On $p$-Sylow Subgroups of Groups of Self Homotopy Equivalences of Sphere Bundles over Spheres

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Dedicated to Professor Nobuo Shimada on his 60th birthday

Introduction

The set $\mathcal{E}(X)$ of homotopy classes of homotopy equivalences of a space $X$ to itself forms a group under composition of maps. This group $\mathcal{E}(X)$ has been investigated by several authors (e.g. [2], [7], and [12]).

In the case where $X$ is an $S^m$-bundle over $S^n$, the group $\mathcal{E}(X)$ has been investigated for $X = V_{n,2}$ and $W_{n,2}$ by Y. Nomura [10] and for $X$ with $3 < m + 1 < n < 2m - 2$ by S. Sasao [13], where $V_{n,2} = O(n)/O(n-2)$ and $W_{n,2} = U(n)/U(n-2)$ are the real and complex Stiefel manifolds respectively.

In this note, we study the $p$-Sylow subgroup of $\mathcal{E}(X)$ for an $S^m$-bundle $X$ over $S^n$ with a mod $p$ $H$-structure such that $i(p): S^m_{(p)} \rightarrow X_{(p)}$ is an $H$-map, where $m$ and $n$ are odd integers, $S^m_{(p)}$ and $X_{(p)}$ are localizations of $S^m$ and $X$ at $\{p\}$ respectively and $i(p)$ is the localization of the inclusion $i: S^m \subset X$ at $\{p\}$. Our main result is as follows:

**Theorem 4.5.** Let $m$ and $n$ be odd integers such that $3 < m < n - 1$, and let $S^m \rightarrow X \rightarrow S^n$ be an $S^m$-bundle over $S^n$. Let $p$ be an odd prime. If $S^m_{(p)}$ and $X_{(p)}$ are $H$-spaces such that $i(p): S^m_{(p)} \rightarrow X_{(p)}$ is an $H$-map, then the group $\mathcal{E}(X)$ is a finite group with a unique $p$-Sylow subgroup $\tilde{S}_p$ given by the semi direct product

$$\tilde{S}_p \cong \pi_{m+n}(X; p) \times_T \pi_n(S^m; p),$$

where $\alpha T \beta = \alpha + l \circ \beta \circ q \circ \alpha$ for $\alpha \in \pi_{m+n}(X; p)$ and $\beta \in \pi_n(S^m; p)$.

In Section 1, we determine the $p$-Sylow subgroup of $\mathcal{E}(S^m \cup e^n)$ (Proposition 1.3). In Section 2, we define a homomorphism $j^1: \mathcal{E}(X) \rightarrow \mathcal{E}(K)$ and study the $p$-Sylow subgroup of $\text{Im } j^1$ (Lemma 2.7). In Section
3, we prepare three lemmas and the above theorem is proved in Section 4. In the last section, Section 5, we calculate the $p$-Sylow subgroup of $\mathcal{E}(X)$ of some $S^n$-bundles $X$ over $S^n$ for any odd prime $p$ and determine the group $\mathcal{E}(X)$ as a group extension of a certain group by a 2-group.

Throughout this note, all spaces have base points and all maps and homotopies preserve base points. For given spaces $X$ and $Y$, we denote by $[X, Y]$ the set of (based) homotopy classes of maps of $X$ to $Y$ and by the same letter a map $f: X \to Y$ and its homotopy class $f \in [X, Y]$.

§ 1. The $p$-Sylow subgroup of $\mathcal{E}(S^m \cup e^n)$

Let $f \in \pi_{n+1}(S^m)$ ($2 \leq m < n - 1$) be a given element and let $K = S^m \cup f e^n$ denote the mapping cone of $f$. Let $\ell_1: K = S^m \cup e^n \to (S^m \cup e^n) \vee S^n = K \vee S^n$ be the coaction defined by shrinking the equator $S^n-1 \times \{1/2\}$ of $e^n$ in $S^m \cup e^n$ to the base point. Then we can define a map

$$A: \pi_n(K) \to [K, K] \quad \text{by} \quad A(\alpha) = \mathcal{V} \circ (1 \vee \alpha) \circ \ell_1,$$

where $\mathcal{V}$ is the folding map and $1$ is the class of the identity map of $K$. Let $i': S^m \subset K$ be the inclusion. Then by composing $i'_*: \pi_n(S^m) \to \pi_n(K)$ with $A$, we obtain a homomorphism (cf. [11, Lemmas 1.4 and 1.8])

$$\lambda_i: \pi_n(S^m) \to \mathcal{E}(K) \quad \text{by} \quad \lambda_i(\alpha) = \mathcal{V} \circ (1 \vee i' \circ \alpha) \circ \ell_1.$$

We put

$$H = \pi_n(S^m)/(f \pi_n(S^{n-1}) + \tau(f)\pi_{m+1}(S^m)).$$

Here $\tau(f)(\eta) = \eta \circ SF + [\tau_m, \eta] \circ Sh(f)$ for $\eta \in \pi_{m+1}(S^m)$ where $\tau_m$ is the class of the identity map of $S^m$, $[\tau_m, \eta] \in \pi_{2m}(S^m)$ is the Whitehead product of $\tau_m$ and $\eta$ and $h(f) \in \pi_{n-1}(S^{2m-1})$ is the generalized Hopf invariant of $f$ due to P. J. Hilton [3]. Then the homomorphism $\lambda_i$ induces a monomorphism: $H \to \mathcal{E}(K)$ and we have

$$\lambda_i: \pi_n(S^m) \to \mathcal{E}(K) \quad \text{by} \quad \lambda_i(\alpha) = \mathcal{V} \circ (1 \vee i' \circ \alpha) \circ \ell_1.$$
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\[ \mathbb{Z}_2 = \{1, -1\} \] on \( H \) by \((-1) \cdot \alpha = -\alpha \) for \( \alpha \in H \) and \( \alpha(f) = f + (-1) \circ f = [\epsilon_m, \epsilon_m] \circ h(f) \).

