Note on Stable Homotopy Types of Stunted Quaternionic Spherical Space Forms

Teiichi Kobayashi

§ 1. Introduction

Let $H_m = \{x, y \mid x^{2^m-1} = y^2, xyx = y\}$ be the generalized quaternion group of order $2^m + 1$ ($m \geq 2$). An element of $H_m$ is uniquely expressed as $x^u y^v$ for $0 \leq u < 2^m$, $v = 0, 1$. Let $d_1: H_m \to S^3 = Sp(1) = SU(2)$ be the natural inclusion map defined by $d_1(x) = \exp(2\pi i/2^m)$, $d_1(y) = j$. Then $H_m$ acts freely on the unit sphere $S^{4n+3}$ in the quaternion $(n+1)$-space $H^{n+1}$ by the diagonal action $(n+1)d_1: H_m \to Sp(n+1)$. The quotient manifold $S^{4n+3}/H_m$ is called the quaternionic spherical space form and is denoted by $N_n(m)$. If $n \geq 0$, we have the natural inclusion map $N_{n-1}(m) \subset N_n+k(m)$, and denote by $N_{n+k}(m)$ the quotient space $N_n+k(m)/N_{n-1}(m)$.

The purpose of this note is to study the stable homotopy types of the stunted quaternionic spherical space forms $N_{n+k}(m)$. We have

Theorem 1.1. If $N_{n+k-1}(m)$ and $N_{n+k}(m)$ are of the same stable homotopy type, then $j \equiv k \mod 2^{2n-2}$.

This is proved in the way of H. Ōshima [10, Theorem 8.4] (cf. [8, Theorem 1.1]), and is a generalization of the Ōshima’s result. As for the converse, we obtain

Theorem 1.2. If $j \equiv k \mod 2^{2n+m-2+\varepsilon}$, then $N_{n+k-1}(m)$ and $N_{n+k}(m)$ are of the same stable homotopy type, where $\varepsilon = 1$ if $n$ is odd, and $\varepsilon = 0$ if $n$ is even.

This is a consequence of the results of M. F. Atiyah [2, Proposition 2.6], H. Ōshima [10, Theorem 2.1 and Proposition 8.2] and [7, Corollary 1.7], and is also a generalization of Theorem 8.3 (iii) in [10].

We recall in Section 2 the representation rings $R_F(H_m)$ of $H_m$, where $F$ denotes the field $R$ of the real numbers or the field $C$ of the complex numbers, according to [4], [5], [6] and [11]. In Section 3 we study Adams operations [1] in $K_F(N^n(m))$. The proofs of Theorems 1.1 and 1.2 depend

Received December 13, 1984.
on the orders of the canonical elements of $\bar{K}_r(N^n(m))$ (cf. [7], [9] and [10]), and are given in Section 4 and Section 5.

§ 2. The representation rings $R_r(H_m)$

We first recall the complex representation ring $R_C(H_m)$ of $H_m$ (cf. [4, § 47], [11, § 1] and [5, § 3]).

**Proposition 2.1.** $R_C(H_m)$ is generated as a free abelian group by $1, a, b, c$ and $d_r (r=1, 2, \ldots, 2^m-1)$ defined below

\[
\begin{align*}
1(x) &= 1, & a(x) &= 1, & b(x) &= -1, & c(x) &= -1, \\
1(y) &= 1, & a(y) &= -1, & b(y) &= 1, & c(y) &= -1,
\end{align*}
\]

\[
d_r(x) = \begin{bmatrix} \omega^r & 0 \\ 0 & \omega^{-r} \end{bmatrix}, \quad d_r(y) = \begin{bmatrix} 0 & (-1)^r \\ 1 & 0 \end{bmatrix},
\]

where $\omega$ is a primitive $2^m$-th root $\exp(2\pi i/2^m)$ of unity. The characters of these representations are

\[
\begin{align*}
\chi(1)(x^uy^v) &= 1, \quad \chi(a)(x^uy^v) = (-1)^v, \quad \chi(b)(x^uy^v) = (-1)^u, \\
\chi(c)(x^uy^v) &= (-1)^{u+v}, \quad \chi(d_r)(x^uy^v) = (\omega^{ur} + \omega^{-ur})(1-v),
\end{align*}
\]

where $u=0, 1, \ldots, 2^m-1$, $v=0, 1$.

