, i.e., $M\cap \mathrm{Sing}(D)=\varnothing$, then B_1 consists of a single irreducible component which is a cross-section of ϕ.

Suppose now that D' is not a minimal ruled surface. Hence $B_1\neq 0$ and $B_2\neq 0$. Let C be the irreducible component of B_1 such that $(X_1', C)\neq 0$. Let E' be a (-1) curve on D'. We claim that $C\cap E'=\varnothing$, i.e., $(C, E')=0$. Suppose, on the contrary, that $(C, E')\neq 0$. Since C is a cross-section of ϕ, we have $(C, E')=1$. Let G be the fiber of ϕ containing E'. Then there exists another (-1) curve, say E_1', in G. As shown above, E_1' must meet B_1 and cannot meet C. Therefore, $B_1\neq C$. On the other hand, since $(l', B_1-C)=0$, B_1-C is contained in a fiber which must be G because $(E_1', B_1-C)\neq 0$. Thus we find an irreducible component C_1 of B_1 such that $(C_1, C)\neq 0$. Then we have

$$1=(G, C)\geqq(E'+C_1, C)\geqq 2,$$

which is a contradiction.

Now E' meets both B_1-C and B_2. This implies that the fiber G containing E' has a unique (-1) curve. In terms of the dual graph, the fiber G is given as one of those in Figure 11:

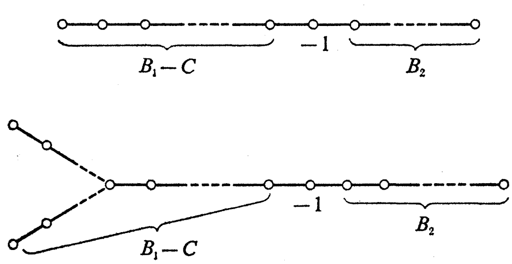

Figure 11

Note that the pair $(D, (F-D)|_D)$ is an almost minimal model of $(D', (F'-D')|_{D'})$, where F' is the fiber of $\pi\colon \mathscr{X} \to \Delta$ corresponding to F. Since $(D, (F-D)|_D)$ is a log-del Pezzo surface, the surface $D'-(F'-D')|_{D'}$ has logarithmic Kodaira dimension $-\infty$. Hence the divisor $(F'-D')|_{D'}$ does not contain a loop. Thus the second dual graph of G does not occur in the present case.

6.2. We shall now list up various examples of possible singular points on $\mathscr{Y}$. Examples 1 and 2 are given above, which are singularities of twig type. Examples 3–8 are singularities of rod type. To simplify the explanations, we visualize the situations by pictures. In the picture, every component except the one denoted by Y is a rational surface and contracts down to a point.

Example 3 (cf. Figures 12 and 13).

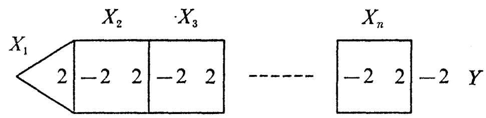

Figure 12

where $X_1 \cong F_0 \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $X_2 \cong X_3 \cong \cdots \cong X_n \cong F_2$, or

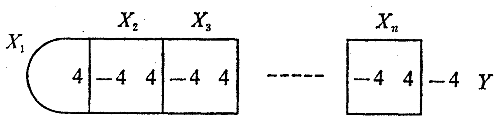

Figure 13

where $X_1 \cong \boldsymbol{P}^2$ and $X_2 \cong X_3 \cong \cdots \cong X_n \cong F_4$. In both cases, we assume

$$\operatorname{rk}(\operatorname{Im}(\operatorname{Pic}(\mathscr{X}) \otimes_{\boldsymbol{Z}} \boldsymbol{Q} \longrightarrow \operatorname{Pic}(X_1) \otimes_{\boldsymbol{Z}} \boldsymbol{Q})) = 1.$$

Example 4 (cf. Figure 14).

Here (1) X is a ruled surface for which $(B(1)_2 + \cdots + B(1)_{r(1)})_X + l + (B(2)_{r(2)} + \cdots + B(2)_1)_X$ is a unique singular fiber and $B(1)_1|_X$ is a cross-section;

(2) $B(1) = B(1)_1 + B(1)_2 + \cdots + B(1)_{r(1)}$ and $B(2) = B(2)_1 + \cdots + B(2)_{r(2)}$ contract to singular points of twig type.

Example 5 (cf. Figure 15).

