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Hadamard Manifolds

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As the starting point in the study of Riemannian manifolds of nonpositive curvature, we first recall the Cartan-Hadamard theorem (cf. [9], [17], [68]).

Theorem. *Let H be an n-dimensional simply connected complete Riemannian manifold of nonpositive curvature. Then H is diffeomorphic to the n-dimensional Euclidean space $\boldsymbol{R}^n$. More precisely, at any point $p \in H$, the exponential mapping* $\exp_p$*: $H_p \to H$ is a diffeomorphism.*

This theorem presents a clear contrast to Meyer's theorem (cf. [9], [17], [68]): if a complete Riemannian manifold M is of strictly positive Ricci curvature, i.e., Ricci curvature $\geqq k > 0$ for some k, then M is compact.

A simply connected complete Riemannian manifold of nonpositive curvature is called a *Hadamard manifold* or a *Cartan-Hadamard manifold* after the Cartan-Hadamard theorem. Unless otherwise mentioned, H will always denote a Hadamard manifold throughout this report.

From the Cartan-Hadamard theorem, there follow several basic properties of Riemannian manifolds of nonpositive curvature. For example, any pair of distinct points of a *Hadamard manifold* can be joined uniquely by a geodesic segment. It also follows that the fundamental group of a compact Riemannian manifold of nonpositive curvature is an infinite group.

The primary object of this survey article is to investigate the behavior of geodesics of a Hadamard manifold. Then we apply these investigations to the study of the isometry groups, discrete subgroups of the isometry groups of Hadamard manifolds and the fundamental groups of compact Riemannian manifolds of nonpositive curvature.

The geodesic behavior in respect to the ergodicity of the geodesic flows on compact Riemannian manifolds of negative curvature has been investigated by many authors. For this subject, there is Sunada's report [86] in this proceeding.

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The behavior of geodesics is controlled by the curvature. In fact, the effect of curvature appears typically in the second variational formula of the lengths of curves. From this formula, we obtain three kinds of convex functions on a Hadamard manifold: distance functions, displacement functions and Busemann functions. The convexity of these functions plays an essential role in the study of a Hadamard manifold. The relation between the lengths of the sides and the angles at the vertices of a geodesic triangle, called the law of cosines will also be of essential use in our arguments.

In [31], Eberlein and O'Neill defined an ideal boundary $H(\infty)$ of a Hadamard manifold H. $H(\infty)$ is defined to be the set of all asymptotic classes of geodesic rays of H. The space $\bar{H}=H\cup H(\infty)$ has a natural topology called the cone topology. With this topology, $\bar{H}$ is homeomorphic to the closed ball, and $\bar{H}$ is a compactification of H (cf. 2.2). The cone topology has several natural properties. One of them is the following: every isometry of H is extended as a homeomorphism of $\bar{H}$, hence the isometry group $I(H)$ can be extended as a group of homeomorphisms of $\bar{H}$. Using this compactification efficiently, we shall study Hadamard manifolds.

Typical examples of Hadamard manifolds are the Euclidean space $\boldsymbol{R}^n$ with the standard flat metric and the hyperbolic space form, i.e., a simply connected complete Riemannian manifold with negative constant curvature. The behavior of geodesics of these spaces differs considerably. For instance, in $\boldsymbol{R}^n$, a point at infinity can be joined by a complete geodesic with only one point at infinity. In the hyperbolic space form, any pair of points at infinity can be joined by a complete geodesic, which is also the case for Hadamard manifolds of strictly negative curvature. A Hadamard manifold is called a Visibility manifold if it has this property.

It can occur, in general, that two distinct complete geodesics join the same pair of points at infinity. However, it is known, by Green [39] and others, that, in this case, the Riemannian structure is strongly restricted. In fact, if two distinct complete geodesics have the same end points, then they bound a totally geodesic flat surface, called a flat strip (Theorem 8). Owing to this fact, Wolf [88] proved an important splitting theorem, saying that a Hadamard manifold has nontrivial flat de Rham factor providing that it has a Clifford translation (Theorem 14). These theorems imply several consequences on the isometry group of a Hadamard manifold (Theorem 15, 30) and the fundamental group of a compact manifold of nonpositive curvature (Theorem 36, 40).

Let M be a complete Riemannian manifold of nonpositive curvature and H, a Hadamard manifold, be the universal covering manifold of M. Then M can be represented as the quotient H/Γ with Γ, some fixed point free properly discontinuous subgroup of $I(H)$. Following Eberlein and O'Neill [31], complete Riemannian manifolds of nonpositive curvature are classified into three classes: axial, parabolic and fuchsian manifolds. A manifold belonging to the first two classes has relatively simple structure (Theorems 21, 22). Γ acts on $\bar{H}$ as a group of homeomorphisms. For Visibility manifolds, this extension is useful to describe the structure of Γ. In fact, the above three classes of fixed point free properly discontinuous subgroup of $I(H)$ are characterized in terms of the limit set of Γ, i.e., the set of all accumulation points in $H(\infty)$ of an orbit of Γ (Theorems 25, 26, 27).

A fixed point free properly discontinuous subgroup Γ of $I(H)$ is called a lattice, if the quotient H/Γ is of finite volume. It is well-known that a symmetric space of noncompact type admits a lattice (Borel [11]). The question what kind of Hadamard manifolds admits a lattice, has been studied by Heintze [55], Goto and Goto [36], Chen and Eberlein [21], Eberlein [28] and others. Let Γ be a lattice of a Hadamard manifold H. Then, from the theory of geodesic flows, we can see that Γ satisfies some density condition, called the duality condition. The duality condition restricts strongly the structure of a Hadamard manifold. In fact, Chen and Eberlein [21] proved that if $I(H)$ satisfies the duality condition, then any nontrivial abelian normal subgroup of $I(H)$ consists of Clifford translations (Proposition 29). Hence, if H has not nontrivial flat de Rham factor, then $I(H)$ is discrete or the identity component $I_0(H)$ of $I(H)$ is semi-simple (Theorem 30). Using this fact, we can describe the structure of a Hadamard manifold which admits a lattice (Theorem 34).

The fundamental group of a compact Riemannian manifold of nonpositive curvature $M=H/\Gamma$ is isomorphic to Γ, and we can investigate it by the action of Γ on H. Since every element of Γ is axial, we can see, by the structure theorem of axial isometries (Proposition 12), Preismann's theorem: any nontrivial abelian subgroup of the fundamental group of a compact Riemannian manifold is infinite cyclic. For finitely generated group, Milnor [70] introduced the growth function which, roughly speaking, measures the abundance of the relations among generators of the group, and he showed that the fundamental group of a compact manifold of negative curvature is of exponential growth, the most rapid possible growth (Theorem 38). Eberlein [26] generalized Milnor's result by an entirely different formulation (Theorem 39). In the case of nonpositive curvature, abelian subgroups and the center of the fundamental group give informations about totally geodesic flat submanifolds.

In the last part of this report, we shall rewiew the study of Kähler Hadamard manifolds in the geometric function theory. This part is somewhat independent of the other parts in this report. Our concern is to see how the geometric structure of a Kähler manifold is reflected in the function theoretic properties.

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§ 1. Fundamental properties of Hadamard manifolds

In this section, we shall review some fundamental properties of Hadamard manifolds. First we shall recall the second variational formula of the lengths of curves and then the convexity of some naturally defined functions. Also we shall discuss briefly manifolds without conjugate points and manifolds without focal points. At the end of this section, we shall give several examples of Hadamard manifolds which might be model spaces in our arguments.