From \( \pi_k(S^{k-1}) = \mathbb{Z}_2 \) for \( k \geq 4 \) and (1.2), we have immediately

**Proposition 1.3.** Let \( 3 \leq m < n - 1 \) and let \( p \) be an odd prime. In the case where \( n \) \( m \) is even, we assume \( n \neq 2m - 1 \). Then, for the mapping cone \( K = S^m \cup_J e^n \) of \( f \in \pi_{n-1}(S^m) \), the group \( \mathcal{E}(K) \) is a finite group with a unique \( p \)-Sylow subgroup \( S_p \) given by

\[ S_p = \beta_1 \pi_n(S^m; p) \cong \pi_n(S^m; p), \]

where \( \pi_n(S^m; p) \) denotes the \( p \)-primary component of \( \pi_n(S^m) \).

§ 2. The \( p \)-Sylow subgroup of \( \operatorname{Im} j^1 \)

Let \( X \) denote an \( S^m \)-bundle over \( S^n \) \((2 \leq m < n - 1) \). Then by James-Whitehead [6], \( X \) has a cell structure given by

\[ X = K \cup e^{m+n}, \quad K = S^m \cup_J e^n. \]

Since the inclusion \( j: K \subset X \) induces a bijection \( j_*: [K, K] \to [K, X] \), the homomorphism

\[ j^1: \mathcal{E}(X) \to \mathcal{E}(K) \]

can be defined by the restriction to \( \mathcal{E}(X) \) of the composite

\[ [X, X] \stackrel{j^*}{\longrightarrow} [K, X] \stackrel{j_*^{-1}}{\longrightarrow} [K, K]. \]

We define the coaction

\[ \ell_2: X = K \cup e^{m+n} \longrightarrow (K \cup e^{m+n}) \vee S^{m+n} = X \vee S^{m+n} \]

by shrinking the equator \( S^{m+n-1} \times \{1/2\} \) of \( e^{m+n} \) to the base point. Since \( \pi_{m+n}(S^m) \) and \( \pi_{m+n}(S^n) \) for \( 2 \leq m < n - 1 \) are finite, \( \pi_{m+n}(X) \) is finite by the exact sequence associated with the \( S^m \)-bundle over \( S^n \):

\[ S^m \xrightarrow{i} X \xrightarrow{q} S^n. \]

Therefore, by the Blakers-Massey theorem and the exact sequence of the pair \((X, K)\) we have

\[ j_*: \pi_{m+n}(K) \longrightarrow \pi_{m+n}(X) \]

is epimorphic.

Hence, similarly to the way that we defined \( A \) in Section 1, we can define
a homomorphism (cf. [11, Lemmas 1.4 and 1.8])

\[ 2 \pi_{m+n}(X) \xrightarrow{j_*} \pi_{m+n}(K) \xrightarrow{\ell_1} \mathfrak{C}(X) \]

where \( \ell_1 : \mathfrak{C}(X) \to \mathfrak{C}(X) \) is the folding map and 1 is the class of the identity map of \( X \). Also, since the attaching element \( g \in \pi_{m+n-1}(K) \) of \( e^{m+n} \) in \( X = K \cup e^{m+n} \) is of infinite order, by Barcus-Barratt [2, Theorem 6.1], J. W. Rutter [12, Theorem 3.1*] and (2.4) we have the following exact sequence:

\[ 0 \to \lambda_1(\pi_{m+n}(X)) \to \mathfrak{C}(X) \xrightarrow{j_1} G \to 1, \]

where \( G = \{ h \in \mathfrak{C}(K) \mid h \circ g = \varepsilon g \ (\varepsilon = \pm 1) \in \pi_{m+n-1}(K) \} \subset \mathfrak{C}(K) \).

**Lemma 2.7.** (i) For \( \alpha \in \pi_n(S^m) \), \( \lambda_1(\alpha) \in \mathfrak{C}(K) \) given in (1.1) can be extended to an element of \( \mathfrak{C}(X) \) if and only if \( i'_*[\alpha, \iota_m] = 0 \), where \( i' : S^m \subset K \) is the inclusion and \( \iota_m \) is the class of the identity map of \( S^m \).

(ii) Let \( m \) be an odd integer. Then for any odd prime \( p \) the subgroup \( G \) of \( \mathfrak{C}(K) \) in the above sequence is a finite group with a unique \( p \)-Sylow subgroup \( S_p \approx \pi_n(S^m; p) \) given in Proposition 1.3.

