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Stiefel-Whitney Homology Classes and Riemann-Roch Formula

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§ 1. Introduction

In this note, we give a Riemann-Roch type theorem for certain maps between Euler spaces. These are the cases where Halperin's conjecture holds, although it is not true in general [6].

Let X be a locally compact n-dimensional polyhedron. For a point x in X, let $\chi(X, X-x)$ denote the Euler number of the pair $(X, X-x)$. The polyhedron X is called a *mod 2 Euler space* or simply an *Euler space* if for each x in X, $\chi(X, X-x) \equiv 1 \pmod 2$ (Halperin and Toledo [3]).

Let K' denote the barycentric subdivision of a triangulation K of a polyhedron X. If X is an Euler space, the sum of all k-simplexes in K' is a mod 2 cycle and defines an element $s_k(X)$ in $H_k(X; Z_2)$ (cf. [3]). The element $s_k(X)$ is called the k-th *Stiefel-Whitney homology class* of X.

In the book [2], Fulton and MacPherson defined the notion of a homologically normally nonsingular map. As an analogy to the Riemann-Roch formula for singular algebraic spaces, they introduced Halperin's conjecture ([2, p. 112]):

If $\phi\colon X \to Y$ is a homologically normally nonsingular map of Euler spaces, then

$$s_*(X) = \phi^! s_*(Y) \cap (wN_\phi)^{-1},$$

where $(wN_\phi)^{-1}$ is the inverse of the cohomology Stiefel-Whitney class of the normal space of ϕ defined by Thom's formula using the Steenrod squares.

If Y is an Euclidean space and ϕ is an embedding, then ϕ is homologically normally nonsingular if and only if X is a Z_2-homology manifold. In this case, Halperin's conjecture is equal to the equation

$$s_*(X) = [X] \cap w^*(X),$$

which is proved by Taylor [8], Veljan [9] and Matsui [4].

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But we have shown in [6] that, in general, there are many examples where the conjecture does not hold.

The main result of this paper is the following.

Theorem. *Let X and Y be Euler spaces and $\phi: X \to Y$ be an embedding such that it has a normal block bundle ν (Rourke-Sanderson [7]). Then the following Riemann-Roch theorem holds,*

$$s_*(X) = \phi^! s_*(Y) \cap w(\nu)^{-1}$$

where $w(\nu)^{-1}$ is the inverse of the Stiefel-Whitney cohomology classes of ν.

A similar result is announced in [2, p. 67]. By virtue of a result of Taylor [8], this theorem will probably hold when ϕ has a normal $\boldsymbol{Z}_2$-homology bundle.

In this paper, homologies and cohomologies are always with $\boldsymbol{Z}_2$ coefficient.

§ 2. Characterization of Stiefel-Whitney homology class

In this section, we give a characterization of the Stiefel-Whitney homology classes of an Euler space. Let X be an Euler space embedded in a Euclidean space $\boldsymbol{R}^\alpha$. Let R be a regular neighborhood of X, $\bar{R}$ its boundary, and $\phi: X \to R$ be the embedding. Let $\mathfrak{N}_*(R, \bar{R})$ denote the unoriented differentiable bordism group (cf. [1]). We define a homomorphism

$$e_\phi: \mathfrak{N}_*(R, \bar{R}) \longrightarrow \boldsymbol{Z}_2$$

as follows ([4, p. 322]).

Let $f: (M, \partial M) \to (R, \bar{R})$ be an element in $\mathfrak{N}_*(R, \bar{R})$. Then there exist a triangulation of M and a PL-embedding $g: (M, \partial M) \to (R \times D^\beta, \bar{R} \times D^\beta)$, where D^β is the disc of sufficiently large dimension such that $g \simeq f \times \{0\}$ and $(\phi \times \mathrm{id})(X \times D^\beta)$ is block transverse to g by Transversality Theorem [4]. Put $Z = (\phi \times \mathrm{id})(X \times D^\beta) \cap g(M)$. Then Z is an Euler space. We define $e_\phi(f, M)$ to be the modulo 2 Euler number $e(Z)$ of Z. This definition is independent of the choice of the representative (f, M) by Transversality Theorem.