For later use, we fix some notations. Let M be a Riemannian manifold. We denote by $\langle \cdot, \cdot \rangle$ or ds^2 the Riemannian metric on M. When it is necessary to specify the underlying manifold M, it is denoted by $\langle \cdot, \cdot \rangle_M$ or ds^2_M. We use similar conventions for other notations. For a point $p \in M$, M_p denotes the tangent space of M at p. We denote by ∇ the covariant derivative and by R the curvature tensor field defined by $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. Let $X, Y \in M_p$ be linearly independent tangent vectors. Then the sectional curvature $K(X, Y)$ of the tangent plane spanned by X and Y is defined by

$$K(X, Y) = \langle R(X, Y)Y, X \rangle / (\|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2),$$

where $\|X\| = \langle X, X \rangle^{1/2}$. If $K(X, Y) < 0$ (resp. $K(X, Y) \leqq 0$) for any pair X and Y, then M is called a Riemannian manifold of negative (resp. non-positive) curvature. If $K(X, Y) < -c < 0$ in particular, then M is called a Riemannian manifold of strictly negative curvature.

1.1. The second variational formula

The second variational formula of the lengths of curves is one of the fundamental tools in the study of the behavior of geodesics. We recall the first and the second variational formulas (cf. [9], [17], [68]).

Let $\gamma\colon [a, b]\to M$ be a geodesic segment. Unless otherwise mentioned, we always assume that a geodesic is parametrized by its arc length. Let $\dot{\gamma}(t)$ denote the tangent vector of γ at $\gamma(t)$. A variation of γ is by definition a differentiable mapping $r(t, u)\colon [a, b]\times(-\varepsilon, \varepsilon)\to M$ satisfying $r(t, 0)=\gamma(t)$. The vector field along γ defined by $Y(t)=r_*(\partial/\partial u)_{(t,0)}$ is called the variational vector field of r. We denote by $l(u)$ the length of the curve $r(\cdot, u)$. The first variational formula is then given as follows:

$$\frac{dl}{du}(0)=\langle Y(b), \dot{\gamma}(b)\rangle-\langle Y(a), \dot{\gamma}(a)\rangle.$$

This formula implies that a geodesic segment which attains the distance from a point p to a submanifold N intersects N perpendicularly.

The second derivative of $l(u)$ at 0 is given by the second variational formula:

$$\frac{d^2l}{du^2}(0)=\int_a^b \{\|\nabla_{\dot{\gamma}} Y^{\perp}\|^2-K(Y, \dot{\gamma})(\|Y\|^2-\langle Y, \dot{\gamma}\rangle^2)\}dt+\langle\nabla_Y Y, \dot{\gamma}\rangle|_a^b$$

where $Y^{\perp}$ stands for the normal component of Y to γ.

Let us define Jacobi fields. A smooth vector field Y along a geodesic γ is called a Jacobi field if it satisfies the following differential equation:

$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}} Y+R(Y, \dot{\gamma})\dot{\gamma}=0.$$

It is well-known that a Jacobi field is in fact a variational vector field of a variation $r\colon [a, b]\times(-\varepsilon, \varepsilon)\to M$ such that every $r(\cdot, u)$ is a geodesic parametrized by a parameter proportional to its arc length. Rauch's comparison theorem (cf. [9], [17], [68]) shows that the growth of a Jacobi field is controlled by the curvature.

1.2. A lemma of Synge

The second variational formula implies a classical theorem, usually called a lemma of Synge [87] (see also [77]). Let $\gamma\colon (a, b)\to M$ be a geodesic and S a 2-dimensional submanifold of M containing γ. Take real numbers $a<a'<b'<b$ and consider a variation $r\colon [a', b']\times(-\varepsilon, \varepsilon)\to S$ such that $r(a', u)=\gamma(a')$ and $r(b', u)=\gamma(b')$. Then we may regard r as a variation either in M and in S.

If r is regarded as a variation in M, then the second variational formula reads:

$$\frac{d^2l}{du^2}(0)=\int_{a'}^{b'} \{\|\nabla_{\dot{\gamma}}^M Y^{\perp}\|^2-K_M(Y, \dot{\gamma})(\|Y\|^2-\langle Y, \dot{\gamma}\rangle^2)\}dt.$$

If r is regarded as a variation in S, then we have

$$\frac{d^2l}{du^2}(0)=\int_{a'}^{b'}\{\|\nabla_{\dot\gamma}^S Y^\perp\|^2-K_S(Y,\dot\gamma)(\|Y\|^2-\langle Y,\dot\gamma\rangle^2)\}dt.$$

Since $\|\nabla_{\dot\gamma}^M Y^\perp\|^2\geqq\|\nabla_{\dot\gamma}^S Y^\perp\|^2$, it follows that

$$\int_{a'}^{b'}K_S(Y,\dot\gamma)(\|Y\|^2-\langle Y,\dot\gamma\rangle^2)dt\leqq\int_{a'}^{b'}K_M(Y,\dot\gamma)(\|Y\|^2-\langle Y,\dot\gamma\rangle^2)dt$$

for any variation of γ in S. This implies that $K_S(Y,\dot\gamma)\leqq K_M(Y,\dot\gamma)$ which proves the following:

Theorem 1. *Let $\gamma\colon(a,b)\to M$ be a geodesic and S be a 2-dimensional submanifold of M containing γ. Then the Gaussian curvature of S along γ is not greater that the corresponding sectional curvature of M. Furthermore, both are equal iff the tangent planes of S are parallel along γ.*

With a typical application of Theorem 1, we arrive at the following fact. Suppose that a piece of surface S in a Hadamard manifold H is parametrized by $r(u,v)$ such a way that $r(\cdot,v)$ is a geodesic in H for every v. Then the Gaussian curvature of S is nonpositive. Furthermore, if S is flat, then H is flat along S and S is totally geodesic.

1.3. The law of cosines

Let H be a Hadamard manifold. We denote by SH_p the set of all unit tangent vectors at a point $p\in H$. For $X, Y\in SH_p$, the angle $\theta=\measuredangle(X,Y)$ is the unique number $0\leqq\theta\leqq\pi$ such that $\langle X,Y\rangle=\cos\theta$. For $p\neq q\in H$, we denote by γ_{pq} the geodesic segment joining p and q. If p, q and r are distinct points of H, then we define the angle $\measuredangle_p(q,r)$ by $\measuredangle_p(q,r)=\measuredangle(\dot\gamma_{pq}(0),\dot\gamma_{pr}(0))$.

Let us consider a geodesic triangle whose vertices are p_1, p_2 and p_3. Let θ_i be the angle at p_i, $i=1,2,3$, and $l_1=d(p_2\ p_3)$, $l_2=d(p_1,p_3)$ and $l_3=d(p_1,p_2)$. Since the sectional curvature is nonpositive, it is well-known that $\theta_1+\theta_2+\theta_3\leqq\pi$.

Furthermore, the law of cosines states

$$l_1^2\geqq l_2^2+l_3^2-2l_2l_3\cos\theta_1.$$

The law of cosines plays a fundamental role in investigations of the behavior of geodesics.

1.4. Convex functions

Let M be a Riemannian manifold. A function $f\colon M\to\boldsymbol{R}$ on M is called a convex function if the composition $f\circ\gamma$ is convex for any geodesic γ, i.e.,

$$f \circ \gamma(ta+(1-t)b) \leqq tf(\gamma(a))+(1-t)f(\gamma(b)), \qquad \text{for } 0 \leqq t \leqq 1.$$

Given a C^2-function f, $\nabla^2 f$ denotes the Hessian of f defined by $\nabla^2 f(X, Y) = X(Yf) - (\nabla_X Y)f$, which is a symmetric covariant 2-tensor field on M. Then f is convex iff $\nabla^2 f \geqq 0$. We say f is strictly convex if $\nabla^2 f > 0$.