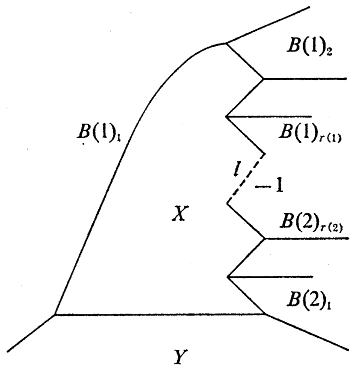

Figure 14

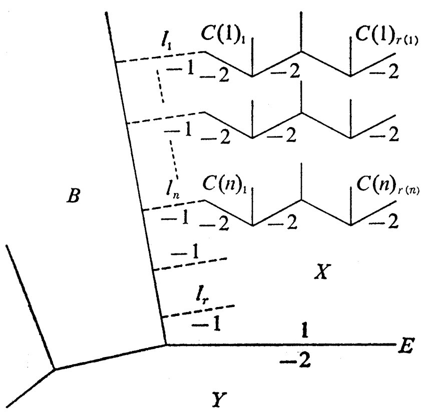

Figure 15

Here (1) $C(i)_1+\cdots+C(i)_{r(i)}$ $(1\leqq i\leqq n)$ contracts to a singular point of rod type;

(2) X becomes $\boldsymbol{P}^2$ after $l_i+(C(i)_1+\cdots+C(i)_{r(i)})|_X$ $(1\leqq i\leqq n)$ and l_j $(n+1\leqq j\leqq r)$ are contracted down;

(3) B contracts to a singular point of twig type.

Example 6 (cf. Figure 16).

Here (1) $B(1):=B(1)_1+\cdots+B(1)_{k(1)}$ and $B(2):=B(2)_1+\cdots+B(2)_{k(2)}$ contract to singular points of twig type, and $C(j)$ $(1\leqq j\leqq n)$ contracts to a singular point of rod type;

(2) X is a ruled surface for which $(B(1)_3+\cdots+B(1)_{k(1)})|_X+l+(B(2)_{k(2)}+\cdots+B(2)_1)|_X$ and $(B(1)_1)|_X+\sum_{i=1}^n(l_i+C(i)|_X)+\sum_{j=n+1}^r l_j$ are only singular fibers, and $B(1)_2|_X$ and $Y|_X$ are cross-sections;

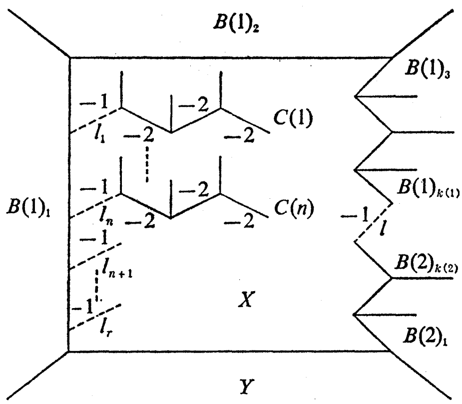

Figure 16

(3) After $B(1)$, $B(2)$ and $C(i)$ $(1 \leqq i \leqq n)$ are contracted, every irreducible curve generates one and the same extremal ray.

Example 7 (cf. Figure 17).

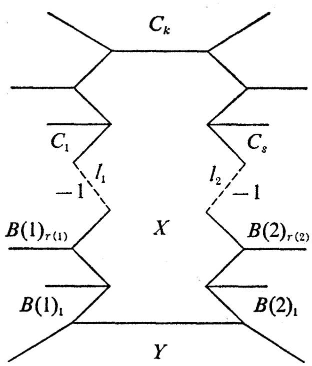

Figure 17

Here (1) $B(1)$ and $B(2)$ contract to singular points of twig type, and $C := C_1 + \cdots + C_s$ contracts to a singular point of rod type;

(2) X is a rational ruled surface for which $(B(1)_1 + \cdots + B(1)_{r(1)})|_X + l_1 + (C_1 + \cdots + C_{k-1})|_X$ and $(B(2)_1 + \cdots + B(2)_{r(2)})|_X + l_2 + (C_{k+1} + \cdots + C_s)|_X$ are all singular fibers, and $C_k|_X$ and $X|_Y$ are cross-sections.

Example 8. The same as in Example 7 except that the double curve $B(1)_1 \cap X$ has some more branches (cf. Figure 18):

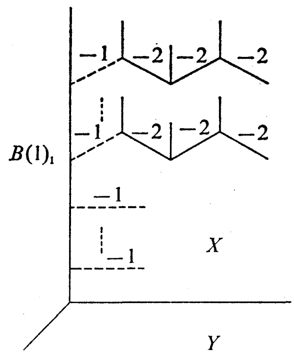

Figure 18

6.3. Let f: $\mathscr{X}=\mathscr{Y}_n \xrightarrow{f_n} \mathscr{Y}_{n-1} \longrightarrow \cdots \longrightarrow \mathscr{Y}_1 \xrightarrow{f_1} \mathscr{Y}_0=\mathscr{Y}$ be a good resolution of singularities on an $\mathscr{S}$-regular 3-fold $\mathscr{Y}$ over Δ. We call f *an* $\mathscr{S}$*-resolution* if, for every singular point P of $\mathscr{Y}$ and for an irreducible component D_i of $\pi_i^{-1}(\pi_0(P))$, which is contracted to a point by f_i, the proper transform X of D_i on $\mathscr{X}$ together with the fiber F of containing X fits to one of the situations appearing in Examples 1–8. We also say that f: $\mathscr{X} \to \mathscr{Y}$ is *an* $\mathscr{S}$*-contraction.*

Lemma. *A singular point of rod type on an* $\mathscr{S}$*-regular* 3*-fold* $\mathscr{Y}$ *has an* $\mathscr{S}$*-resolution.*