**Proof.** (i) Let \( g \) be the attaching element of \( e^{m+n} \) in \( X = K \cup e^{m+n} \) given in (2.1). Then we have \( j_* g = \pm [\sigma, i'] \), where \( j_* : \pi_{m+n-1}(K) \to \pi_{m+n-1}(K, S^m) \) and \( \sigma \in \pi_n(K, S^m) \) is an element such that \( \partial \sigma = f \), the attaching element of \( e^n \) in \( K = S^m \cup e^n \). So, by [4, Lemma 5.4], we have

\[ \ell_1 \circ g = k_1 g \pm [k_n, k_m], \]

where \( \ell_1 : K \to K \vee S^m \) is the coaction given in Section 1, \( k : K \to K \vee S^n \) and \( k_r : S^r \to K \vee S^n \) \((r = m, n)\) are obvious inclusions. Therefore, for \( \lambda_1(\alpha) \ (\alpha \in \pi_n(S^m)) \) given in (1.1), we have

\[
\lambda_1(\alpha) \circ g = \bar{\ell}_1 \circ (1 \vee (i' \circ \alpha)) \circ \ell_1 \circ g \\
= \bar{\ell}_1 \circ (1 \vee (i' \circ \alpha)) \circ (k_1 g \pm [k_n, k_m]) \\
= g \pm [i' \circ \alpha, i'] \\
= g \pm i'_*[\alpha, \iota_m].
\]

Since \( g \) is of infinite order and \([\alpha, \iota_m]\) is of finite order, the above equalities imply that \( \lambda_1(\alpha) \circ g \neq -g \) for any \( \alpha \in \pi_n(S^m) \) and that \( \lambda_1(\alpha) \circ g = g \) if and only if \( i'_*[\alpha, \iota_m] = 0 \).

(ii) If \( m \) is an odd integer, then \([\alpha, \iota_m] = 0 \) for any \( \alpha \in \pi_n(S^m; p) \) and so by (i) and Proposition 1.3, \( G \) has a unique \( p \)-Sylow subgroup \( S_p \approx \pi_n(S^m; p) \). q.e.d.
§ 3. Some lemmas for an $H$-structure on $X(p)$

Let $m$ and $n$ be odd integers such that $3 < m < n - 1$ and $S^m \to X \to S^n$ be an $S^m$-bundle over $S^n$. In this section we assume that $p$ is an odd prime such that the localized space $X(p)$ at $\{p\}$ is an $H$-space. Then we have the following lemmas which will be used in the next section.

**Lemma 3.1.** Let $\alpha \in \pi_n(X;p)$, $\beta \in \pi_n(X;p)$, $\gamma \in \pi_{m+n}(X;p)$ and $\xi \in \pi_n(S^m;p)$, and let $\pi: X \to X/K = S^{m+n}$ be the collapsing map, where $K$ is the subcomplex of $X$ given in (2.1). Then we have

(i) $$(1 + \alpha(p) \circ q(p) + \beta(p) \circ q(p)) = 1 + (\alpha(p) \circ q(p) + \beta(p) \circ q(p)),$$

(ii) $$(1 + \alpha(p) \circ q(p) + \gamma(p) \circ \pi(p)) = 1 + (\alpha(p) \circ q(p) + \gamma(p) \circ \pi(p)),$$

(iii) $$i(p) \circ \xi(p) + \gamma(p) \circ \pi(p) = \gamma(p) \circ \pi(p) + i(p) \circ \xi(p) \circ q(p),$$

where $+$ denotes the multiplication induced from the $H$-structure on $X(p)$.

**Proof.** Since $[Y, X(p)]$ is an algebraic loop for any CW-complex $Y$ by [5, Theorem 1.1], we can define an obstruction $\phi \in [X(p) \times X(p) \times X(p), X(p)]$ for the multiplication to be homotopy associative by

$$(p_1 + p_2 + p_3) = (p_1 + (p_2 + p_3)) + \phi,$$

where $p_i: X(p) \times X(p) \times X(p) \to X(p)$ ($i = 1, 2, 3$) is the $i$-th projection. We put $L = (X(p) \times X(p) \times \{1\}) \cup (X(p) \times \{1\} \times X(p)) \cup (\{1\} \times X(p) \times X(p))$. Using the Puppe exact sequence associated with the cofibering $L \to X(p) \times X(p) \times X(p) \to X(p) \times X(p) \times X(p)$, we see that there exists an element $\phi'$ such that

$$(p_1 + p_2 + p_3) = (p_1 + (p_2 + p_3)) + \phi,$$

where $d: X(p) \to X(p) \times X(p)$ is the diagonal map. Since, in the above equalities, $1 \circ q(p) \circ q(p) = q(p) \circ q(p)$, the constant map for dimensional reasons, we have the equality of (i).

The proof of (ii) is similar to that of (i) and so we omit it.