Proposition 1 (characterization of Stiefel-Whitney homology class). *Let X be an Euler space embedded in a Euclidean space $\boldsymbol{R}^\alpha$. Let R be a regular neighborhood of X in $\boldsymbol{R}^\alpha$, and let $\phi: X \to R$ be the embedding. Then the Stiefel-Whitney homology class $s_*(X)$ is the unique homology class in $H_*(X)$ satisfying the relation*

$$\langle([R]\cap)^{-1}\phi_*(s_*(X)), f_*([M]\cap w^*(M))\rangle = e_\phi(f, M),$$

for any $(f, M) \in \mathfrak{N}_*(R, \bar{R})$.

This is a conjunction of Lemma 6 and Lemma 7 of Matsui [5]. In [4], this is proved when X is an Euler space satisfying the Poincaré duality. In Veljan [9], this is proved when X is an Euler manifold, a little narrower category.

§ 3. Characteristic classes of block bundles

Let ξ be a q-block bundle over a complex K [7]. In this paper, we write

$$\xi = (E, B, \phi),$$

where $B = |K|$, E is the total space and $\phi: B \to E$ is the inclusion. We write $\bar{E}$ for the total space of the $(q-1)$-sphere bundle of E. Let $\mathfrak{B}_*(E, \bar{E})$ denote the unoriented bordism group consisting of PL-maps from Euler space pairs to $(E, \bar{E})$ (see [4]). We will define a homomorphism

$$e_\xi: \mathfrak{B}_*(E, \bar{E}) \longrightarrow \boldsymbol{Z}_2$$

as follows ([4, p. 326]).

Let R be a regular neighborhood of the polyhedron B embedded in $\boldsymbol{R}^\alpha$, for α sufficiently large. Let $i: B \hookrightarrow R$ be the inclusion and let $p: R \to B$ be the retraction. Let $p^*\xi = (p^*E, R, \phi_R)$ be the induced bundle. For each $(g, N) \in \mathfrak{B}_*(E, \bar{E})$, we can choose an embedding $h: (N, \partial N) \to (p^*E, p^*\bar{E})$ such that $h \simeq i \circ g$. By Transversality theorem, we can assume that $h(N)$ is block transverse to $\phi_R: R \to p^*E$. We define $e_\xi(g, N)$ to be the modulo 2 Euler number $e(Z)$ of the intersection $Z = \phi_R(R) \cap h(N)$. This is independent of the choice of (g, N) by Transversality Theorem.

Let $U_\xi \in H^q(E, \bar{E})$ be the Thom class of ξ and let $T_\xi^*: H^*(B) \to H^{*+q}(E, \bar{E})$ be the Thom isomorphism defined by $T_\xi^*(x) = (\phi^*)^{-1}(x) \cup U_\xi$.

Proposition 2. *Let* $\xi = (E, B, \phi)$ *be a block bundle over a polyhedron* B. *Then the inverse Stiefel-Whitney cohomology class* $w(\xi)^{-1}$ *is the unique cohomology class in* $H^*(B)$ *satisfying the relation*

$$\langle T_\xi^*(w(\xi)^{-1}), g_*(s_*(N))\rangle = e_\xi(g, N),$$

for any $(g, N) \in \mathfrak{B}_*(E, \bar{E})$.

Proof. There exists a unique cohomology class Φ in $H^*(E, \bar{E})$ satisfying the relation $\langle\Phi, g_*(s_*(N))\rangle = e_\xi(g, N)$ for any $(g, N) \in \mathfrak{B}_*(E, \bar{E})$