§ 7. Partial proof of Main Lemma

To exhibit Tsunoda's ideas to prove Main Lemma, we shall indicate how to treat the cases (1) and (2) of Main Lemma. The case (3) in Main Lemma is treated in Tsunoda [13]. In the following, Δ is assumed to have positive genus. We shall begin with the following:

7.1. Lemma. *Let* π_0: $\mathscr{Y} \to \Delta$ *be an* $\mathscr{S}$*-regular threefold and let* X *be a reduced divisor with simple normal crossings contained in a fiber* F *of* π_0 *such that* $\mathrm{Sing}(\mathscr{Y}) \cap X=\emptyset$ *and* $(K_{\mathscr{Y}}+X, C)<0$ *for an irreducible curve* C, *i.e.,* $K_{\mathscr{Y}}+X$ *is not nef. Assume that there exists an extremal rational curve* M_0 *with respect to* $K_{\mathscr{Y}}+X$ *such that* $M_0 \cap X \neq \emptyset$. *Then there exists an irreducible curve* M *such that* $[M] \in R:=\boldsymbol{R}_+[M_0] \subseteq \overline{NE}(\mathscr{Y})$ *and* $M \subseteq X$. *Let* L *be a nef Cartier divisor on* $\mathscr{Y}$ *such that* $L|_{X_i}$ *is big for each irreducible component* X_i *of* X *and* $|mL|$ *is free for an integer* $m \gg 0$. *Suppose, moreover, that* $(L, C)=0$ *for the above* C. *Then one can find the above* M *so that* $(L, M)=0$. *Furthermore, if* M *is any irreducible curve such that* $[M] \in R$, $(M, L)=0$, $M \subseteq X_i$ *for some irreducible component* X_i *of* X *and* M *is not a* *double curve, then M is a (-1) curve.*

Proof. Let $R:=\boldsymbol{R}_+[M_0]$. Assume that $M\not\subseteq X$ for any irreducible curve $M\in R$. Then $(X, M)>0$ because $(X, M_0)>0$. Hence $(K_{\mathscr{Y}}+X, M)<0$ implies $(K_{\mathscr{Y}}, M)<0$. Hence R is an extremal ray with respect to $K_{\mathscr{Y}}$, and every curve belonging to R is contained in the fiber F because M_0 is rational, $(M_0, X)>0$ and Δ has positive genus. Therefore M falls into one of the cases (1), (2), (3.1), (3.2.1) and (3.2.2) in Main Lemma. We check each of these cases and derive a contradiction.

Case (1). Let D be the irreducible divisor determined by R. Since $M_0\cap X\neq\varnothing$ by the hypothesis, $X\cap D\neq\varnothing$. By the assumption, $D\not\subseteq X$. Hence $D\cap X$ is an effective 1-cycle. Let M be any irreducible component of $D\cap X$. Then $M\subseteq X$ and $[M]\in R$. This is a contradiction.

Case (2). With the notations of Main Lemma, let M' be a general fiber of $g|_D\colon D\to g(D)$. Since D is an irreducible component of the fiber F of an $\mathscr{S}$-regular threefold $\mathscr{Y}$ and $M'\in R$, M' is a nonsingular rational curve. Since

$$0>(K_{\mathscr{Y}}, M')=(K_D+(F-D)_D, M')=-2+(X, M')+((F-X-D)|_D, M')$$

and $(X, M')>0$, we have $(K_{\mathscr{Y}}, M')=-1$. Then $(K_{\mathscr{Y}}+X, M')<0$ implies $(X, M')=0$. This is a contradiction.

Case (3.1). Let M' be a double curve such that $[M']\in R$, i.e, $M'\subseteq Y\cap Z$, Y and Z being irreducible components of F. Since $M'\not\subseteq X$, M' meets an irreducible curve $C\subseteq Y\cap X$. Therefore we have

$$0>(K_{\mathscr{Y}}, M')\geqq(K_{\mathscr{Y}}+M'+C, M')\geqq-1.$$

On the other hand, since $(K_{\mathscr{Y}}+X, M')<0$ and $(X, M')>0$, we have $(K_{\mathscr{Y}}, M')<-1$. Note that (X, M') is a positive integer because X is a Cartier divisor. This is a contradiction.

Case (3.2.1). There exists an irreducible curve M' such that $[M']\in R$ and M' meets a double curve. Hence we have $(K_{\mathscr{Y}}, M')\geqq-1$. Since $(X, M')\geqq1$, we have

$$0>(K_{\mathscr{Y}}+X, M')\geqq-1+1=0,$$

which is a contradiction.

Case (3.2.2). Since $(X, M_0)>0$, this case does not occur.