(iii) Let $\omega \in [X(p) \times X(p), X(p)]$ be an obstruction for the multiplication to be homotopy commutative defined by

$$(p_1 + p_3) = (p_2 + p_3) + \omega,$$
where \( P_i: X(p) \times X(p) \to X(p) \) \((i=1, 2)\) is the \( i \)-th projection. Using the Puppe exact sequence associated with the cofibering

\[
 X(p) \vee X(p) \to X(p) \times X(p) \to X(p) \wedge X(p),
\]

we see that there exists an element \( \omega'' \) such that

\[
(3.5) \quad \omega = \omega'' \circ \pi'', \quad \omega'' \in [X(p) \wedge X(p), X(p)].
\]

Therefore, by (3.4) and (3.5),

\[
i(p) \circ \xi(p) \circ q(p) + \gamma(p) \circ \pi(p) = (p_1 + p_2) \circ (i(p) \circ \xi(p) \circ q(p) \times \gamma(p) \circ \pi(p)) \circ d
\]

\[
= ((p_2 + p_1) + \omega) \circ (i(p) \circ \xi(p) \circ q(p) \times \gamma(p) \circ \pi(p)) \circ d
\]

\[
= (\gamma(p) \circ \pi(p) + i(p) \circ \xi(p) \circ q(p)) + \omega'' \circ ((i(p) \circ \xi(p)) \wedge \gamma(p)) \circ (q(p) \wedge \pi(p)) \circ \pi'' \circ d,
\]

where \( d: X(p) \to X(p) \times X(p) \) is the diagonal map. Since, in the above equalities, \((q(p) \wedge \pi(p)) \circ \pi'' \circ d: X(p) \to S^n(p) / S^{n+n} \) is homotopic to the constant map for dimensional reasons, we have the equality of (iii). q.e.d.

**Lemma 3.6.** Let \( \alpha \in \pi_n(S^n; p) \) and \( \lambda_i(\alpha) \) be an element of \( \delta(K) \) given in (1.1), and let \( j: K \subset X \) be the inclusion and \( \pi: X \to X/K = S^{m+n} \) be the collapsing map. Then we have

(i) \( (1+i(p) \circ \alpha(p) \circ q(p)) \circ j(p) = j(p) \circ \lambda_i(\alpha(p)) \),

(ii) \( \pi(p) \circ (1+i(p) \circ \alpha(p) \circ q(p)) = \pi(p) \).

**Proof.** (i) Let \( m: X(p) \times X(p) \to X(p) \) be the multiplication on \( X(p) \).

Then we have the following homotopy commutative diagram:

\[
\begin{array}{c}
X(p) \xrightarrow{d} X(p) \times X(p) \\
\downarrow j(p) \downarrow f(p) \times f(p) \\
K(p) \xrightarrow{d} K(p) \times K(p) \\
\downarrow \ell_1(p) \downarrow K(p) \vee S^n(p) \\
K(p) \xrightarrow{\ell_1(p)} K(p) \vee K(p) \vee S^n(p) \xrightarrow{\gamma} K(p)
\end{array}
\]

where \( \pi: K \to K/S^n = S^n \) is the collapsing map. Therefore we have the equality of (i), since \( \lambda_i(\alpha)(p) = \gamma \circ (1 \vee \ell_1(p) \circ \alpha(p)) \circ \ell_1(p) \).

(ii) Consider the following diagram:

\[
\begin{array}{c}
K(p) \xrightarrow{j(p)} X(p) \xrightarrow{\pi(p)} S^{n+n}(p) \\
\downarrow \lambda_2(\alpha)(p) \downarrow h \downarrow \varepsilon \\
K(p) \xrightarrow{j(p)} X(p) \xrightarrow{\pi(p)} S^{n+n}(p)
\end{array}
\]
where $h=1+i(p)\circ\alpha(p)\circ q(p)$. The left square of this diagram is homotopy commutative by (i) and so is the right square for some map $\varepsilon:S^{m+n}_{(p)}\to S^{m+n}_{(p)}$. On the other hand, since $\alpha$ induces the trivial homomorphism in reduced cohomology group, we have

$$(1+i(p)\circ\alpha(p)\circ q(p))^*=1:H^*(X(p);\mathbb{Z}(p))\to H^*(X(p);\mathbb{Z}(p)),$$

where $\mathbb{Z}(p)$ denotes the ring of integers localized at $\{p\}$. Hence the above map $\varepsilon$ is homotopic to the identity. q.e.d.

Lemma 3.7. Let $\alpha\in\pi_n(S^m; p)$ and $\gamma\in\pi_{m+n}(X; p)$, and let $\pi:X\to X/K=S^{m+n}$ be the collapsing map. If $S^m_{(p)}$ and $X_{(p)}$ are $H$-spaces such that $i(p):S^m_{(p)}\to X_{(p)}$ is an $H$-map, then we have

$$q(p)\circ((1+\gamma(p)\circ\pi(p))+i(p)\circ\alpha(p)\circ q(p))=q(p)+q(p)\circ\gamma(p)\circ\pi(p).$$