A subset of a Riemannian manifold is called totally convex if for arbitrary points p and q in the set, any geodesic segment joining p and q is contained in the set. It is easily seen that if f is convex then the set $\{p \in M : f(p) \leqq c\}$ is totally convex for any $c \in \boldsymbol{R}$. Hence the existence of a nontrivial convex function places very strong restrictions on the topology of a manifold (cf. Cheeger and Gromoll [18], Gromoll and Meyer [49], Shiohama [83]).

In this section, we will review the construction of three kinds of convex functions on a Hadamard manifold. For a detailed treatment, see Bishop and O'Neill [10].

1.4.1 The distance function from a totally geodesic submanifold

Let γ_1 and γ_2 be geodesics of H, then the function $g(t) = d(\gamma_1(t), \gamma_2(t))$ is convex. In fact, define a variation r of the geodesic segment joining $\gamma_1(t_0)$ and $\gamma_2(t_0)$ for a fixed t_0 so that $r(\cdot, t)$ is the geodesic segment joining $\gamma_1(t_0+u)$ and $\gamma_2(t_0+u)$. Then, applying the second variational formula, we can obtain $g''(t_0) \geqq 0$.

Now, let N be a closed totally geodesic submanifold of H. For a point $p \notin N$, let $\gamma : [0, l] \to H$ be the unique geodesic segment which attains the distance from p to N, $\gamma(0) = p$, $l = d(p, N)$. The point $\gamma(l) \in N$ is called the foot of p. Let $\sigma : (-\varepsilon, \varepsilon) \to H$ be another geodesic such that $\sigma(0) = p$. For $u \in (-\varepsilon, \varepsilon)$, let $r(\cdot, u)$ be a geodesic segment joining $\sigma(u)$ and its foot on N. Then r is a variation of γ and, applying the second variational formula, it follows that $d^2/du^2\, d(\sigma(u), N) \geqq 0$ at $u = 0$.

Proposition 2. *Let N be a closed totally geodesic submanifold of H. Then the function $d(\cdot, N)$ is convex on H and its square $d(\cdot, N)^2$ is smooth and convex. Furthermore, if the sectional curvature of H is negative, $d(\cdot, N)^2$ is strictly convex on $H - N$. If N is a point, $d(\cdot, N)^2$ is smooth and convex on H.*

More generally, it is known that if $S \subset H$ is a totally convex subset, then the distance function $d(\cdot, S)$ is a continuous convex function.

1.4.2. The displacement function

Given an isometry ϕ of H, the function $d_\phi(p) = d(p, \phi(p))$ is called the displacement function of ϕ.

Proposition 3. *The displacement function d_ϕ of an isometry ϕ of a Hadamard manifold is convex and its square d_ϕ^2 is smooth and convex. Furthermore, if the sectional curvature of H is negative, then d_ϕ is strictly convex outside the minimum set $C_\phi=\{p \in H: d_\phi(p)=\inf d_\phi\}$ of d_ϕ.*

The convexity of d_ϕ follows from the arguments in 1.4.1. In fact, it is sufficient to notice that $d_\phi \circ \gamma(t)=d(\gamma(t), \phi\circ\gamma(t))$ and $\phi\circ\gamma$ is a geodesic for any geodesic γ.

The displacement functions of isometries will be used in Section 3 to classify isometries of a Hadamard manifold.

1.4.3. Busemann functions

Let $\gamma: [0, \infty)\to H$ be a geodesic ray. For a point $p \in H$, the function $t\mapsto d(p, \gamma(t))-t$ is monotone nonincreasing and bounded from below, as easily seen by the triangle inequality. Hence we can define a function f_γ by $f_\gamma(p)=\lim_{t\to\infty} d(p, \gamma(t))-t$. Since the function $d(\cdot, \gamma(t))-t$ is convex for each fixed t, the function f_γ is also convex. We call f_γ the Busemann function relative to a geodesic ray γ (Busemann [14]). Properties of Busemann functions will be discussed in 2.4.

1.5. Manifolds without conjugate points and manifolds without focal points

Let M be a complete Riemannian manifold and $\gamma: [0, l]\to M$ be a geodesic segment. We call $q=\gamma(l)$ a conjugate point of $p=\gamma(0)$ along γ if there is a nontrivial Jacobi field along γ which vanishes both at p and at q. Suppose that a point $p \in M$ has no conjugate points along any geodesic issuing from p. Then the exponential mapping $\exp_p$: $M_p\to M$ is a covering map. If every point of M has the above property, we call M a manifold without conjugate points. Note that a simply connected Riemannian manifold M without conjugate points is diffeomorphic to $\boldsymbol{R}^n$ ($n=\dim M$), and any pair of points p and q of M can be joined by the unique geodesic segment.

Let us recall the definition of an N-Jacobi field for a submanifold N of M. Let $\tau: [0, l]\to M$ be a geodesic segment such that $\tau(0) \in N$ and $\dot\tau(0)\perp N_{\tau(0)}$. A perpendicular Jacobi field Y with $Y(0) \in N_{\tau(0)}$ along τ is called an N-Jacobi field if the vector $A_{\dot\tau(0)}Y(0)+\nabla_{\dot\tau}Y(0)$ is normal to N, where A_ξ denotes the second fundamental tensor relative to a normal vector ξ to N, defined by $\langle A_\xi X, Y\rangle=\langle \xi, \nabla_X Y\rangle$. It is known that an N-Jacobi field is in fact a variational vector field of a variation of τ consisting of a family of geodesics which are perpendicular to N. We say $q=\tau(l)$ is a focal point of N along τ if there is a nontrivial N-Jacobi field which vanishes at q.

We call M a manifold without focal points if any geodesic in M, regarded as a submanifold, has no focal points in M.

The following proposition characterizes manifolds without focal points in terms of the growth of order Jacobi fields.

Proposition 4 (O'Sullivan [76]). *Let M be a complete Riemannian manifold. Then:*

(1) *M is of nonpositive curvature iff $d^2/dt^2\langle Y, Y\rangle \geqq 0$ for any Jacobi field Y along any geodesic and for all t.*

(2) *M is without focal points iff $d/dt\langle Y, Y\rangle > 0$ for any nontrivial initially vanishing Jacobi field Y and for all $t > 0$.*

(3) *M is without conjugate points iff $\langle Y, Y\rangle > 0$ for any nontrivial initially vanishing Jacobi field Y and for all $t > 0$.*

From this proposition, it follows that if M is a complete Riemannian manifold of nonpositive curvature, then M is a manifold without focal points and that if M is without focal points, then M is without conjugate points. These three classes of Riemannian manifolds are actually different, as was shown in Gulliver [52].

Another characterization of manifolds without focal points is given as follows.

Proposition 5 (Eschenburg [32], Innami [59]). *A simply connected complete Riemannian manifold M is without focal points iff the function $d(\cdot, p)$ is convex for any point p of M.*

Remark here that a complete Riemannian manifold of nonnegative curvature has no conjugate points if and only if it is flat.

Recently, studies have been made on the similarity of manifolds without focal points to manifolds of nonpositive curvature (cf. [32], [34], [35], [59], [75], [76]).

1.6. Examples of Hadamard manifolds

We shall present here some typical examples of Hadamard manifolds which will be model spaces in our later considerations.

1.6.1. Symmetric spaces

A simply connected complete flat Riemannian manifold of dimension n is isometric to $\boldsymbol{R}^n$ with the standard metric. So $\boldsymbol{R}^n$ is the unique flat Hadamard manifold.

Let H be the set $\{(x_1, \cdots, x_n) \in \boldsymbol{R}^n : \sum x_i^2 < 1\}$ with the metric $ds^2 = 4\sum dx_i^2/(1-\sum x_i^2)^2$. Then H is a complete Riemannian manifold with constant negative curvature, which is called the hyperbolic space form.