Now assume that $(L, C)=0$ for a nef Cartier divisor L as described above. Applying the cone theorem to K_Y+X+mL with $m\gg0$, one can find an extremal rational curve M_0 such that $(K_{\mathscr{Y}}+X, M_0)<0$ and $(L, M_0)=0$. Hence $(M, L)=0$ for every irreducible curve M belonging to R. Writing $X=\sum_{i=1}^n X_i$ (the irreducible decomposition), we may assume that

$M \subseteq X_1$. Since $(L^2, X_1) > 0$ and $(L, M) = 0$, we have $(M^2)_{X_1} < 0$ by the Hodge index theorem. We also have

$$0 > (K_{\mathscr{Y}} + X, M) = (K(X_1) + \sum_{i \neq 1} X_i|_{X_1}, M).$$

Hence $(K(X_1), M) < 0$ provided M is not a component of $\sum_{i \neq 1} X_i|_{X_1}$. Namely, if M is not a double curve, M is a (-1) curve. Even if $M \subseteq X_1 \cap X_2$, M is a (-1) curve both on X_1 and X_2 provided M meets at most 2 triple points. In fact, this follows from

$$0 = (M^2)_{X_1} + (M^2)_{X_2} + \#(\text{triple points on } M) \leqq (-1) + (-1) + 2 = 0.$$

Q.E.D.

7.2. Let us look into the proof of Main Lemma. As already explained at the end of Section 5, the proof will proceed by induction on $\rho(\mathscr{X}) - \rho(\mathscr{Y})$. We consider first the case (1). We employ mostly the notations in Main Lemma. Since $\pi_0(D)$ is a point on Δ, we consider the fiber F of π_0 containing D. Let X be a fiber component of $\mathscr{Y}$ such that $X \neq D$ and $X \cap D \neq \varnothing$. Let X' and D' be the proper transforms of X and D on $\mathscr{X}$. Let $l := X|_D$. Then l is a reduced curve on D and it is ample because every irreducible curve on D belongs to one and the same ray in $\overline{NE}(\mathscr{Y})$. We shall verify:

Claim 1. *l is irreducible.*

Proof. Write $l = l_1 + \cdots + l_r$, where l_i's are irreducible components. If $r \geqq 2$, index them so that $l_1 \cap l_2 \neq \varnothing$. Let $q \in l_1 \cap l_2$. Since q is a "triple point", D is nonsingular at q by the assumption that $\mathscr{Y}$ is $\mathscr{S}$-regular. Hence $\mathscr{Y}$ is nonsingular at q and the morphism $f: \mathscr{X} \to \mathscr{Y}$ is an isomorphism in a neighborhood of q. Hence X' and D' intersect on $\mathscr{X}$ as in Figure 19.

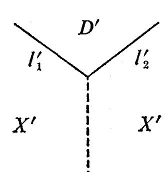

Figure 19

Namely, X' has a self-intersection. This is a contradiction, and l is irreducible. This implies that $l' := D'|_{X'}$ is also irreducible.

Claim 2. *If we choose the component X appropriately, l' is a (-1) curve on X'.*

Proof. Concerning $g: \mathscr{Y} \to \mathscr{X}$, we easily see that:

(i) $\mathscr{X}$ is $\boldsymbol{Q}$-factorial (cf. [2]),

(ii) the induced mapping $\pi_0 \cdot g^{-1}: \mathscr{X} \to \Delta$ is a morphism,

(iii) $g(D)$ is a point,

(iv) $-K(\mathscr{Y})|_D$ is ample and the image of the restriction homomorphism $\mathrm{Pic}(\mathscr{Y}) \otimes \boldsymbol{Q} \to \mathrm{Pic}(D) \otimes \boldsymbol{Q}$ has rank 1.

Since $\mathscr{X}$ is not $\mathscr{S}$-regular, the condition that $f: X' \to X$ is a minimal resolution breaks down for an irreducible component X of F with $X \cap D \neq \emptyset$. Note that l' is a nonsingular rational curve with $(l'^2)_{X'} < 0$. Hence l' is a (-1) curve.

Claim 3. *Let $l := X|_D$ be as above. Suppose that l' is a (-1) curve on X'. Let $l \cap \mathrm{Sing}(\mathscr{Y}) = \{P_1, \cdots, P_s\}$ and let $\{G_1, \cdots, G_s\}$ be the set of all double curves on X meeting l. Then $r+s \leqq 2$.*

Proof. We have:

$$0 > (K_{\mathscr{Y}}, l) = (K_X + l + G_1 + \cdots + G_s, l)$$
$$= (K_X + l, l) + s = s - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{a_i}\right),$$

where a_i $(\geqq 2)$ is the order of the cyclic group associated to a cyclic quotient singular point P_i $(1 \leqq i \leqq s)$ (cf. the proof of Lemma 6.1). Hence either $s=1$ and $r \leqq 1$ or $s=0$ and $r \leqq 3$. Suppose $s=0$ and $r=3$. Then we have

$$\sum_{i=1}^{r} \left(1 - \frac{1}{a_i}\right) < 2.$$

Hence $\{a_1, a_2, a_3\} = \{2, 2, n\}$ $(n \geqq 2)$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$ up to permutation. Since $f_{X'}: X' \to X$ is a minimal resolution of singularities, the inverse image $(f_{X'})^{-1}(l')$ is a non-admissible rational fork on X' with the central component l' (cf. [6]), which is contractible to a point $g(D)$ on the surface $g(X)$ in $\mathscr{X}$. This is a contradiction because the intersection matrix of $(f_{X'})^{-1}(l')$ is not negative definite. Hence $r \leqq 2$ if $s=0$. Thus $r+s \leqq 2$.