Evaluating the characters, one has the following relations.

**Proposition 2.2.** $a^2 = b^2 = c^2 = 1$, $ab = c$, $d_r d_s = d_{r+s} + d_{r-s}$, $bd_r = d_{2^{m-1} - r}$, where $d_0 = 1 + a$, $d_{2^{m-1}} = b + c$, $d_{-r} = d_r$, $d_{2^{m-1} + r} = d_{2^{m-1} - r}$.

Define the elements $\alpha$, $\beta$, $\gamma$ and $\delta_r$ of the reduced representation ring $\bar{R}_C(H_m)$ by

\[
\alpha = a - 1, \quad \beta = b - 1, \quad \gamma = a + b + c - 3, \quad \delta_r = d_r - 2 \quad (1 \leq r < 2^{m-1}).
\]

**Proposition 2.3.** $\bar{R}_C(H_m)$ is generated as a free abelian group by $\alpha$, $\beta$, $\gamma$ and $\delta_r (r=1, 2, \ldots, 2^{m-1} - 1)$ with relations:

\[
\begin{align*}
\alpha^2 &= -2\alpha, \quad \beta^2 = -2\beta, \quad \gamma = \alpha \beta + 2\alpha + 2\beta, \quad \alpha \delta_1 = -2\alpha, \\
\beta \delta_1 &= -2\beta + \delta_{2^{m-1} - 1} - \delta_1, \quad \gamma \delta_1 + \delta_{-1} - \gamma = 2\delta_r + 2\delta_1,
\end{align*}
\]

where $\delta_{2^{m-1}} = -\alpha$, $\delta_0 = \alpha$, $\delta_{-r} = \delta_r$, $\delta_{2^{m-1} + r} = \delta_{2^{m-1} - r}$. Thus $\bar{R}_C(H_m)$ is generated by $\alpha$, $\beta$ and $\delta_1$ as a ring.
Let $R_R(\mathbb{H}_m)$ be the real representation ring of $\mathbb{H}_m$ (cf. [11, Proposition 1.5]). Since the complexification $c: R_R(\mathbb{H}_m) \to R_C(\mathbb{H}_m)$ is monomorphic, in what follows we identify $R_R(\mathbb{H}_m)$ with the image in $R_C(\mathbb{H}_m)$ under $c$.

**Proposition 2.4.** $R_R(\mathbb{H}_m)$, considered as the subring $c(R_R(\mathbb{H}_m))$ of $R_C(\mathbb{H}_m)$, is generated by $1, a, b, c, d_r (r = 1, 2, \ldots, 2^m - 2 - 1), 2d_{2^r+1} (r = 0, 1, \ldots, 2^m - 2 - 1)$.

§ 3. **Adams operations in $R_F(\mathbb{H}_m)$ and $K_F(\mathbb{N}^n(m))$**

Let $\Phi^\prime_F: R_F(\mathbb{H}_m) \to R_F(\mathbb{H}_m)$ be the Adams operation, where $F = C$ or $R$.

**Lemma 3.1.** Let $i$ be odd. Then $\Phi^\prime_C: R_C(\mathbb{H}_m) \to R_C(\mathbb{H}_m)$ is given as follows. For $\theta = 1, a, b, c \in R_C(\mathbb{H}_m)$,

$$\Phi^\prime_C(\theta) = \theta.$$

For $d_r \in R_C(\mathbb{H}_m) (r = 1, 2, \ldots, 2^m - 1)$,

$$\Phi^\prime_C(d_r) = d_r.$$

**Proof.** First, we prove a relation in $\mathbb{H}_m$:

$$\langle x^u \rangle^{2k+1} = x^{2^m - 1k + u} y,$$

where $k$ is any non-negative integer. This is clear for $k = 0$. Suppose that this is true for $k$. Then

$$\langle x^u \rangle^{2(k+1)+1} = \langle x^u \rangle^{2k+1} \langle x^u \rangle \langle x^u \rangle = x^{2^m - 1(k+1) + u} y(x^u y)x^u y$$

$$= x^{2^m - 1k + u} y x^{2^m - 1k + u} y = x^{2^m - 1(k+1) + u} y,$$

since $x^u y x^u = x^u^{-1}(xy)x x^u^{-1} = x^u y x x^u^{-1} = \cdots = y$.