([4, Lemma 3.2]). Since the natural map σ: $\mathfrak{N}_*(E, \bar{E}) \to H_*(E, \bar{E})$ defined by $\sigma(g, N) = \sum_i g_* s_i(N)$ is surjective, we can suppose that N is a triangulation of a smooth manifold. Then, as in the definition of e_ξ, we can choose an embedding h: $(N, \partial N) \to (p^*E, p^*\bar{E})$, $h \simeq i \circ g$, such that $Z = \phi_R(R) \cap h(N)$ is a PL-manifold. Since $s_*(h(N)) = [h(N)] \cap w^*(h(N))$, we have

$$\begin{aligned}
&\langle T_\xi^*(w(\xi)^{-1}), g_*(s_*(N))\rangle \\
&\quad = \langle T_{p^*\xi}^*(w(p^*\xi)^{-1}), h_*(s_*(N))\rangle \\
&\quad = \langle (\phi_R^*)^{-1}(w(p^*\xi)^{-1}) \cup U_{p^*\xi}, [h(N)] \cap w^*(h(N))\rangle \\
&\quad = \langle (\phi_R^*)^{-1}(w(p^*\xi)^{-1}) \cup w^*(h(N)), [h(N)] \cap U_{p^*\xi}\rangle \\
&\quad = \langle (\phi_R^*)^{-1}(w(p^*\xi)^{-1}) \cup w^*(h(N)), (\phi_R)_*[Z]\rangle \\
&\quad = \langle w^*(Z), [Z]\rangle \\
&\quad = e(Z),
\end{aligned}$$

which completes the proof.

Remark. When ξ is a vector bundle, Proposition 2 is proved in [9], [4] using the axioms of Stiefel-Whitney cohomology classes. The proposition will still hold for Z_2-homology bundles by a result of Taylor [8].

As a special case of Proposition 2, we have the following.

Corollary 3. *Let $\xi = (E, B, \phi)$ be a block bundle. If the base space B is an Euler space, then*

$$\langle T_\xi^*(w(\xi)^{-1}), s_*(E)\rangle = e(B).$$

Proof. Since B is an Euler space, so is E. Thus (id, E) is an element of $\mathfrak{B}_*(E, \bar{E})$. Let i: $B \hookrightarrow R$ be the inclusion. The composition $i \circ \mathrm{id}$ is already transverse to ϕ_R. Consequently the intersection Z is equal to B. Thus $e_\xi(\mathrm{id}, E) = e(B)$, which completes the proof.

§ 4. Proof of Theorem

In order to prove the theorem, it is sufficient to consider the case when Y itself is the total space of a block bundle $\nu = (Y, X, \phi)$ over an Euler space X. Then Y is an Euler space with boundary. The definition of an Euler space with boundary is a natural extension of the definition of an Euler space (without boundary), and is given, e.g., in [4]. The Stiefel-Whitney homology class $s_*(Y)$ is an element in $H_*(Y, \partial Y)$. Let ψ: $Y \to \boldsymbol{R}_+^\alpha$ be an embedding for α sufficiently large and let R be a relative regular neighborhood of $(Y, \partial Y)$ in $(\boldsymbol{R}_+^\alpha, \partial \boldsymbol{R}_+^\alpha)$. Put $\bar{R} = \partial R$. We may suppose that R is also a regular neighborhood of X in $\boldsymbol{R}_+^\alpha$. We regard ψ as an embedding

$$\psi\colon (Y, \partial Y) \longrightarrow (R, \bar{R}).$$

Put

$$W = (\phi_*)^{-1}(s_*(Y) \cap U_\nu) \cap w(\nu)^{-1} \in H_*(X),$$

where U_ν is the Thom class of the bundle ν as before. Since $\phi^! s_*(Y) = (\phi_*)^{-1}(s_*(Y) \cap U_\nu)$, it suffices to show that $s_*(X) = W$. By Proposition 1, this is equivalent to prove that

$$\langle ([R] \cap)^{-1} (\psi\phi)_* W, f_*([M] \cap w^*(M)) \rangle = e_\phi(f, M),$$

for any $(f, M) \in \mathfrak{N}_*(R, \bar{R})$. Note that $W = (\phi_*)^{-1}(s_*(Y) \cap T_\nu(w(\nu)^{-1})$. We may assume that $f\colon (M, \partial M) \to (R, \bar{R})$ is an embedding which is already block transverse to $\psi\phi(X)$ and $\psi(Y)$. Let $\xi = (E, M, f_E)$ be the normal block bundle of M in R such that the restriction $\xi|_{\partial M}$ is the normal block bundle of ∂M in $\bar{R}$. Here $f_E\colon M \to E$ is equal to f with the restricted target space. Put