By Claim 3, we know that the three cases in Figure 20 exhaust all possible cases of the configuration of l', D' and X', where the double curves meeting l' come out in both ways, i.e., the exceptional curves of the resolution of singular points on l and the proper transforms of double curves on Y meeting l.

In the last case, D' and D are isomorphic around l' and l. Since l is ample, there are no double curves on D. We shall show that D has no

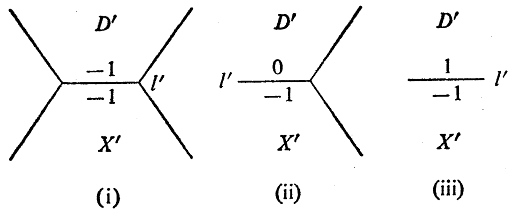

Figure 20

singular points. Suppose that P is a singular point. Consider a linear pencil Λ on D' consisting of members of $|l'|$ which pass through a fixed point P_1 on l'. Then there exists an irreducible curve C' on D' such that $C'+(f_{D'})^{-1}(P)$ supports a member B' of Λ. Let $\sigma: D''\to D'$ be the blowing-up of P_1. Then $l'':=\sigma'(l')\sim B'':=\sigma'(B')$, and $C'':=\sigma'(C')$ is a (-1) curve. Since $|l''|$ defines a $\boldsymbol{P}^1$-fibration on D'', B'' must contain another (-1) curve. This is not the case. So, D has no singular points. Hence $D'\cong D\cong \boldsymbol{P}^2$. In this case, it is easy to see that l' is an extremal rational curve and D' can be contracted algebraically to a point (cf. Theorem 3.2). Let $\mathscr{X}_1$ be the contraction of D'. Then $\rho(\mathscr{X}_1)=\rho(\mathscr{X})-1$, and we are done.

In the first two cases, let A be an ample divisor on $\mathscr{X}$ and let $L:=h^*(mA)$ with $m\gg 0$. Note that $(K_{\mathscr{X}}+X', l')=(K_{X'}, l')=-1<0$ and $(L, l')=0$. Hence there exists an extremal rational curve M with respect to $K_{\mathscr{X}}+X'+L$. There are two cases to consider. Namely, either (1) $M\cap X'\neq\emptyset$ or (2) $M\cap X'=\emptyset$. In the case (1), one may assume $M\subseteq X'$ (cf. Lemma 7.1). We verify:

Claim 4. *In the case* (1), *we have* $M=l'$.

Proof. Since $(K_{\mathscr{X}}+X'+L, M)<0$, we have $(h^*(mA), M)=0$. Hence $h(M)$ is a point. Moreover, $(K_{\mathscr{X}}+X', M)=(K_{X'}, M)<0$ and $(M^2)_{X'}<0$ since $h(M)$ is a point. Thus M is a (-1) curve. Recall that $h_{X'}: X'\to h(X')$ contracts l' and the components of the exceptional locus of minimal resolutions of quotient singularities. Hence $M=l'$.

We consider the remaining cases (i) and (ii) under the hypothesis $M=l'$. In the case (i), we perform a modification of type II along the curve l' as in Figure 21. Since l' is an extremal rational curve, $\mathscr{X}'$ is projective (cf. 4.4). Then $h':=h\cdot\phi^{-1}: \mathscr{X}'\to\mathscr{X}$ is a birational morphism. Note that Y' and Z' are contracted under the morphism $f: \mathscr{X}\to\mathscr{Y}$. For, otherwise, $l:=f(l')$ is not ample on $D:=f(D')$. Therefore $\tilde{Y}'$ and $\tilde{Z}'$ are contracted by h'. Let $\mathscr{Y}'$ be an $\mathscr{S}$-regular threefold with a good contraction $f': \mathscr{X}'\to\mathscr{Y}'$ and a morphism $g':=h'\cdot f'^{-1}: \mathscr{Y}'\to\mathscr{X}$; we assume

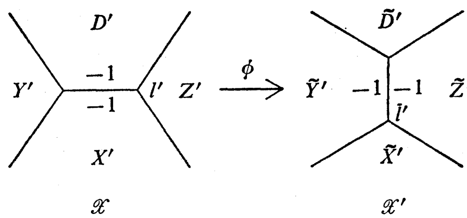

Figure 21

that f' is not extended further. Then $\tilde{Y}'$ and $\tilde{Z}'$ are not contracted under f' because $(\tilde{l}'^2)_{\tilde{Y}'}=(\tilde{l}'^2)_{\tilde{Z}'}=-1$. Hence we must have $\rho(\mathscr{X}')-\rho(\mathscr{Y}')\leqq\rho(\mathscr{X})-\rho(\mathscr{Y})-1$. So, we can apply the induction hypothesis of Main Lemma. In the case (ii) with $M=l'$,

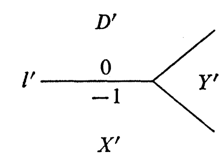

Figure 22

the component Y' is contracted under f (cf. Figure 22). For, otherwise, $l:=f(l')$ is not ample on D. Since $(Y'^2, D')\leqq-2$ and since the point $f(l'\cap Y')$ is a cyclic quotient singular point on D, we must have $(Y', D'^2)\geqq 0$ by the triple-point formula. The contraction $\mathscr{X}\to\mathscr{X}'$ of $R:=\boldsymbol{R}_+[l']$ is then the contraction of D' onto the curve $D'\cap Y'$. Hence $\mathscr{X}'$ is smooth and $\rho(\mathscr{X}')=\rho(\mathscr{X})-1$. So, we are done.