Next we consider the unit open ball in $\boldsymbol{C}^n$ with the Bergman metric. Let H be the unit open ball $\{(z_1, \cdots, z_n) \in \boldsymbol{C}^n : \sum |z_i|^2 < 1\}$ with the metric

$ds^2=4(1-\sum z_i\bar{z}_i)^{-2}\{(1-\sum z_i\bar{z}_i)(\sum dz_i d\bar{z}_i)-(\sum \bar{z}_i dz_i)(\sum z_i d\bar{z}_i)\}$. Then the holomorphic sectional curvature of H is negative constant -1 and the sectional curvature varies between $-1/4$ and -1.

The last two examples are known to be rank 1 symmetric spaces of noncompact type. It is known that rank 1 symmetric spaces of noncompact type are manifolds of strictly negative curvature.

More generally, it is known that a symmetric space of noncompact type is of nonpositive curvature. If the rank l of a symmetric space is greater than 2, then it contains a totally geodesic l-dimensional flat submanifold. So its sectional curvature attains zero at some tangent plane.

1.6.2. Homogeneous Hadamard manifolds

Let H be a Hadamard manifold and $I(H)$ the group of all isometries of H. If $I(H)$ acts transitively on H, we call H a homogeneous Hadamard manifold. Wolf [88] showed that if H is a homogeneous Hadamard manifold, then there is a solvable subgroup of $I(H)$ which acts transitively on H. Heintze [55] showed that in the same situation, it is actually possible to find a solvable subgroup which acts simply transitively. So every homogeneous Hadamard manifold can be described as a simply connected solvable Lie group G with a left invariant metric, whose curvature is nonpositive. Identifying the Lie algebra of G and the tangent space of G at the identity element, it is sufficient to consider a solvable Lie algebra with a positive definite inner product. Heintze [55] has characterized solvable Lie algebras which admit metrics with negative curvature, and Azencott and Wilson [4], [5] have generalized his characterization to the case of nonpositive curvature.

1.6.3. Warped product

Take two Riemannian manifolds M and N with metrics ds_M^2 and ds_N^2, respectively. Let $f\colon M\to \boldsymbol{R}$ be a positive smooth function on M. Then the product manifold $M\times N$ with the metric $ds^2=ds_M^2+f^2 ds_N^2$ is called a warped product and is denoted by $M\times_f N$. From the curvature formula of a warped product (cf. [10]), we obtain the following fact. If N is a Riemannian manifold of nonpositive curvature and f is a convex function on $\boldsymbol{R}$, then $\boldsymbol{R}\times_f N$ is of nonpositive curvature. Furthermore, if f is strictly convex without minimum, then the sectional curvature of $\boldsymbol{R}\times_f N$ is negative everywhere.

§ 2. Ideal boundaries of Hadamard manifolds

Eberlein and O'Neill [31] defined the set of points at infinity $H(\infty)$ of a Hadamard manifold, and defined a natural topology on $\bar{H}=H\cup H(\infty)$, called the cone topology. This section is devoted to a brief survey of some properties of the cone topology.

2.1. Asymptotic classes of geodesic rays

Let H be a Hadamard manifold, and $\gamma_1, \gamma_2: [0, \infty) \to H$ be geodesic rays. We say γ_1 and γ_2 are asymptotic if the function $d(\gamma_1(t), \gamma_2(t))$ is bounded on $[0, \infty)$. Since $d(\gamma_1(t), \gamma_2(t))$ is convex, $d(\gamma_1(t), \gamma_2(t))$ is monotone nonincreasing if γ_1 and γ_2 are asymptotic. It is immediate that the asymptote relation is an equivalence relation on the set of all geodesic rays in H.

We denote by $H(\infty)$ the set of all asymptotic classes of geodesic rays and by $\gamma(\infty)$ the asymptotic class containing a geodesic ray γ. For a complete geodesic γ, we denote by $\gamma(-\infty)$ the asymptotic class containing the reversed geodesic $t \mapsto \gamma(-t)$ of γ. Let x be a point of $H(\infty)$ and γ be a geodesic ray such that $\gamma(\infty)=x$. Let $p \in H$ and for each n take a geodesic γ_n such that $\gamma_n(0)=p$ and γ_n passes through $\gamma(n)$. Passing to a subsequence if necessary, we may assume that $\{\dot{\gamma}_n(0)\}$ converges to a unit vector v at p. Then $\{\gamma_n\}$ converges to γ_v, the geodesic ray with initial velocity v. Because of the convexity of the distance function between two geodesics γ and γ_v, we can conclude that γ_v is asymptotic to γ. Thus we have:

Proposition 6. *For any $p \in H$ and $x \in H(\infty)$, there is a unique geodesic ray γ such that $\gamma(0)=p$ and $\gamma(\infty)=x$.*

The uniqueness also follows from the convexity of the distance function between two geodesics. We denote by γ_{px} the geodesic ray in the proposition.

2.2. The cone topology

Following Eberlein and O'Neill [31], we shall define a topology on $\bar{H}=H \cup H(\infty)$ compatible with the manifold topology on H. It is sufficient to define a fundamental neighbourhood system of x for each $x \in H(\infty)$. Take a point p of H and a positive number $\varepsilon>0$ and put $C(p, x, \varepsilon)=\{q \in \bar{H}: \measuredangle_p(q, x)<\varepsilon\}$, which is called the cone of vertex p with axis γ_{px} and angle ε. Then take $\{C(p, x, \varepsilon): \varepsilon>0, p \in H\}$ as a fundamental neighbourhood system of x. It is not a trivial matter that the manifold topology together with the above fundamental neighbourhood systems define a topology on $\bar{H}$. Refer to Eberlein and O'Neill [31] for details. We call this topology the cone topology of $\bar{H}$.

We can prove the existence of another natural topology on $\bar{H}$ which can be considered as an extension of the metric topology, called the horocycle topology. The reader should refer Eberlein and O'Neill [31] and Eberlein [24] for this topology.

In the following, we always consider $\bar{H}$ as a topological space with the cone topology.

Now we recall some basic facts about the cone topology. Take a point $p \in H$ and put $\bar{B}_p = \{v \in H_p : \|v\| < 1\}$. Define a mapping $\Phi_p : \bar{B}_p \to \bar{H}$ by $\Phi_p(v) = \exp(v/1 - \|v\|)$ if $\|v\| < 1$ and $\Phi_p(v) = \gamma_v(\infty)$ if $\|v\| = 1$. Then the mapping Φ_p defines a homeomorphism of $\bar{B}_p$ and $\bar{H}$. In particular $\bar{H}$ is compact.

The following properties can be derived easily from the definition of the topology.

1. Let $\gamma : [0, \infty) \to H$ be a geodesic ray. Then the extension of γ, $\bar{\gamma} : [0, \infty) \to \bar{H}$ defined by $\bar{\gamma}(\infty) = \gamma(\infty)$ is continuous.

2. Let $\phi \in I(H)$. Then letting $\phi(\gamma(\infty)) = (\phi \circ \gamma)(\infty)$, we define an extension of ϕ. Then $\phi : \bar{H} \to \bar{H}$ is a homeomorphism.

For a simply connected Riemannian manifold without focal points, we can also define both the set of points at infinity and the cone topology (Goto [35], Innami [59]). But we do not know whether they can be generalized to a manifold without conjugate points.

2.3. Visibility manifolds

For a pair of distinct points p and q of H, there is a unique geodesic joining p and q. In the case where $p \in H$ and $x \in H(\infty)$, there is also a unique geodesic ray γ_{px} joining p and x, as we have seen in Proposition 6. But for $x, y \in H(\infty)$, $x \neq y$, there is not always a complete geodesic joining x and y. For example, if $H = \boldsymbol{R}^n$, then for any point $x \in H(\infty)$, the point $y \in H(\infty)$ which can be joined with x is uniquely determined by x. In [31], Eberlein and O'Neill formulated the possibility of joining points at infinity in the following fashion.