Proof. Using the Puppe exact sequence associated with the cofiber $X_{(p)}/S^m_{(p)}\to X_{(p)}\times S^m_{(p)}\to X_{(p)}\wedge S^m_{(p)}$, we see that there exists $\omega_1\in[X_{(p)}\wedge S^m_{(p)}, S^m_{(p)}]$ such that $q(p)\circ(p_1+p_2)\circ(1\times i(p))=q(p)\circ p_1\circ(1\times i(p))+\omega_1\circ\pi_1$, where $+$ denotes the multiplication induced from the $H$-structure on $X_{(p)}$ or $S^m_{(p)}$ and $p_i\ (i=1, 2)$ is the projection. Furthermore, since $i(p):S^m_{(p)}\to X_{(p)}$ is an $H$-map by the assumption, we see easily that there exists $\omega_2\in[(S^m_{(p)}\cup e^{m+n}_{(p)})\wedge S^m_{(p)}, S^m_{(p)}]$ such that $\omega_1=\omega_2\circ\pi_2$, where $\pi_2:X_{(p)}\wedge S^m_{(p)}\to(X_{(p)}/S^m_{(p)})\wedge S^m_{(p)}=(S^m_{(p)}\cup e^{m+n}_{(p)})\wedge S^m_{(p)}$ is the collapsing map. Consequently we have

$$q(p)\circ(p_1+p_2)\circ(1\times i(p))=q(p)\circ p_1\circ(1\times i(p))+\omega_2\circ\pi_2\circ\pi_1.$$  

Using this equality, by the similar way to the proof of Lemma 3.1, we have

$$q(p)\circ((1+\gamma(p)\circ\pi(p))+i(p)\circ\alpha(p)\circ q(p))=q(p)\circ(1+\gamma(p)\circ\pi(p)).$$

Furthermore, since $1+\gamma(p)\circ\pi(p)=\lambda_2(\gamma)(p)$, where $\lambda_2$ was defined in (2.5),

$$q(p)\circ(1+\gamma(p)\circ\pi(p))=q(p)\circ(\mathcal{V}(1+\gamma)\circ\delta_2)(p)=(\mathcal{V}(q\vee q)\circ(1+\gamma)\delta_2)(p)$$

$$=(p_1+p_2)\circ(q(p)\times q(p)\circ\gamma(p)\circ\pi(p))\circ d$$

$$=q(p)+q(p)\circ\gamma(p)\circ\pi(p).$$

Hence we have the equality of this lemma. q.e.d.

§ 4. Main result

In this section we study the $p$-Sylow subgroup of $\mathcal{B}(X)$ for an $S^m$-bundle $X$ over $S^n$ by using an $H$-structure on $X_{(p)}$ and the method of
localization.

The following result due to Lieberman-Smallen will be used to obtain our main result.

(4.1) (Lieberman-Smallen [7, Theorem 1.3]) Let \( P \) and \( \bar{P} \) denote sets of primes such that \( P \cup \bar{P} = \{ \text{all primes} \} \) and \( P \cap \bar{P} = \emptyset \). Let \( Y \) be a simple finite CW-complex. Then localization induces the isomorphism

\[
L : \mathcal{E}(Y) \cong \{(h, h') \in \mathcal{E}(Y_P) \times \mathcal{E}(Y_{\bar{P}}) \mid h_{(p)} = h'_{(p)}\},
\]

where \( h_{(p)} \) and \( h'_{(p)} \) are localizations of \( h \) and \( h' \) at the rational numbers \( \mathbb{Q} \) respectively.

Let \( m \) and \( n \) be odd integers such that \( 3 \leq m < n - 1 \) and \( X \) denote an \( S^m \)-bundle over \( S^n \) in (2.3). Let \( p \) be an odd prime. If \( S^m_{(p)} \) and \( X_{(p)} \) are \( H \)-spaces such that \( i_{(p)} : S^m_{(p)} \rightarrow X_{(p)} \) is an \( H \)-map, then we have

**Proposition 4.2.** The \( p \)-Sylow subgroup \( S_p \cong \pi_n(S^m; p) \) of \( G \) given in (ii) of Lemma 2.7 splits partly the exact sequence (2.6), that is, there exists a homomorphism

\[
\psi : S_p \cong \pi_n(S^m; p) \rightarrow \mathcal{E}(X)
\]

such that \( j^1 \circ \psi = 1 \).

**Proof.** In Proposition 1.3 we identify \( \pi_n(S^m; p) \) with \( S_p \) by means of the homomorphism \( \lambda_1 \). We put \( h = 1 + i_{(p)} \circ \alpha_{(p)} \circ q_{(p)} \) for \( \alpha \in \pi_n(S^m; p) \), where \( + \) is the multiplication induced from the \( H \)-structure on \( X_{(p)} \). Then \( h \in \mathcal{E}(X_{(p)}) \) by Lemma 3.6, and \( h_{(0)} = 1 \) since \( \alpha \) is of finite order. Therefore, using the isomorphism \( L \) in (4.1), we can define a map

\[
(4.3) \quad \psi : S_p \cong \pi_n(S^m; p) \rightarrow \mathcal{E}(X) \quad \text{by} \quad \psi(\lambda_1(\alpha)) = L^{-1}(1 + i_{(p)} \circ \alpha_{(p)} \circ q_{(p)}, 1),
\]

where \( \alpha \in \pi_n(S^m; p) \) and \( \lambda_1(\alpha) \in S_p \). First we show that \( \psi \) is an homomorphism. For \( \alpha \in \pi_n(S^m; p) \) and \( \beta \in \pi_n(S^m; p) \),

\[
(1 + i_{(p)} \circ \alpha_{(p)} \circ q_{(p)}) \circ (1 + i_{(p)} \circ \beta_{(p)} \circ q_{(p)})
\]

\[
= (1 + i_{(p)} \circ \beta_{(p)} \circ q_{(p)}) + i_{(p)} \circ \alpha_{(p)} \circ q_{(p)} \circ (1 + i_{(p)} \circ \beta_{(p)} \circ q_{(p)})
\]

\[
= (1 + i_{(p)} \circ \beta_{(p)} \circ q_{(p)}) + i_{(p)} \circ \alpha_{(p)} \circ q_{(p)} \quad \text{by Lemma 3.7}
\]

\[
= 1 + (i_{(p)} \circ \beta_{(p)} + i_{(p)} \circ \alpha_{(p)}) \circ q_{(p)} \quad \text{by (i) of Lemma 3.1}
\]

\[
= 1 + i_{(p)} \circ (\alpha + \beta)_{(p)} \circ q_{(p)}.
\]