Second, we recall a formula due to J. F. Adams [1, Theorem 4.1 (vi)]:

$$\chi(\Phi^\prime_C(\theta))(g) = \chi(\theta)(g^\prime), \quad \text{for } \theta \in R_C(\mathbb{H}_m) \text{ and } g \in \mathbb{H}_m.$$

For the proof of the lemma it suffices to check that in each equation, the characters of the two sides agree. Obvious calculations, based on (3.1.2), (3.1.1) and Proposition 2.1, show that

$$\chi(\Phi^\prime_C(\theta))(x^u y^v) = \chi(\theta)(x^u y^v), \quad \text{for } \theta = 1, a, b, c,$$

$$\chi(\Phi^\prime_C(d_r))(x^u y^v) = \chi(d_r)(x^u y^v),$$

where $x^u y^v \in \mathbb{H}_m$. q.e.d.
Lemma 3.2. \( \Psi_R^{2m+1} \) is the identity on \( R_F(H_m) \).

Proof. We first consider the case \( F = C \). By Lemma 3.1, it remains only to prove that \( d_r = d_r \) for \( i = 2^m + 1 \). We may assume \( r > 0 \). Then, by Proposition 2.2,

\[
d_{2^m+1+r} = d_{2^m+1+2^m-1(2^m-1)}+r = d_{2^m+1-2^m-1(2^m-1)}-r = d_{2^m(1-2^m-1)-r} = d_{r}.\]

The fact that \( \Psi_R^{2m+1} = 1 \) follows from the following commutative diagram [1, Theorem 4.1 (iv)]:

\[
\begin{array}{ccc}
R_R(H_m) & \xrightarrow{\Psi_R^i} & R_R(H_m) \\
\downarrow c & & \downarrow c \\
R_C(H_m) & \xrightarrow{\Psi_C^i} & R_C(H_m),
\end{array}
\]

since the complexification \( c \) is monomorphic. q.e.d.

Let \( \Psi_F: K_F(N^n(m)) \to K_F(N^n(m)) \) be the Adams operation, where \( F = C \) or \( R \).

Lemma 3.3. \( \Psi_F^{2m+1} \) is the identity on \( K_F(N^n(m)) \).

Proof. The result follows from the following commutative diagram:

\[
\begin{array}{ccc}
R_F(H_m) & \xrightarrow{\xi_F} & K_F(N^n(m)) \\
\downarrow \Psi_F^i & & \downarrow \Psi_F^i \\
R_F(H_m) & \xrightarrow{\xi_F} & K_F(N^n(m)),
\end{array}
\]

where \( \xi_F \) is the natural projection (cf. [5, § 4], [6, § 3], [11, Theorem 2.5]). q.e.d.

§ 4. Proof of Theorem 1.1

Let \( i: N^{k-1}(m) \to N^{n+k}(m) \) be the inclusion and \( p: N^{n+k}(m) \to N^{n+k}(m) = N^{n+k}(m)/N^{k-1}(m) \) be the projection. Consider the Puppe exact sequence:

\[
\begin{array}{ccc}
\tilde{K}_G^{-1}(N^{n+k}(m)) & \xrightarrow{i^*} & \tilde{K}_G^{-1}(N^{k-1}(m)) \to \tilde{K}_G(N^{n+k}(m)) & \xrightarrow{p^*} & \tilde{K}_G(N^{n+k}(m)).
\end{array}
\]

Using Atiyah-Hirzebruch spectral sequences, we see that \( \tilde{K}_G^{-1}(N^i(m)) \cong Z \) and that \( i^* \) is trivial. Hence \( p^* \) is monomorphic on Tor \( \tilde{K}_G(N^{n+k}(m)) \). We prove
Lemma 4.1. \( \tilde{\Psi}^{2m+1} = (2^m + 1)^t \) on Tor \((\tilde{K}_c(S^{2t}N^{n+k}_k(m)))\).