Definition. A Hadamard manifold H is said to satisfy

Axiom 1 if for any points $x \neq y$ in $H(\infty)$, there exists at least one complete geodesic joining x and y.

Axiom 2 if for any points $x \neq y$ in $H(\infty)$, there exists at most one complete geodesic joining x and y.

Note that $\boldsymbol{R}^n$ satisfies neither Axiom 1 nor Axiom 2.

Proposition 7 (Eberlein and O'Neill [31]). *A Hadamard manifold H of strictly negative curvature satisfies both Axiom 1 and Axiom 2.*

Axiom 1 is equivalent to the following Visibility axiom, which is often more convenient (Eberlein and O'Neill [31]).

Visibility axiom. For any point $p \in H$ and $\varepsilon > 0$, there exists a positive number $R(p, \varepsilon)$ with the property that $\measuredangle_p(\sigma(a), \sigma(b)) < \varepsilon$, for any geodesic segment σ: $[a, b] \to H$ such that $d(p, \sigma([a, b])) > R(p, \varepsilon)$.

This axiom can be interpreted as follows. If a geodesic segment σ is sufficiently far from p, then no matter how long σ is, any two of its points subtend an arbitrarily small angle at p. Roughly speaking, distant geodesics look small.

A Hadamard manifold satisfying the Visibility axiom is called a *Visibility manifold.* Eberlein and O'Neill [31] gave a sufficient condition for the Visibility axiom in terms of decays of the sectional curvature. In the case of H having a compact quotient, it is known that H satisfies the Visibility axiom iff there is no isometrically imbedded totally geodesic flat submanifold of dimension 2 (Eberlein [23]).

If two complete geodesics γ_1 and γ_2 satisfy $\gamma_1(\infty) = \gamma_2(\infty)$ and $\gamma_1(-\infty) = \gamma_2(-\infty)$, we say that γ_1 and γ_2 are biasymptotic. Suppose that two complete geodesics γ_1 and γ_2 are biasymptotic. We reparametrize γ_1 and γ_2 so that the foot of $\gamma_1(0)$ on γ_2 is $\gamma_2(0)$. Since the function $d(\gamma_1(t), \gamma_2(t))$ is a bounded convex function on $\boldsymbol{R}$, it is constant, say l. Let F: $[0, l] \times \boldsymbol{R} \to H$ be a mapping such that $F(\cdot, t)$ is the geodesic segment joining $\gamma_1(t)$ and $\gamma_2(t)$. Endow the flat metric on $[0, l] \times \boldsymbol{R}$. Then, using the convexity of the length of a Jacobi field and the first variational formula, we can show that F is an isometric imbedding. By a lemma of Synge (1.2), we see that the image of F is totally geodesic. By a flat strip, we mean a totally geodesic isometric imbedding F: $[0, l] \times \boldsymbol{R} \to H$. Then we obtain the following theorem.

Theorem 8 (Flat strip theorem; Green [39], Eberlein and O'Neill [31], O'Sullivan [75]). *Let γ_1 and γ_2 be distinct biasymptotic geodesics in H. Then there exists a flat strip F: $[0, l] \times \boldsymbol{R} \to H$ such that $F(0, t) = \gamma_1(t)$ and $F(l, t) = \gamma_2(t)$ after a suitable reparametrization of γ_i.*

From the above theorem, we see that H satisfies Axiom 2 iff H contains no flat strips.

Some generalization of this theorem will be needed later, so let us state it here. Let B be a closed totally geodesic submanifold of H. Define the set of points at infinity of B by $B(\infty) = \{\gamma(\infty) : \gamma$ is a geodesic ray in $B\}$. Then the following is due to Eberlein [29].

Theorem 9 (Sandwich lemma). *Let B_1 and B_2 be distinct closed totally geodesic submanifolds of H such that $B_1(\infty) = B_2(\infty)$. Then there exists a totally geodesic isometric imbedding F: $[0, l] \times B_1 \to H$ such that $F(0, b) = b$ for all $b \in B_1$ and $F(\{l\} \times B_1) = B_2$.*

This can be proved by a careful use of the flat strip theorem.

2.4. Busemann functions

The Busemann function f_γ relative to a geodesic ray γ is defined by $f_\gamma(p)=\lim_{t\to\infty} d(p,\gamma(t))-t$, which is convex, as was seen in 1.4.3. We shall discuss here the dependence on γ of the function f_γ.

Let γ: $[0, \infty)\to H$ be a geodesic ray, and $\{p_n\}$ a sequence in H which converges to $x=\gamma(\infty)$. Then the Busemann function f_γ can be written as $f_\gamma(p)=\lim_{n\to\infty}\{d(p,p_n)-d(\gamma(0),p_n)\}$ (cf. [24]).

Let γ, τ: $[0,\infty)\to H$ be asymptotic geodesic rays and x be their asymptotic class. Compare the two functions f_γ and f_τ by the above descriptions of these functions and notice that $f_\gamma(\gamma(t))=-t$. Then we obtain:

Proposition 10. *Let γ and τ be asymptotic geodesic rays. Then:*

(1) $f_\gamma-f_\tau=f_\gamma(\tau(0))$

(2) $f_\gamma(\tau(t))-f_\gamma(\tau(s))=s-t$.

This proposition implies that the Busemann function is determined by the asymptotic class up to a constant. Hence it is reasonable to call f_γ a Busemann function at $x=\gamma(\infty)$. Let f be a Busemann function at x. Given a point $p\in H$, we define two subsets $L(p,x)$, $B(p,x)$ respectively by $L(p,x)=\{q\in H: f(q)=f(p)\}$ and $B(p,x)=\{q\in H: f(q)\leqq f(p)\}$. It is easy to see that these sets do not depend on the choice of a Busemann function at x. We call $L(p,x)$ the limit sphere or horosphere at x through p and $B(p,x)$ the limit ball or horoball at x determined by p.

In Heintze and Im Hof [57] it is shown that a Busemann function is a C^2-function. Given a point $x\in H(\infty)$, we define the vector field $V(\cdot,x)$ by $V(p,x)=\dot\gamma_{px}(0)$. Then it holds that grad $f=-V(\cdot,x)$ for any Busemann function at x.

Let L be a limit sphere at $x\in H(\infty)$. For any point $q\in H$, the complete geodesic γ through q and belonging to x intersects L just one time. Define η_L: $H\to L$ by $\eta_L(q)=\gamma\cap L$. Then the point $\eta_L(q)$ is the nearest point in L from q.

Proposition 11. *Let L and L' be limit spheres at $x\in H(\infty)$, and f be a Busemann function at x. Then:*

(1) *$d(L,L')=|f(p)-f(p')|$ for any $p\in L$ and $p'\in L'$.*

(2) *$\eta_L\times f$: $H\to L\times \boldsymbol{R}$ is a homeomorphism.*

The properties mentioned above mean that a Busemann function may be regarded as a distance function from a point at infinity.

On a complete noncompact Riemannian manifold, we can define the Busemann function for a geodesic ray in a similar manner. The Busemann functions play an essential role in the study of the topology of a noncompact complete Riemannian manifold. For this subject, see Shiohama's report [83].

§ 3. Classification of isometries and Wolf's splitting theorem

3.1. Classification of isometries

Following Bishop and O'Neill [10], we classify isometries of a Hadamard manifold into three classes.

Let ϕ be an isometry of a Hadamard manifold H and d_ϕ be the displacement function of ϕ. Then three cases occur:

(1) d_ϕ takes zero at some point.