Hence \( \psi(\lambda_1(\alpha)) \circ \psi(\lambda_1(\beta)) = \psi(\lambda_1(\alpha + \beta)) \). Next we show \( j^1 \circ \psi = 1 \). The naturality of localization gives the following commutative diagram:
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\[ \delta'(X) \xrightarrow{L} \{(h, h') \in \delta'(X(p)) \times \delta'(X(p)) | h(h_0) = h'(h_0)\} \]
\[ \delta'(K) \xrightarrow{L} \{(h, h') \in \delta'(K(p)) \times \delta'(K(p)) | h(h_0) = h'(h_0)\} , \]

where both \( L \)'s are isomorphisms in (4.1), and \( j^1(p) \) and \( j^1(p) \) are defined by the same way as \( j^1 \) in (2.2). For \( \alpha \in \pi_n(S^m; p) \) we have

\[ j^1 \circ (\psi(\lambda_\alpha)) = L^{-1} \circ (j^1(p) \times j^1(p)) \circ L(\psi(\lambda_\alpha)) \]
\[ = L^{-1} \circ (j^1(p) \times j^1(p)) \circ (1 + i(p) \circ \alpha(p) \circ q(p), 1) \]
\[ = L^{-1}(\lambda_\alpha(p), 1) \]
\[ = \lambda_\alpha(p), \]

because \( \lambda_\alpha(p) = 1 \). Hence \( j^1 \circ \psi = 1 \).

q.e.d.

Now we consider the \( p \)-Sylow subgroup of \( \delta'(X) \).

**Theorem 4.5.** Let \( m \) and \( n \) be odd integers such that \( 3 \leq m < n - 1 \), and let \( X \) denote an \( S^m \)-bundle over \( S^n \) in (2.3). Let \( p \) be an odd prime. If \( S^m(p) \) and \( X(p) \) are H-spaces such that \( i(p): S^m(p) \rightarrow X(p) \) is an H-map, then the group \( \delta'(X) \) is a finite group with a unique \( p \)-Sylow subgroup \( S^p \) given by the semi direct product

\[ S^p \cong \pi_m + n(X; p) \times \pi_n(S^m; p), \]

where \( \alpha T \beta = \alpha + i \circ \beta \circ q \circ \alpha \) for \( \alpha \in \pi_m + n(X; p) \) and \( \beta \in \pi_n(S^m; p) \).

**Proof.** By (2.6), Lemma 2.7 and Proposition 4.2, we have the exact sequence

\[ 0 \rightarrow \lambda_\alpha(\pi_{m+n}(X)) \times \psi(S_p) \rightarrow \delta'(X) \rightarrow G/S_p \rightarrow 1, \]

where \( S_p \cong \pi_n(S^m; p) \), \( G \) is given in (2.6) and \( G/S_p \) has the order prime to \( p \). Let \( p' \) be a prime with \( (p, p') = 1 \). Then, using the isomorphism \( L \) in (4.1), for \( \gamma \in \pi_{m+n}(X; p') \) and \( \beta \in \pi_n(S^m; p) \) we have

\[ \lambda_\alpha(\gamma) T \psi(\lambda_\beta) = \psi(\lambda_\alpha(\beta)) \circ \lambda_\gamma \circ \psi(\lambda_\beta) = \lambda_\gamma(\alpha). \]

Noticing that \( \lambda_\alpha(p) = 1 + \alpha(p) \circ \pi(p) \) for \( \alpha \in \pi_{m+n}(X; p) \), by Lemmas 3.6 and 3.7 and the similar way to the proof of Lemma 3.1, we have

