§ 1. Introduction

In [3], Gromov introduced the notion of the bounded cohomology $H^b_\partial(M, \mathbb{R})$ of a manifold $M$. This is the cohomology of the complex of singular cochains $\phi$ which have the property:

There exists a constant $c$ such that $|\phi(\sigma)| < c$ for any singular simplex $\sigma$.

Let $S$ be a closed oriented surface of genus $\geq 2$. In [1] and [5], it is shown that $H^3_b(S, \mathbb{R})$ is infinitely generated.

In this paper, we shall show

**Theorem 1.** $H^3_b(S, \mathbb{R})$ is infinitely generated.

Our method is an application of Thurston's theory of pleated (un-crumpled) surfaces in hyperbolic 3-manifolds ([7]).

§ 2. A construction of elements of $H^3_b(S, \mathbb{R})$

For a convenience, we choose and fix a complete hyperbolic structure on $S$.

Let $f$ be a pseudo Anosov diffeomorphism of $S$. Let $M_f$ be the mapping torus of $f$. It is the identification space obtained from $S \times [0, 1]$ by equivalence relation $(x, 0) \sim (f(x), 1)$ ($x \in S$). $M_f$ admits a complete hyperbolic structure which is unique up to isometry ([6]). The projection onto the second factor $S \times [0, 1] \to [0, 1]$ induces a fibering $p: M_f \to S^1$. Let $\tilde{M}_f$ be the infinite cyclic regular covering space of $M_f$, defined by the pull-back by $p$ of $e: \mathbb{R} \to S^1$, where $e(t) = \exp 2\pi \sqrt{-1} t$, $t \in \mathbb{R}$. The hyperbolic structure on $M_f$ can be lifted to the hyperbolic structure on $\tilde{M}_f$. There is a natural inclusion $S \times [0, 1] \subset \tilde{M}_f$, and let $j: S \to \tilde{M}_f$ be the embedding defined by $j(x) = (x, 0) \in S \times [0, 1] \subset \tilde{M}_f$.

Let $\Delta$ be the standard 3-simplex in $\mathbb{R}^4$. Let $\sigma: \Delta \to S$ be a singular 3-simplex of $S$. Then $j\sigma: \Delta \to \tilde{M}_f$ is a singular 3-simplex of $\tilde{M}_f$. The universal covering space of $\tilde{M}_f$ is isometric to the hyperbolic 3-space $H^3$.
and there is a covering projection \( q: H^3 \to \tilde{M}_f \). There is a map \( \tilde{\sigma}: \mathcal{A} \to H^3 \) such that \( q \tilde{\sigma} = j\sigma \). Let straight \((j\sigma)\) be the geodesic 3-simplex in \( H^3 \) with the same vertices as \( \tilde{\sigma} \). The isometry class of straight \((j\sigma)\) depends only on \( j\sigma \). We define a singular 3-cochain \( \phi_f \) of \( S \) by

\[
\phi_f(\sigma) = \varepsilon \ \text{vol}(\text{straight}(j\sigma)),
\]

for each 3-simplex \( \sigma \), where \( \text{vol} \) denotes the hyperbolic volume and \( \varepsilon = 1 \) if \( \tilde{\sigma} \) maps \( \mathcal{A} \) into \( H^3 \) orientation preservingly and \( \varepsilon = -1 \) otherwise. Since the volume of geodesic 3-simplices in \( H^3 \) has a finite upper bound ([7]), \( \phi_f \) defines a bounded 3-cocycle of \( S \).

§ 3. Linear independence of \( \phi_f \)

Let \( \mathcal{A} \) be the space of all the geodesic laminations on \( S \) with geometric topology ([7] § 8). \( \mathcal{A} \) is compact. Any homeomorphism of \( S \) induces a homeomorphism of \( \mathcal{A} \). For a pseudo Anosov diffeomorphism \( f \) of \( S \), there are two mutually transverse geodesic laminations \( \lambda^s_f \) and \( \lambda^u_f \) such that they are invariant by \( f \), and for each simple closed geodesic \( \gamma \) on \( S \), \( f^k(\gamma) \to \lambda^s_f \) and \( f^{-k}(\gamma) \to \lambda^u_f \) as \( k \to +\infty \) ([2] [7]). \( \lambda^s_f \) and \( \lambda^u_f \) are called as the stable and the unstable geodesic lamination of \( f \) respectively.