(2) d_ϕ takes a nonzero minimum value.

(3) d_ϕ does not attain the minimum value on H.

According to whether d_ϕ is (1), (2) or (3), we call ϕ elliptic, axial or parabolic respectively. It is immediate that ϕ is elliptic if and only if ϕ has a fixed point. First we investigate an axial isometry. Let ϕ be an axial isometry and ω be the minimum value of d_ϕ. Then the minimum set $C_\phi=\{p \in H: d_\phi(p)=\omega\}$ is not empty. Take a point $p \in C_\phi$. Then $\gamma_{p\phi(p)}(\omega)=\phi(p)=\phi\gamma_{p\phi(p)}(0)$. If $\dot{\gamma}_{p\phi(p)}(\omega)\neq\phi_*\dot{\gamma}_{p\phi(p)}(0)$, then, by the triangle inequality, $d(\gamma_{p\phi(p)}(t), \phi\circ\gamma_{p\phi(p)}(t))<d(p, \phi(p))$ for $0<t<\omega$. This implies that $\phi\circ\gamma_{p\phi(p)}(t)=\gamma_{p\phi(p)}(t+\omega)$, i.e., ϕ translates the geodesic through p and $\phi(p)$. Conversely, we assume that a complete geodesic γ is translated by ϕ. Let q be a point outside γ and p be the foot of q on γ. Then the geodesic segment γ_{pq} intersects γ perpendicularly. Since ϕ is an isometry, $\phi\circ\gamma_{qp}$ intersects γ perpendicularly also. From the convexity of $d(\gamma_{qp}(t), \phi\circ\gamma_{qp}(t))$, we obtain that $d_\phi(q)=d(q, \phi(q))\geqq d(p, \phi(p))=d_\phi(p)$. That is, d_ϕ takes the minimum value on γ. Hence it follows:

Proposition 12. *If ϕ is an axial isometry of H, then the minimum set of d_ϕ is the union of all geodesics which are translated by ϕ.*

A complete geodesic which is translated by an axial isometry ϕ is called an axis of ϕ. If two distinct complete geodesics are axes of the same isometry, then they are biasymptotic and bound a flat strip. Hence if H satisfies Axiom 2, for example, when H is of negative curvature, then an axis of an axial isometry is unique up to a parametrization.

Let ϕ be an isometry of H. Then ϕ induces a homeomorphism of $\bar{H}$ (cf. 2.2). Since $\bar{H}$ is homeomorphic to a closed ball, ϕ has a fixed point in $\bar{H}$ by Brouwer's fixed point theorem. For example, a translation of $\boldsymbol{R}^n$ fixes every point at infinity. Under the assumption of the Visibility axiom, axial and parabolic isometries are characterized in terms of the fixed point sets as follows:

Theorem 13 (Eberlein and O'Neill [31]). *Let H be a Hadamard manifold satisfying the Visibility axiom. Then any non elliptic isometry ϕ has at most two fixed points on $H(\infty)$. If ϕ has only one fixed point, then ϕ is parabolic. If ϕ has two fixed points $\{x, y\}$ in $H(\infty)$, then ϕ is axial and ϕ translates a complete geodesic joining x and y.*

In order to see the necessity of the Visibility axiom in the theorem, we shall give a sketch of the proof. Let $x \in H(\infty)$ be one of the fixed points of ϕ. Then, by the definition, for any geodesic ray γ with $\gamma(\infty)=x$, γ and $\phi\circ\gamma$ are asymptotic, and d_ϕ is monotone nonincreasing on γ. The convexity of d_ϕ means that there is a sequence $\{p_n\}$ such that $\{p_n\}$ converges to x and $\{d_\phi(p_n)\}$ converges to $\inf d_\phi$. Now we assume that ϕ has two fixed points x and y in $H(\infty)$. Then we can choose sequences $\{p_n\}$ and $\{q_n\}$ so that $p_n \to x$, $q_n \to y$ and $d_\phi(p_n) \to \inf d_\phi$, $d_\phi(q_n) \to \inf d_\phi$. Take an arbitrary point p of H. Since $\sphericalangle_p(p_n, q_n)$ converges to $\sphericalangle_n(x, y) > 0$ and H satisfies the Visibility axiom, there is a compact set K such that $\{\gamma_{p_nq_n}\}$ intersects K for large n. Then we can choose a subsequence of $\{\gamma_{p_nq_n}\}$ so that it converges to a complete geodesic, say γ. From the convexity of d_ϕ, we observe that d_ϕ takes the minimum value on γ. That is, ϕ is an axial isometry and γ is an axis of ϕ. Now we assume that ϕ has another fixed point $z \in H(\infty)$ other than $\{x, y\}$. Since d_ϕ is monotone nonincreasing on γ_{qz} for arbitrary $q \in \gamma$, d_ϕ takes the minimum value on γ_{qz}. Then for any point r on γ_{qz}, there is an axis of ϕ through r. By the flat strip theorem we obtain a flat half plane bounded by γ, i.e., a totally geodesic isometric imbedding $F\colon [0, \infty) \times \boldsymbol{R} \to H$ such that $F(0, t) = \gamma(t)$. The existence of a flat half plane contradicts to our assumption. In fact, any Hadamard manifold containing a flat half plane does not satisfy the Visibility axiom. And this implies that ϕ has at most two fixed points.

3.2. A splitting theorem of Wolf

In [88], Wolf proved that the existence of a Clifford translation implies the existence of a nontrivial flat de Rham factor. We shall review briefly his result.

We begin with fixing some definition. A Clifford translation is by definition a nontrivial isometry ϕ whose displacement function d_ϕ is constant. Since d_ϕ is convex, ϕ is a Clifford translation if d_ϕ is bounded on H.

Recall that, by the de Rham decomposition theorem, any Hadamard manifold H splits uniquely into $H_0 \times H_1$, where H_0 is a Euclidean space and H_1 contains no nontrivial flat splitting components. We call H_0 the flat de Rham factor of H. The splitting theorem due to Wolf [88] is:

Theorem 14. *If there is a Clifford translation ϕ of H, then H splits into $H_0 \times H_1$, where H_0 is the nontrivial flat de Rham factor of H and ϕ acts trivially on H_1 and as a translation on H_0.*

Associated with a Clifford translation ϕ, we define a vector field X_ϕ by $\phi(p) = \exp_p X_\phi(p)$. Take a geodesic such that $\dot{\tau}(0) \perp X_\phi$. Since d_ϕ is constant, τ and $\phi \circ \tau$ are biasymptotic and bound a flat strip. Considering every such a geodesic τ and using the fact a flat strip is totally geodesic, we see that X_ϕ is a parallel vector field. This implies that H splits into $\boldsymbol{R} \times H'$ and that ϕ acts on H as a translation of $\boldsymbol{R}$. In consequence, H has the nontrivial flat de Rham factor H_0 and ϕ acts as a translation of H_0.

As an application of Theorem 14, Wolf [88] proved:

Theorem 15. *A homogeneous Riemannian manifold of nonpositive curvature must split into a flat torus $\times$ a homogeneous Hadamard manifold.*

Hence, in order to classify homogeneous Riemannian manifolds of nonpositive curvature, it is sufficient to consider homogeneous Hadamard manifolds. This problem has been studied by Chen [19], Heintze [55] and Azencott and Wilson [4], [5] (cf. 1.6).

§ 4. Properly discontinuous groups

Let M be a complete Riemannian manifold of nonpositive curvature and H be the universal covering manifold of M. Then H is a Hadamard manifold and M can be represented as $M = H/\Gamma$ for some fixed point free properly discontinuous subgroup Γ of $I(H)$. In this section, we shall study the structure of these Γ. We shall divide fixed point free properly discontinuous groups into three classes and give characterizations of these classes in terms of the limit sets for Visibility manifolds.