$$Z = f(M) \cap X, \qquad Y_E = Y \cap E.$$

Then Y_E is the total space of the Whitney sum $\xi|_Z \oplus \nu|_Z$ over Z. We write

$$\psi_E\colon Y_E \longrightarrow E, \qquad j_E\colon Y_E \longrightarrow Y$$

for the inclusions. Let $\bar{E}$ be the boundary of E and let $q\colon R \to E/\bar{E}$ be the Thom map defined by collapsing $R - E$ to the one point $\{\bar{E}\}$ in $E/\bar{E}$. Since $w^*(M) = w(\xi)^{-1}$, we have the following;

$$\begin{aligned}
&\langle ([R] \cap)^{-1} (\psi\phi)_* W, f_*([M] \cap w^*(M)) \rangle \\
&\quad = \langle f^*([R] \cap)^{-1} (\psi\phi)_* W \cup w(\xi)^{-1}, [M] \rangle \\
&\quad = \langle f^*([R] \cap)^{-1} (\psi\phi)_* W \cup w(\xi)^{-1}, (f_E^*)^{-1}([E]_\cap U_\xi) \rangle \\
&\quad = \langle (f_E^*)^{-1}(f^*([R] \cap)^{-1} (\psi\phi)_* W \cup w(\xi)^{-1}) \cup U_\xi, [E] \rangle \\
&\quad = \langle (f_E^*)^{-1} w(\xi)^{-1} \cup U_\xi, [E] \cap (f_E^*)^{-1} f^*([R] \cap)^{-1} (\psi\phi)_* W \rangle \\
&\quad = \langle (f_E^*)^{-1} w(\xi)^{-1} \cup U_\xi, q_*(\psi\phi)_* W \rangle \\
&\quad = \langle T_\xi^*(w(\xi)^{-1}), (q\psi)_*(s_*(Y) \cap T_\nu^*(w(\nu)^{-1})) \rangle \\
&\quad = \langle T_\xi^*(w(\xi)^{-1}), (\psi_E)_*(s_*(Y_E) \cap j_E^* T_\nu^*(w(\nu)^{-1})) \rangle \\
&\quad = \langle \psi_E^*(T_\xi^*(w(\xi)^{-1}) \cup j_E^* T_\nu^*(w(\nu)^{-1}), s_*(Y_E) \rangle \\
&\quad = \langle T^*_{(\xi|_Z \oplus \nu|_Z)}(w(\xi|_Z \oplus \nu|_Z)^{-1}), s_*(Y_E) \rangle \\
&\quad = e(Z) \qquad \text{by Corollary 3.}
\end{aligned}$$

The proof is complete.

References

[1] P. Conner and E. Floyd, Differentiable periodic maps, Ergebnisse der Math. **33**, Springer, Berlin, Heiderberg, New York, 1964.
[2] W. Fulton and R. MacPherson, Categorical framework for the study of singular spaces, Memoirs Amer. Math. Soc., **243** (1981).
[3] S. Halperin and D. Toledo, Stiefel-Whitney homology classes, Ann. of Math., **96** (1972), 511–525.
[4] A. Matsui, Stiefel-Whitney homology classes of Z_2-Poincaré-Euler spaces, Tôhoku Math. J., **35** (1983), 321–339.
[5] ——, Axioms for Stiefel-Whitney homology classes of Z_2-Euler spaces, Tôhoku Math. J., **37** (1985), 27–32.
[6] —— and H. Sato, Stiefel Whitney homology classes and homotopy type of Euler spaces, J. Math. Soc. Japan, **37** (1985), 437–453.
[7] C. P. Rourke and B. J. Sanderson, Block bundles, I, II, III, Ann. of Math., **87** (1968) 1–28, 255–277, 431–483.
[8] L. Taylor, Stiefel-Whitney homology classes, Quart. J. Oxford, **28** (1977), 381–387.
[9] D. Veljan, Axioms for Stiefel-Whitney homology classes of some singular spaces, Trans. Amer. Math. Soc., **277** (1983), 285–305.

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