Proof. Consider the following diagram:

\[
\begin{array}{ccc}
\tilde{K}_c(N^{n+k}_k(m)) & \xrightarrow{\tilde{p}^*} & \tilde{K}_c(S^{2t}N^{n+k}_k(m)) \\
\downarrow \psi^t_c & & \downarrow \psi^t_c \\
\tilde{K}_c(N^{n+k}_k(m)) & \xrightarrow{\tilde{p}^*} & \tilde{K}_c(S^{2t}N^{n+k}_k(m)),
\end{array}
\]

where \( I \) is the Bott isomorphism. Then, by [1, Corollary 5.3], we have \( \psi^t_c I^t = i^t I^t \psi^t_c \). If \( i = 2^m + 1 \), \( \psi^t_c^{2m+1} \) is the identity on Tor \((\tilde{K}_c(N^{n+k}_k(m)))\) by Lemma 3.3, since \( p^* \) is monomorphic on Tor \((\tilde{K}_c(N^{n+k}_k(m)))\). Hence we have the desired result. q.e.d.

Next, we recall the order of the canonical element of \( \tilde{K}_c(N^n(m)) \).

Let \( \lambda \) be the canonical complex plane bundle over the quaternion projective space \( H^m = S^{4m+3}/S^3 \), and let \( \pi : N^n(m) = S^{4m+3}/H_m \rightarrow H^m \) be the natural projection. Let \( \xi_c : \tilde{K}_c(H_m) \rightarrow \tilde{K}_c(N^n(m)) \) be the natural projection defined in [5, § 4] and put \( \delta = \xi_c(\delta_i) \). Then we have

\( \delta = \pi^* \lambda - 2 \)

(cf. [5, Lemma 4.4]). The order \( \# \delta^t \) of \( \delta^t \in \tilde{K}_c(N^n(m)) \) is determined by H. Oshima in [10, Proposition 5.2] and by T. Mormann in [9, Chapter 2, Theorem 4.52] as follows.

Proposition 4.2. \( \# \delta^t = 2^{m+m+1-2i} \) (1 \( \leq i \leq n \)).

Proof of Theorem 1.1. If \( N^{n+j}_j(m) \) is stably homotopy equivalent to \( N^{n+k}_k(m) \), we may assume that there exists a homotopy equivalence

\( g : S^{2t}N^{n+j}_j(m) \rightarrow S^{2t+4j-4k}N^{n+k}_k(m) \)

for some integer \( t \geq 0 \). Consider the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{K}_c(S^{2t+4j-4k}N^{n+k}_k(m)) & \xrightarrow{\tilde{g}^*} & \tilde{K}_c(S^{2t+j}(m)) \\
\downarrow \psi^t_c & & \downarrow \psi^t_c \\
\tilde{K}_c(S^{2t+4j-4k}N^{n+k}_k(m)) & \xrightarrow{\tilde{g}^*} & \tilde{K}_c(S^{2t+j}(m)),
\end{array}
\]

where \( i = 2^m + 1 \). Then, by Lemma 4.1,

\( (2^m+1)^{t+2j-2k} g^* = (2^m+1)^t g^* \) on Tor \((\tilde{K}_c(S^{2t+4j-4k}N^{n+k}_k(m)))\),

and so, for any torsion element \( \alpha \in \tilde{K}_c(S^{2t+j}(m)) \), \( (2^m+1)^{t+2j-2k} \alpha = (2^m+1)^t \alpha \). Now we have
\[ \tilde{K}_c(S^{2n}N^+_g(m)) \cong \tilde{K}_o(N^+_g(m)) \] (by Bott isomorphism)
\[ \cong \tilde{K}_c((N^o(m))^{fr*1}) \] (by Ōshima [10, Theorem 2.1])
\[ \cong K_c(N^o(m)) \] (by Thom isomorphism [3, Theorem 7.2]),

where \( X^\zeta \) denotes the Thom complex of a vector bundle \( \zeta \) over \( X \). Hence, by Proposition 4.2, there is an element of order \( 2^{2n+m-1} \) in \( \tilde{K}_c(S^{2n}N^+_g(m)) \).