4.1. Limit set

Let G be a subgroup of $I(H)$. Take a point $p \in H$ and consider the closure $\overline{G}p$ in $\overline{H}$ of the orbit $Gp = \{\phi(p) : \phi \in G\}$. Put $L(G) = \overline{G}p \cap H(\infty)$. It follows from the law of cosines that the set $L(G)$ is independent of the choice of p. We call $L(G)$ the limit set of G.

Definition. We say two points x and y in $H(\infty)$, not necessarily distinct, are dual relative to G if there exists a sequence $\{\phi_n\} \subset G$ such that $\phi_n p \to x$ and $\phi_n^{-1} p \to y$ as $n \to \infty$, for some point $p \in H$.

It turns out from the law of cosines that if x and y are dual relative to G, then $\phi_n p \to x$ and $\phi_n^{-1} p \to y$ for any point $p \in H$. If H is a Visibility manifold, this is equivalent to a more stronger statement. That is:

Proposition 16 (Eberlein and O'Neill [31]). *Let H be a Visibility manifold. Points x and y in $H(\infty)$ are dual relative to G if and only if, for any neighbourhoods U of x and V of y, there exists an element ϕ of G such that $\phi(\bar{H}-U)\subset V$ and $\phi^{-1}(\bar{H}-V)\subset U$.*

This can be seen from the following lemma which is proved by arguments similar to those in the proof of Theorem 13.

Lemma 17 ([31]). *Let H be a Visibility manifold and x be a point of $H(\infty)$. Then for any neighbourhood W of x and for any sequence $\{p_n\}\subset H$ which converges to x, $\sphericalangle_{p_n}(\bar{H}-W)\to 0$ as $n\to\infty$.*

In the following, we denote by Γ a fixed point free properly discontinuous subgroup of $I(H)$, and by π: $H\to H/\Gamma$ be the projection map.

A point of $O(\Gamma)=H(\infty)-L(\Gamma)$ is called an ordinary point. For a Visibility manifold, the limit set is characterized by the action of Γ on $\bar{H}$ as follows:

Proposition 18 (Eberlein and O'Neill [31]). *Let H be a Visibility manifold and Γ be a fixed point free properly discontinuous subgroup of $I(H)$. A point $x\in H(\infty)$ is an ordinary point if and only if there is a neighbourhood U of x such that $\phi U\cap U$ is empty, for any $1\neq\phi\in\Gamma$.*

This proposition can be interpreted as follows. We say a point $x\in H(\infty)$ is ultimately Γ-minimizing if for any geodesic ray belonging to x, there exists a number $a>0$ such that $\pi\circ\gamma|_{[a,\infty)}$ is a miminizing geodesic in $M=H/\Gamma$. Then the Proposition 18 implies that every ordinary point relative to Γ is ultimately Γ-minimizing if H is a Visibility manifold.

We observe another basic property of a Visibility manifold.

Proposition 19 (Eberlein and O'Neill [31]). *Let H be a Visibility manifold and Γ be a fixed point free porperly discontinuous subgroup of $I(H)$. If points $x\neq y\in H(\infty)$ are dual relative to Γ, then for any neighbourhoods U of x and V of y, there is an axial element $\phi\in\Gamma$ such that an axis γ of ϕ has the property that $\gamma(\infty)\in U$ and $\gamma(-\infty)\in V$.*

To see this proposition, we may assume that U and V are cones such that $\bar{U}\cap\bar{V}$ is empty. By Proposition 16, there exists an element $\phi\in\Gamma$ such that $\phi(\bar{H}-\bar{U})\subset V$ and $\phi^{-1}(\bar{H}-\bar{V})\subset U$. This implies that $\phi(\bar{V})\subset\bar{V}$ and $\phi^{-1}(\bar{U})\subset\bar{U}$. Since $\bar{U}$ and $\bar{V}$ are n-cells, ϕ has fixed points in $\bar{U}$ and $\bar{V}$. Since ϕ fixes two points at infinity, ϕ is an axial isomety.

Proposition 19 corresponds to the fact that if M is a compact

Riemannian manifold of negative curvature, then the set of all closed geodesics is dense in the set of all complete geodesics in M.

4.2. Axial groups

We shall divide the fixed point free properly discontinuous subgroups of $I(H)$ into three classes: axial groups, parabolic groups and fuchsian groups.

Definition. A fixed point free properly discontinuous subgroup Γ of $I(H)$ is said to be an axial group if there exist points x and y of $H(\infty)$ such that any element $1 \neq \phi \in \Gamma$ translates a complete geodesic joining x and y. The quotient manifold $M = H/\Gamma$ is called an axial manifold when Γ is axial.

If the curvature of H is negative or more generally if H satisfies Axiom 2, then the complete geodesic joining x and y is unique. Then every element of an axial group Γ translates the same geodesic, say γ. This implies Γ is an infinite cyclic group whose generator is an element that translates γ with minimum period.

This fact can be generalized to Hadamard manifolds by more involved arguments, namely:

Theorem 20 (Eberlein and O'Neill [31]). *An axial group of a Hadamard manifold is infinite cyclic.*

The topology of an axial manifold is rather simple. Let ϕ be a generator of Γ, and γ be an axis of Γ. Then any element of Γ translates γ and $\pi \circ \gamma$ is a closed geodesic of M with the minimum period. Furthermore, we can prove:

Theorem 21 (Eberlein and O'Neill [31]). *Let Γ be an axial group. Then the following hold.*

(1) *Every closed geodesic in $M = H/\Gamma$ is simply closed.*

(2) *Let γ be a closed geodesic with the minimum period, then γ is totally convex. Hence M is diffeomorphic to a vector bundle, the normal bundle of γ, over S^1.*

Remarks. (1) Closed geodesics of an axial manifold are not unique in general, as is easily seen by considering a cylinder. (2) The periods of closed geodesics of an axial manifold are also not unique in general. To see this, it is sufficient to consider a Möbius band. However, we can prove that the period of a closed geodesic is an integral multiple of the minimum period.

4.3. Parabolic groups

In this subsection, we shall investigate parabolic groups. The property of a parabolic isometry due to Eberlein and O'Neill [31] is fundamental.

Proposition 22. *Let ϕ is a parabolic isometry of a Hadamard manifold H. Then ϕ preserves limit spheres at some fixed point of ϕ in $H(\infty)$.*

Remark that the fixed point of a parabolic isometry is unique, if H is a Visibility manifold.

Now we define parabolic groups.

Definition. A fixed point free properly discontinuous subgroup Γ of $I(H)$ is said to be a parabolic group if there is a point $z \in H(\infty)$ such that z is the unique fixed point of any element of Γ. The quotient manifold $M = H/\Gamma$ is called a parabolic manifold if Γ is a parabolic group.

Let Γ be a parabolic subgroup of $I(H)$ and z be the unique fixed point of Γ. By Proposition 22, every element of Γ preserves limit spheres at z. Then a Busemann function f at z is invariant under the action of Γ.

Let L be a limit sphere at z and $\eta_L \colon H \to L$ be the projection defined in 2.4. We can easily show that the homeomorphism $\eta_L \times f \colon H \to L \times \boldsymbol{R}$ is compatible with the action of Γ. Summarizing these arguments, we arrive at:

Theorem 23 (Eberlein and O'Neill [31]). *Let Γ be a parabolic subgroup of $I(H)$ and z be the fixed point of Γ. Then the following hold:*

(1) *If a complete geodesic γ belongs to z, then $\pi \circ \gamma$ is a minimizing geodesic in $M = H/\Gamma$.*

(2) *A Busemann function at z induces a convex function on M.*

(3) *M is homeomorphic to $F \times \boldsymbol{R}$, where F is a level hypersurface of a convex function* (2) *above.*

The topology of a parabolic manifold is not so simple when compared with that of an axial manifold. To explain this point, we shall look at an example. Let F be a complete Riemannian manifold of nonpositive curvature. Then the warped product $\boldsymbol{R} \times_{e^t} F$ is a parabolic manifold. Clearly this manifold has the same homotopy type as that of F. This implies that the homotopy type of parabolic manifolds has no restrictions other than those of manifolds of nonpositive curvature.