Now we consider the cases (i) and (ii) under the hypothesis $M\cap X'=\emptyset$. Let $f_1\colon \mathscr{X}\to\mathscr{X}_1$ be the contraction of $\boldsymbol{R}_+[M]$. Since $M\cap X'=\emptyset$, $\mathscr{X}$ and $\mathscr{X}_1$ are isomorphic to each other near the components X' and $X_1:=f_1(X')$. If $\mathscr{X}_1$ is not $\mathscr{S}$-regular, we apply the induction hypothesis of Main Lemma. If $\mathscr{X}_1$ is $\mathscr{S}$-regular, we have $(K(\mathscr{X}_1)+X', l')=-1<0$. So, repeat the above arguments all over again, though $\mathscr{X}$ is replaced by an $\mathscr{S}$-regular $\mathscr{X}_1$; the above arguments can be applied. Then at certain step, the image of l' becomes an extremal rational curve.

This completes the proof of Main Lemma in the case (1).

7.3. Next we treat the case (2) in Main Lemma. Let D be as in Main Lemma and let X be an irreducible component of the fiber F containing D such that $X\cap D\neq\emptyset$ and $(X, l)>0$.

Claim 1. *Such a component X is unique.*

Proof. Suppose that there exists another component Y like X. Then we have

$$0>(K_{\mathscr{Y}}, l)=(K_D+X|_D+Y|_D+(F-X-Y)|_D, l).$$

Let D' be the proper transform of D on $\mathscr{X}$ and let $\sigma:=f_{D'}: D'\to D$, which is a minimal resolution of quotient singularities. Then we have

$$\sigma^*(K_D+(F-D)|_D)\equiv K_{D'}+((F'-D')|_{D'})^\#,$$

where F' is the fiber of $\pi: \mathscr{X}\to\Delta$ corresponding to F and $((F'-D')|_{D'})^\#$ is the stripped form of $(F'-D')|_{D'}$ (cf. [6]). Note that $l':=\sigma' l$ is a nonsingular rational curve with $(l'^2)_{D'}=0$ and $(K_{D'}, l')=-2$. Hence we have

$$(K_D+(F-D)|_D, l)=(K_{D'}+((F'-D')|_{D'})^\#, l')\geqq 0.$$

This is a contradiction.

Note that $X|_D$ is a cross-section of the fibration $D\to g(D)$ and its general fiber is a nonsingular rational curve.

Claim 2. *Let C be a double curve on D other than $X|_D$. Then C is a fiber of $D\to g(D)$ and $C\cap \mathrm{Sing}(D)=\varnothing$.*

Proof. Since $[C]\in \boldsymbol{R}_+[l]$ and $(X, l)>0$, one has $(C, X)>0$. Then the point $C\cap X$ is a triple point and it is a smooth point. Hence $(C, X|_D)=1$. This implies that C is a fiber of $D\to g(D)$. Let $C':=\sigma'(C)$ and let L' be the fiber of $D'\to g(D)$ containing C'. If $C\cap \mathrm{Sing}(D)\neq\varnothing$, then L' is a degeneration of $\boldsymbol{P}^1$ and C' is the unique (-1) curve in L'. However, since $(C, X|_D)=1$, L' should contain another (-1) curve. This is a contradiction.

The above proof also verifies:

Claim 2′. *Let L be a fiber of $D\to g(D)$. If the point $L\cap X$ is smooth on D, then L is an irreducible smooth fiber and $L\cap \mathrm{Sing}(D)=\varnothing$.*

Let L' be anew a singular fiber of $D'\to g(D)$ and let $X':=f'(X)$.

Claim 3. *The following assertions hold true:*

(1) *The component G of L' meeing $X'|_{D'}$ is a double curve.*

(2) *Every component E of L' is a (-1) curve provided $f(E)$ is not a point. Moreover, E meets at most two connected components of the double curves $(F'-D')|_{D'}$ on D'.*

(3) *Any (-1) curve E in L' meets the connected component of $(F'-D')|_{D'}$ which includes $X'|_{D'}$.*

(4) *The configuration of L' is exhibited as follows, where the solid*

lines are the double curves. Moreover, all the connected components of $(F'-D')|_{D'}$ *contained in* L' *except one are linear chains consisting of* (-2) *curves* (cf. Figure 23).

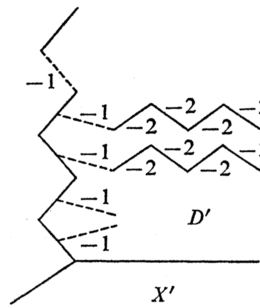

Figure 23

Proof. (1) follows from Claim 2′.