\[ (\psi(\lambda_\beta) \circ \lambda_\alpha(p) \circ \psi(\lambda_\beta) - 1(p) = ((1 + i(p) \circ (-\beta(p) \circ q(p)) + \alpha(p) \circ \pi(p)) \]
\[ + (i(p) \circ \beta(p) \circ q(p) + i(p) \circ \beta(p) \circ q(p) \circ \alpha(p) \circ \pi(p)) = \lambda_\alpha(p). \]
Also, obviously we have
\[(\psi(\lambda(\beta)) \circ \lambda_2(\alpha) \circ \psi(\lambda(\beta))^{-1})_{(p)} = 1 = (\lambda_2(\alpha + i \circ \beta \circ q \circ \alpha))_{(p)}.
\]
Hence, by (4.1) we have
\[(4.8) \lambda_3(\alpha) T \psi(\lambda(\beta)) = \psi(\lambda(\beta)) \circ \lambda_3(\alpha) \circ \psi(\lambda(\beta))^{-1} = \lambda_3(\alpha + i \circ \beta \circ q \circ \alpha)
\]
for $\alpha \in \pi_{m+n}(X; p)$ and $\beta \in \pi_n(S^m; p)$. Next we show that $\lambda_3$ is monomorphic on $\pi_{m+n}(X; p)$. Let $L$ be the isomorphism in (4.1). Then $L\lambda_3(\gamma) = (1 + r_{(p)} \circ \pi(p), 1)$ for $\gamma \in \pi_{m+n}(X; p)$. Therefore, we have
\[
\text{Ker } \lambda_3 \cap \pi_{m+n}(X; p) \cong \text{Ker } \{\pi_n^{(p)}: [S^{m+n}, X_{(p)}] \rightarrow [X_{(p)}, X_{(p)}]\} \\
= \text{Im } \{(Sg)^{\circ (p)}: [S\pi(p), X_{(p)}] \rightarrow [S^{m+n}, X_{(p)}]\} \\
= 0,
\]
where $g$ is the attaching element of $e^{m+n}$ in $X = K \cup e^{m+n}$, because the middle equality is obtained by the Puppe exact sequence and the next one is obtained by the fact due to C. A. McGibbon [8, Theorem 1] that $Sg$ has order at most 2. Hence, by (4.6), (4.7) and (4.8) we have the desired result.

Let $P$ be a set of odd primes $p$ such that $S_{(p)}^n$ and $X_{(p)}$ are $H$-spaces and $i^{(p)}: S_{(p)}^n \rightarrow X_{(p)}$ is an $H$-map, and we put $S_p = \sum_{p \in P} S_p$ for the $p$-Sylow subgroup $S_p$ given in Proposition 1.3. Then $S_p$ is the normal subgroup of $G$ by Lemma 2.7 and (1.2) (see also e.g. [11, Theorem 2.11]), and the splitting homomorphism $\psi$ in Proposition 4.2 can be extended to $S_p$. So by the same way as in (4.6) we have the exact sequence
\[
0 \rightarrow \lambda_3(\pi_{m+n}(X)) \times T \psi(S_p) \rightarrow \mathcal{E}(X) \rightarrow G/S_p \rightarrow 1.
\]
By Theorem 4.5 and (4.7), we have immediately
\[
\lambda_3(\pi_{m+n}(X)) \times T \psi(S_r) \cong (\sum_{p \in P} \tilde{S}_p) \oplus \sum_{p \in P} \lambda_3(\pi_{m+n}(X; r)),
\]
where $\tilde{S}_p$ is given in Theorem 4.5. By (1.2) (see also e.g. [11, Theorem 2.11]), we have the exact sequence
\[
0 \rightarrow H \rightarrow G/S_p \rightarrow \mathbb{Z}_4 \oplus \mathbb{Z}_4.
\]
Here
\[
H = \sum_{p \in P} \pi_n(S^m; p) \oplus H_1 / (\langle f \circ \pi_n(S^{n-1}) + \gamma(f) \pi_{m+1}(S^m) \rangle \cap H_1,
\]
where \( P_i = \overline{P \cup \{2\}} \), \( H_1 = \{ \alpha \in \pi_n(S^m; 2) \mid \iota_m[\alpha, \iota_m] = 0 \} \) and \( \gamma(f) \) is given in Section 1. Hence we have

**Theorem 4.9.** Let \( m \) and \( n \) be odd integers such that \( 3 \leq m < n - 1 \) and \( X \) denote an \( S^m \)-bundle over \( S^n \) in (2.3). Let \( P \) be a set of odd primes \( p \) such that \( S^m_p \) and \( X_p \) are \( H \)-spaces and \( i_p : S^m_p \to X_p \) is an \( H \)-map. Then we have the following exact sequences:

\[
0 \to (\sum_{p \in P} \mathcal{S}_p) \oplus \sum_{p \in P} \lambda_p(\pi_{m+n}(X; r)) \to \varepsilon(X) \to G/S_p \to 1,
\]

\[
0 \to H \to G/S_p \to \mathbb{Z}_2 \oplus \mathbb{Z}_2.
\]

Here \( \mathcal{S}_p \) and \( G \) are given in Theorem 4.5 and (2.6) respectively, \( S_p \equiv \sum_{p \in P} \pi_n(S^m; p) \) and \( H \) is given as above.

§ 5. Two examples

In this section, we give the following two examples in which the group \( \varepsilon(X) \) is determined as a group extension of a certain group by a 2-group.