Therefore
\[ (2^n+1)^j+2j-2k \equiv (2^n+1)^j \equiv 0 \pmod{2^{2n+m-1}}, \]

that is, \((2^n+1)^j+2j-2k \equiv 0 \pmod{2^{2n+m-1}}\). Set \( 2j-2k=u \cdot 2^n \), where \( u \) is odd. Then, according to [8, Lemma 3.1], \((2^n+1)^{u \cdot 2^n}-1 \equiv u \cdot 2^{m+n} \pmod{2^{n+m+1}}\). Thus \( u+m \geq 2n+m-1 \), and so \( u \geq 2n-1 \). Hence we have \( j-k \equiv 0 \pmod{2^{2n-2}} \).

**§ 5. Proof of Theorem 1.2**

Let \( r: \tilde{K}_o(X) \to \tilde{K}_R(X) \) and \( c: \tilde{K}_R(X) \to \tilde{K}_o(X) \) be the real restriction and the complexification, respectively. Let \( \xi_R: \tilde{R}_R(H_m) \to \tilde{R}_R(N^o(m)) \) be the natural projection defined in [6, (3.9)] (or in [11, Theorem 2.5]). Then, for the elements \( v \) and \( z \) of \( \tilde{R}_R(H_m) \) defined by \( c^{-1}(2\partial_0) = v \) and \( c^{-1}(\partial_1^2) = z \) (cf. § 2), we have
\[ \xi_R v = r(\pi^* \lambda - 2) \quad \text{and} \quad \xi_R z = c^{-1}(\pi^* \lambda - 2)^2 \]
(cf. [6, Lemma 3.10]), since \( \delta_1 \) is self-conjugate and \( cr=1 \) + conjugation. For simplicity we write \( v \) and \( z \) instead of \( \xi_R v \) and \( \xi_R z \). Then, for the complexification \( c: \tilde{K}_R(N^o(m)) \to \tilde{K}_o(N^o(m)) \), we have
\[ c(v) = 2\delta \quad \text{and} \quad c(z) = \delta^2. \]

The orders of the canonical elements \( z^i \) and \( vz^i \) of \( \tilde{K}_R(N^o(m)) \) are determined in [7, Theorems 1.5 and 1.6]. As a corollary, the order of \( v \) is determined in [7, Corollary 1.7].

**Proposition 5.1.** For \( v \in \tilde{K}_R(N^o(m)) \),
\[ \#v = 2^{2n+m-2+\epsilon}, \quad \text{where} \quad \epsilon = 1 \quad \text{if} \ n \text{ is odd,} \ \epsilon = 0 \quad \text{if} \ n \text{ is even} > 0. \]

**Proof of Theorem 1.2.** Let \( v = \xi_R v = r(\pi^* \lambda - 2) \in \tilde{K}_R(N^o(m)) \). If \( j-k \equiv 0 \pmod{2^{2n+m-2+\epsilon}} \), then, by Proposition 5.1, \( J((j-k)v) = 0 \), where \( J: \tilde{K}_R(N^o(m)) \to \tilde{J}(N^o(m)) \) is the J-homomorphism. Thus \( J(r(\pi^* \lambda - 2)) = J(kr(\pi^* \lambda - 2)) \). According to [2, Proposition 2.6], this implies that the Thom complexes \((N^o(m))^{fr*1}\) and \((N^o(m))^{kr*1}\) are of the same stable
homotopy type. On the other hand, by [10, Theorem 2.1], there are natural homeomorphisms:

\[ N_j^{n+i}(m) \approx (N^n(m))^{ir \times i}, \quad N_k^{n+k}(m) \approx (N^n(m))^{kr \times k}, \]

and hence \( N_j^{n+i}(m) \) and \( N_k^{n+k}(m) \) are of the same stable homotopy type.

q.e.d.

References