Let \( T \) be a (not simplicial) triangulation of \( S \) such that it contains a simple closed geodesic \( \gamma \) and it has only one vertex lying on \( \gamma \). Let \( \tau_\gamma \) be the Dehn twist along \( \gamma \). Let \( T_n = \tau_\gamma^n T \) be the triangulation of \( S \) which is the image of \( T \) by \( \tau_\gamma^n \) for each non-negative integer \( n \) (\( T_0 = T \)). Let \( T_\infty \) be the ideal triangulation of \( S \) which is the limit of \( T_n \) as \( n \to \infty \).

Let \( c, c_n = \tau_\gamma^n c \) and \( c_\infty = \lim c_n \) be the singular 2-chains of \( S \) associated to \( T, T_n \) and \( T_\infty \) respectively which represent the fundamental class of \( S \).

Since \( f^*_uc_n \) is homologous to \( c_n \), there is a singular 3-chain \( d_n \) such that \( \partial d_n = f^*_uc_n - c_n \). We define a sequence of singular 3-chains of \( S \) by

\[
D_n(f)_k = \sum_{i=k}^{k} f^*_x d_n,
\]

for \( k = 1, 2, \ldots \) and \( n = 0, 1, \ldots, \infty \). Then \( \partial D_n(f)_k = f^{k+1}_x c_n - f^{-k}_x c_n \).

**Proposition 1.** Let \( f \) and \( g \) be two pseudo Anosov diffeomorphisms of \( S \). Let \( \lambda^s_f, \lambda^u_f, \lambda^s_g \) and \( \lambda^u_g \) be the stable and the unstable geodesic laminations of \( f \) and \( g \) respectively. If none of \( \lambda^s_f \) and \( \lambda^s_g \) coincides with any of \( \lambda^u_f \) and \( \lambda^u_g \), then \( \phi_f \) and \( \phi_g \) are linearly independent in \( H^3_b(S, \mathbb{R}) \).

**Proof.** Let \( j_f: S \to \tilde{M}_f \) and \( j_g: S \to \tilde{M}_g \) be the embeddings given in Section 2. For each \( n \) and \( k \), the image of the 3-chain \( j_f(D_n(f)_k) \) under the projection \( \tilde{M}_f \to M_f \) gives a singular 3-chain of \( M_f \) representing \((2k+1)\)-times the fundamental class of \( M_f \). Hence we have \( \phi_f(D_n(f)_k) \)
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\[ \lim_{k \to \infty} \frac{1}{2k+1} \phi_f(D_\infty(f)_k) = 0. \]

Next we consider \( \phi_f(D_\infty(g)_k) \). Projecting the chain of the ideal geodesic simplices, straight \((j_\delta(D_\infty(g)_k))\), from \( H^3 \) to \( \tilde{M}_f \), we may consider straight \((j_\delta(D_\infty(g)_k))\) as an ideal singular 3-chain of \( \tilde{M}_f \). The boundary, \( \partial \) straight \((j_\delta(D_\infty(g)_k))\), consists of two pleated surfaces \( S_k \) and \( S_{-k} \) which are the straightenings of the ideal triangulations \( g^{k+1}T_\infty \) and \( g^{-k}T_\infty \) of \( S \) in \( \tilde{M}_f \) respectively. The bending locus \( b(S_k) \) (resp. \( b(S_{-k}) \)) of \( S_k \) (resp. \( S_{-k} \)) is the geodesic lamination which is the straightening of the ideal 1-simplices of \( g^{k+1}T_\infty \) (resp. \( g^{-k}T_\infty \)). Since \( T_\infty \) contains a simple closed geodesic \( \gamma \), \( b(S_k) \) (resp. \( b(S_{-k}) \)) converges in \( \Lambda \) to a geodesic lamination \( \lambda_+ \) (resp. \( \lambda_- \)) which contains \( \lambda_\gamma^\pm \) (resp. \( \lambda_\gamma^\mp \)) as \( k \to \infty \). By assumption, none of \( \lambda_+ \) and \( \lambda_- \) contains any of \( \lambda_\gamma^+ \) and \( \lambda_\gamma^- \). By Thurston’s realization theorem of geodesic laminations in \( \tilde{M}_f \) ([7] 9.3.10), there exist two pleated surfaces \( S_+ \) and \( S_- \) in \( \tilde{M}_f \) whose bending laminations are \( \lambda_+ \) and \( \lambda_- \) respectively. Since \( T_\infty \) is an ideal triangulation of \( S \), both of \( S - \lambda_+ \) and \( S - \lambda_- \) consist of finite ideal triangles. Hence \( S_+ \) and \( S_- \) are uniquely determined, and the pleated surfaces \( S_k \) and \( S_{-k} \) converge to \( S_+ \) and \( S_- \) respectively as \( k \to \infty \) ([7] 9.5.6, 7). Therefore \( \phi_f(D_\infty(g)_k) \) converges to the volume of the compact region bounded by \( S_+ \) and \( S_- \) in \( \tilde{M}_f \) as \( k \to \infty \), and it is bounded. Hence,