(2) If $f(L')$ is irreducible, E is obviously a (-1) curve. Suppose that $f(L')$ is reducible. Then $(\bar{E}^2)_D<0$, where $\bar{E}:=f(E)$. Moreover, since $(K_{\mathscr{Y}}, \bar{E})<0$, one has $(K_{D'}, E)<0$ (cf. the proof of Claim 1). Hence E is a (-1) curve and $\bar{E}$ is a generalized (-1) curve. The contraction of $\bar{E}$ produces a log-terminal surface (cf. 2.2). Then it is known that E meets at most two connected components of the exceptional locus of $\sigma: D'\to D$ (hence at most two connected components of $(F'-D')|_{D'}$) (cf. [6]).

(3) Note that $[\bar{E}]\in \boldsymbol{R}_+[l]$. Hence $(X|_D, \bar{E})>0$. This implies the assertion.

(4) Note that the connected component of double curves in L' which meets $X'|_{D'}$ is the exceptional locus of the minimal resolution of a cyclic quotient singularity. Hence it is a linear chain. The rest of the assertion is verified, if we note that L' is a degeneration of $\boldsymbol{P}^1$.

Claim 4. *Assume that there exists a singular fiber* L' *in the fibration* $D'\to g(D)$. *Then there exists an extremal rational curve* M *on* $\mathscr{X}$ *such that*:

(i) $(K_{\mathscr{X}}+D'+X', M)<0$ *and* $h(M)$ *is a point,*

(ii) *either Case* (a) $M\subseteq D'$, *or Case* (b) $M\cap D'=\emptyset$.

Proof. Let A be an ample divisor on $\mathscr{X}$. Let E be a (-1) curve in L'. Then $(K_{\mathscr{X}}+D'+X'+h^*(mA), E)=(K_{D'}+X'|_{D'}, E)=-1<0$ for any $m\gg 0$. Then, by Lemma 7.1, there exists an extremal rational curve M satisfying the conditions (i) and (ii).

If there are no singular fibers in the fibration $D'\to g(D)$, then D' is a minimal ruled surface and D' can be contracted algebraically onto $g(D)$. Indeed, after the contraction ϕ associated with several extremal rational curves which are disjoint from D', a fiber of $D' \to g(D)$ becomes an extremal rational curve and the contraction ψ associated with it is the contraction of D' onto $g(D)$. Then $\phi^{-1} \cdot \psi \cdot \phi$ is the required projective contraction. Hence $\phi^{-1} \cdot \psi \cdot \phi \colon \mathscr{X} \to \mathscr{X}_1$ produces a smooth projective threefold $\mathscr{X}_1$ with $\rho(\mathscr{X}_1) = \rho(\mathscr{X}) - 1$, and we are done.

So, we assume hereafter that there exists a singular fiber L' in the fibration $D' \to g(D)$. We consider an extremal rational curve M as in Claim 4. If the case (b) occurs, the contraction of $\boldsymbol{R}_+[M]$, $\phi_M \colon \mathscr{X} \to \mathscr{Y}'$, produces a threefold $\mathscr{Y}'$. Under ϕ_M, the component D' and its neighborhood are untouched. So, if $\mathscr{Y}'$ is $\mathscr{S}$-regular, we can repeat the above arguments and obtain threefolds $\mathscr{Y}', \mathscr{Y}'', \cdots, \mathscr{Y}^{(i)}, \cdots$. If $\mathscr{Y}^{(i)}$ is not $\mathscr{S}$-regular, we are done by the induction hypothesis. Then the case (a) occurs on some $\mathscr{Y}^{(m)}$. The above observation in the case where $D' \to g(D)$ has no singular fibers shows that we can consider the case (a) on $\mathscr{X}$ instead of $\mathscr{Y}^{(m)}$.

Since $h(M)$ is a point, M is contained in a fiber of $D' \to g(D)$. If M is a complete fiber then M is an extremal rational curve with respect to $K_{\mathscr{X}}$. Hence $D' \to g(D)$ has no singular fibers, which is a contradiction. Therefore we may assume that M is one of the (-1) curves in L'. We then proceed as follows:

(I) If $M = E$ as in Figure 24 then we can flip the following chain in L' off the component D'.

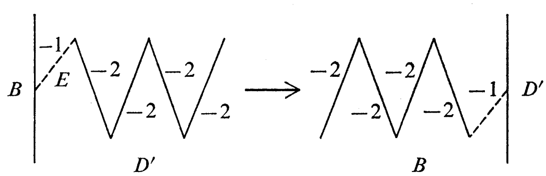

Figure 24

If there is a (-2) tail $\underset{-2}{\circ}-\cdots-\underset{-2}{\circ}$, such a linear chain is determined uniquely by $M = E$. Let $\bar{\mathscr{X}}$ be a threefold obtained from $\mathscr{X}$ by contracting the connected component T to a singular point of rod type, where T is the connected component of the exceptional locus $f \colon \mathscr{X} \to \mathscr{Y}$ such that $T \cap D'$ is the above (-2) tail. Then $\bar{\mathscr{X}}$ is projective and the image $\bar{E}$ of E is an extremal rational curve. Then we can algebraically flip $\bar{E}$ off the component $\bar{D}'$ (=the image of D' on $\bar{\mathscr{X}}$), cf. Lemma 4.3. This implies that the above flipping of $\underset{-1}{\circ}-\underset{-2}{\circ}-\cdots-\underset{-2}{\circ}$ on $\mathscr{X}$ is algebraic. If there is no (-2) tail we may have to flip all $E_1, \cdots, E_p$ simultaneously as long as $[E_1], \cdots, [E_p] \in \boldsymbol{R}_+[M]$. The projectivity of $\mathscr{X}$ is also preserved (cf. Lemma 4.2).