**Example 5.1.** Let \( m \) and \( n \) be odd integers such that \( 3 \leq m < n - 1 \) and let \( X = S^m \times S^n \). Then, for any odd prime \( p \), \( \varepsilon(X) \) is a finite group with a unique \( p \)-Sylow subgroup \( \mathcal{S}_p \) given by the semi direct product

\[
\mathcal{S}_p \cong (\pi_{m+n}(S^m; p) \oplus \pi_{m+n}(S^n; p)) \times \pi_n(S^m; p),
\]

where \( (\alpha', \alpha'')T_1 \beta = (\alpha' + \beta \circ \alpha'' \circ \alpha') \) for \( (\alpha', \alpha'') \in \pi_{m+n}(S^m; p) \oplus \pi_{m+n}(S^n; p) \) and \( \beta \in \pi_n(S^m; p) \). Furthermore, let \( P \) be the set of all odd primes. Then we have the following exact sequence:

\[
0 \to (\sum_{p \in P} \mathcal{S}_p) \oplus \tilde{H} \to \varepsilon(X) \to \tilde{G} \to 1.
\]

Here \( \tilde{H} = \pi_{m+n}(S^m; 2)/[[\iota_m, \pi_{n+1}(S^m)]] \oplus \pi_{m+n}(S^n; 2) \) and \( \tilde{G} \equiv \{ \alpha \in \pi_n(S^m; 2) \mid [\iota_m, \alpha] = 0 \} \times (\mathbb{Z}_2 \oplus \mathbb{Z}_2) \),

where \( \alpha T_3(\varepsilon', \varepsilon'') = \varepsilon' \circ \alpha \circ \varepsilon'' \) for \( (\varepsilon', \varepsilon'') \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{ \pm \iota_m \} \oplus \{ \pm \iota_n \} \).

In fact, since \( S^k \) \((k = m, n)\) is an \( H \)-space for any odd prime \( p \) by J. F. Adams [1] and \( i_p : S^m \to S^m_p \times S^n \) is an \( H \)-map, the first half of this example is obtained from Theorem 4.5. Also, by [14, Theorem 2.6] we have the exact sequence

\[
0 \to [[\iota_m, \pi_{n+1}(S^m)]] \to \pi_{m+n}(S^m \times S^n) \overset{\lambda_g}{\to} \varepsilon(S^m \times S^n) \overset{j_1}{\to} G \to 1,
\]
where $G \cong \{ \alpha \in \pi_n(S^m) | [\iota_m, \alpha] = 0 \} \times \tau_a(Z_2 \oplus Z_2)$. Therefore, by noticing that $[\iota_m, \pi_{n+1}(S^m; p)] = 0$, the latter half of this example is obtained from Theorem 4.9 and the above exact sequence.

Let $r$ be an odd prime and let $\alpha_i$ be a generator of $\pi_{2r}(S^3; r) \cong \mathbb{Z}_r$. Let

$$S^3 \xrightarrow{i} B(r) \xrightarrow{q} S^{2r+1}$$

be the principal $S^3$-bundle over $S^{2r+1}$ with a characteristic element $\alpha_i$. Then we have the following

**Example 5.2.** For any odd prime $p$, $\mathscr{S}(B(r))$ is a finite group with a unique $p$-Sylow subgroup $\tilde{S}_p$ given by the semi direct product

$$\tilde{S}_p \cong \pi_{2r+4}(S^3; p) \oplus \pi_{2r+4}(S^3; p) \quad (p \neq 3 \text{ or } r = 3),$$

$$\tilde{S}_3 \cong (\pi_{2r+4}(S^3; 3) \oplus \pi_{2r+4}(S^{2r+1}; 3)) \times \pi_{2r+4}(S^3; 3) \quad (r \neq 3),$$

where $\pi_{2r+4}(S^{2r+1}; 3) \cong \mathbb{Z}_3$, $(\alpha', \alpha'')T_1 \beta = (\alpha' + \beta \circ \alpha', \alpha'')$ for $(\alpha', \alpha'') \in \pi_{2r+4}(S^3; 3) \oplus \pi_{2r+4}(S^{2r+1}; 3)$ and $\beta \in \pi_{2r+1}(S^3; 3)$. Furthermore, we have the following exact sequences:

$$0 \rightarrow (\sum_{p \in P} \tilde{S}_p) \oplus \pi_{2r+4}(S^3; 2) \oplus \pi_{2r+4}(S^{2r+1}; 2) \rightarrow \mathscr{S}(B(r)) \rightarrow \tilde{G} \rightarrow 1,$$

$$0 \rightarrow \pi_{2r+4}(S^3; 2) \rightarrow \tilde{G} \rightarrow \mathbb{Z}_2 \rightarrow 1,$$

where $P$ is the set of all odd primes and $\pi_{2r+4}(S^{2r+1}; 2) \cong \mathbb{Z}_8$.

In fact, since $(S^3 \times \cdots \times S^{2r-1} \times B(r))_r \cong SU(r+1)_r$ by H. Toda [15] and $B(r)_p \cong (S^3 \times S^{2r+1})_p$ for any prime $p \neq r$, $B(r)_p$ is an $H$-space for any odd prime $p$ and $i_p : S^3_p \rightarrow B(r)_p$ is an $H$-map for dimensional reasons. Therefore we can apply Theorem 4.5 to $B(r)$ for any odd prime $p$, and by the same way as in [9, Example 3.3] the homotopy group $\pi_{2r+4}(B(r))$ is calculated and we have the first half of this example. Also, by [9, Theorem 3.1] we have the exact sequence

$$0 \rightarrow \pi_{2r+4}(B(r)) \xrightarrow{\lambda_2} \mathscr{S}(B(r)) \xrightarrow{j_i} G \rightarrow 1,$$

where $G = \mathscr{S}(S^3 \cup_{a_i} e^{2r+1})$ is given in (1.2). Therefore the latter half of this example is obtained from Theorem 4.9 and the above exact sequence.

**References**

Groups of Self Homotopy Equivalences


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