\[ \lim_{k \to \infty} \frac{1}{2k+1} \phi_f(D_\infty(g)_k) = 0. \]

Exchanging \( f \) and \( g \), we have

\[ \lim_{k \to \infty} \frac{1}{2k+1} \phi_g(D_\infty(f)_k) = 0 \quad \text{and} \quad \lim_{k \to \infty} \frac{1}{2k+1} \phi_g(D_\infty(g)_k) = \text{vol}(M_g). \]

Now suppose that \( a\phi_f + b\phi_g = 0 \) in \( H^3(S, R) \) for some \( a, b \in R \) and \( ab \neq 0 \). Then \( a\phi_f + b\phi_g = \partial \omega \) for some bounded 2-cochain \( \omega \) of \( S \). For each \( 0 \leq n < +\infty \),

\[ (a\phi_f + b\phi_g)(D_n(f)_k) = (\partial \omega)(D_n(f)_k) = \omega(f^{k+1}_*c_n) - \omega(f^{-k}_*c_n). \]

As \( \omega \) is bounded and both of \( f^{k+1}_*c_n \) and \( f^{-k}_*c_n \) are sums of a constant number of simplices for each \( k \), it follows

\[ \lim_{k \to \infty} \frac{1}{2k+1} (a\phi_f + b\phi_g)(D_n(f)_k) = 0. \]
Since $\phi_f$ and $\phi_g$ are continuous cochains, we have

$$\lim_{k \to \infty} \frac{1}{2k+1} (a\phi_f + b\phi_g)(D^k(f)) = 0.$$  

Replacing $D^k(f)$ by $D^k(g)$, the same equality holds. However this contradicts to the above facts. q.e.d.

The above proposition can be generalized in straightforward way as follows,

**Proposition 2.** Let $f_1, \ldots, f_m$ be pseudo Anosov diffeomorphisms of $S$. If the stable and the unstable geodesic laminations of $f_1, \ldots, f_m$ are all distinct from each other, then $\phi_{f_1}, \ldots, \phi_{f_m}$ are linearly independent in $H^1(S, \mathbb{R})$.

Now let $\alpha$ and $\beta$ be two simple closed curves on $S$ such that $S-(\alpha \cup \beta)$ is a disjoint union of open 2-discs. Then $f_m^m = \tau^m_{\alpha} \beta^{-m}$ is a pseudo Anosov diffeomorphism of $S$ for each positive integer $m$ ([8]). In [4], Masur calculates the stable and unstable geodesic laminations $\lambda^s_m$ and $\lambda^u_m$ of $f_m$ (in terms of measured foliations and quadratic differentials), and it is shown that $\lambda^s_m \to \alpha$ and $\lambda^u_m \to \beta$ as $m \to \infty$. Hence we may choose an infinite family $\{f_m\}$ of pseudo Anosov diffeomorphisms such that each finite subset of $\{f_m\}$ satisfies the condition of Proposition 2. This proves Theorem 1.

References