(II) We must also consider the case where the singular fiber L' is a linear chain with a unique (-1) curve E as shown in Figure 25. Then $N_{E/\mathscr{X}}=\mathcal{O}(-1)\oplus\mathcal{O}(-2)$, and one can apply the inverse of the modification of type III along E to get rid of E from the chain L'.

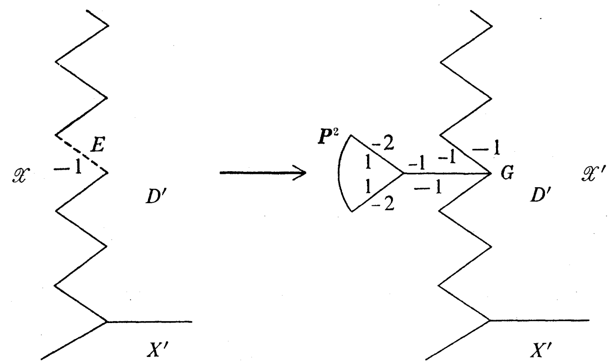

Figure 25

This process preserves the projectivity of $\mathscr{X}$ provided E is extremal on $\mathscr{X}$.

By the operation (I), we can reach the situation treated in (II). Then apply the operation (II) to obtain an $\mathscr{S}$-degeneration π': $\mathscr{X}'\to\Delta$ with a birational morphism h': $\mathscr{X}'\to\mathscr{X}$. Note that $\rho(\mathscr{X}')=\rho(\mathscr{X})+1$. For the sake of simplicity, *we assume that* L' is a unique singular fiber of $D'\to g(D)$. After the operation (II), one of the two irreducible components in L' which are adjacent to E has the image (on $\mathscr{X}'$) as exhibited in Figure 25. After replacing $\mathscr{X}'$ by a new $\mathscr{S}$-degeneration $\mathscr{X}''$ such that $\mathscr{X}'$ and $\mathscr{X}''$ are isomorphic near the components described in Figure 25 and $\rho(\mathscr{X}'')\leqq\rho(\mathscr{X}')$, we may assume that G is extremal on $\mathscr{X}'$. Then we operate the modification of type II along G. Hence we obtain a picture like the one in Figure 26.

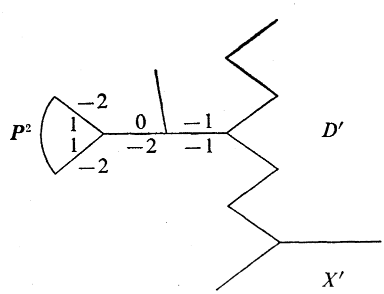

Figure 26

Applying repeatedly the modifications of type II along suitable components of L', we finally reach a situation as in Figure 27.

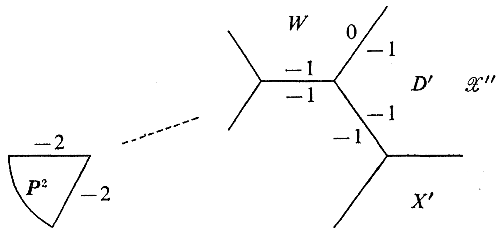

Figure 27

We denote by $\mathscr{X}''$ the $\mathscr{S}$-degeneration on which we have the above situation. One can easily show that, after replacing $\mathscr{X}''$ by a new $\mathscr{S}$-degeneration $\mathscr{X}^{(3)}$, C is an extremal rational curve on $\mathscr{X}''$, where $\mathscr{X}''$ and $\mathscr{X}^{(3)}$ are isomorphic near the component D' and where $\rho(\mathscr{X}'')=\rho(\mathscr{X}^{(3)})=\rho(\mathscr{X}')$. Then the component denoted by W is a minimal ruled surface and contracted smoothly along the $\boldsymbol{P}^1$-fibration $|C|$. Thus we obtain a nonsingular threefold $\mathscr{X}$. Then the component D' on $\mathscr{X}$ is a minimal ruled surface and contracted smoothly to the curve $D'\cap X'$. Thus we obtain a nonsingular projective threefold $\mathscr{X}$, which is an $\mathscr{S}$-degeneration with $\rho(\mathscr{X})\leqq \rho(\mathscr{X})-1$. The above arguments are easily modified even in the case where the fibration $D'\to g(D)$ has more than one singular fibers. This completes the proof of Main Lemma in the case (2).

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