4.4. Trichotomy

Let H be a Visibility manifold and $\Gamma \subset I(H)$ be a fixed point free properly discontinuous subgroup. Γ acts on $\bar{H}$ as a group of homeomorphisms.

Proposition 24 ([31]). *Let H and Γ be as above and $X \subset H(\infty)$ be a subset. If there exists an element $\phi \in \Gamma$, $\phi \neq 1$ such that $\phi X = X$, then X consists of one, two or infinitely many points. If $\sharp X$, the cardinality of X, is finite, then ϕ fixes X pointwisely.*

In fact, if $\sharp X = n$ is finite, ϕ^n acts trivially on X. Since Γ is torsion free, $\phi^n \neq 1$. It then follows, by Theorem 13, that the number of the fixed points of ϕ^n is at most two and this implies that $\sharp X \leqq 2$.

Since the limit set $L(\Gamma)$ of Γ is Γ-invariant, $\sharp L(\Gamma) = 1$, 2 or ∞.

Now we introduce the following:

Definition. A fixed point free properly discontinuous subgroup Γ of $I(H)$ is said to be fuchsian if Γ is neither axial nor parabolic. If Γ is fuchsian, the quotient H/Γ is called a fuchsian manifold.

Summarizing the above arguments, we obtain the characterizations of axial, parabolic and fuchsian groups, which are all due to Eberlein and O'Neill [31].

Theorem 25. *Let H be a Visibility manifold and Γ be a fixed point free properly discontinuous subgroup of $I(H)$. Then the following conditions are equivalent:*

(1) *$L(\Gamma)$ is a singleton $\{x\}$.*
(2) *Γ has a unique fixed point in $H(\infty)$.*
(3) *Γ is a parabolic group.*
(4) *Any element of Γ is parabolic.*

Theorem 26. *Let H and Γ be as in the above theorem. Then the following conditions are equivalent:*

(1) *$L(\infty)$ consists of two points.*
(2) *Γ fixes two points of $H(\infty)$.*
(3) *Γ is an axial group and hence is infinite cyclic.*

Also we obtain the following.

Theorem 27. *Let H and Γ be as above. Then the following conditions are equivalent:*

(1) *$L(\infty)$ is an infinite set.*
(2) *Γ has no fixed points in $H(\infty)$.*
(3) *Γ is a fuchsian group.*

§ 5. The duality condition and symmetric spaces

Let H be a Hadamard manifold. A fixed point free properly discontinuous subgroup Γ of $I(H)$ is called a *lattice* if the quotient manifold

H/Γ is of finite volume. A lattice is said to be uniform or nonuniform according to whether H/Γ is compact or noncompact. It is well-known that a symmetric space of noncompact type admits a lattice (Borel [11]). In [54], Heintze observed that if a homogeneous Hadamard manifold H of negative curvature has a compact quotient, then H is a rank 1 symmetric space. Goto and Goto [36] proved that if a 3-dimensional Hadamard manifold with negatively pinched curvature admits a lattice, then it is a rank 1 symmetric space or its isometry group is discrete. Chen and Eberlein [21] and Eberlein [27], [29] generalized these results.

If Γ is a lattice of H, then Γ and $I(H)$ satisfy some density property, called the duality condition. In the case of symmetric spaces, Selberg's S-property implies the duality condition (Heintze [56]). Under the assumption that $I(H)$ satisfies the duality condition, we can show that any abelian normal subgroup of $I(H)$ consists of Clifford translations. Hence if H has no flat de Rham factor, then $I_0(H)$ is semi-simple unless $I(H)$ is discrete. From this fact, it follows that a Hadamard manifold which admits a lattice splits into a product of a Euclidean space, a Hadamard manifold whose isometry group is discrete and a symmetric space.

5.1. The duality condition

In Section 4.1, we defined the notion of the duality of a pair of points at infinity relative to a subgroup G of $I(H)$. Now we introduce:

Definition. Let G be a subgroup of $I(H)$. We say that G satisfies the duality condition if for any complete geodesic γ in H, $\gamma(\infty)$ and $\gamma(-\infty)$ are dual relative to G.

First we shall observe that a lattice satisfies the duality condition.

Let $SM=\{v \in TM: \|v\|=1\}$ be the unit tangent bundle of a complete Riemannian manifold M. The geodesic flow $\{T_t\}$, $T_t: SM \to SM$ is defined by setting $T_t(v)=\dot{\gamma}_v(t)$. A unit vector $v \in SM$ is said to be a nonwandering point if there exist sequences $\{t_n\} \subset \boldsymbol{R}$ with $t_n \to \infty$ and $\{v_n\}$ with $v_n \to v$ such that $\{T_{t_n} v_n\}$ converges to v. The set $\Omega=\{v \in SM: v$ is a nonwandering point$\}$ is called the nonwandering set of the geodesic flow.

Proposition 28 (Eberlein [23], [24]). *Let Γ be a fixed point free properly discontinuous subgroup of $I(H)$ and $M=H/\Gamma$. Then Γ satisfies the duality condition if and only if the nonwandering set $\Omega=SM$. Furthermore, if H is a Visibility manifold, $\Omega=SM$ if and only if $H(\infty)=L(\Gamma)$.*

It is well-known that if $M=H/\Gamma$ is of finite volume, i.e., if Γ is a lattice, then $\Omega=SM$, hence Γ satisfies the duality condition. This is a consequence of the general theory of geodesic flows.

It follows immediately from the definition that if G, G' are subgroups of $I(H)$ such that $G \subset G'$ and if G satisfies the duality condition, then G' also satisfies the duality condition. Hence if H admits a lattice, $I(H)$ satisfies the duality condition.

5.2. Isometry group

We shall continue to investigate the structure of the isometry group of a Hadamard manifold. In [16], Byer showed that if a Hadamard manifold H of strictly negative curvature admits a uniform lattice, then $I(H)$ is discrete or semi-simple. In this subsection, we shall give a generalization of this result.

Now we assume, more generally, that $I(H)$ satisfies the duality condition. Under this assumption, the existence of a nontrivial abelian normal subgroup of $I(H)$ implies a strong consequence:

Proposition 29 (Chen and Eberlein [21]). *Let $A \subset I(H)$ be a nontrivial abelian subgroup. If the normalizer $G = N_{I(H)}(A)$ of A in $I(H)$ satisfies the duality condition, then every element of A is a Clifford translation.*

We shall give a sketch of the proof. For the details, see Chen and Eberlein [21]. First let $x \in H(\infty)$, and then take a point $y \in H(\infty)$ which can be joined with x. If $z \in H(\infty)$ is a point which can be joined with x, then it is contained in $\overline{Gy}$. In fact, take a geodesics γ and σ such that $\gamma(-\infty) = \sigma(-\infty) = x$ and $\gamma(\infty) = y$, $\sigma(\infty) = z$. Take a point p on γ. Since G satisfies the duality condition, there exists a sequence $\{\phi_n\} \subset G$ such that $\phi_n p \to z$ and $\phi_n^{-1} p \to x$. Since $\measuredangle_p(\phi_n p, \phi_n y) = \measuredangle_{\phi_n^{-1} p}(p, y) \leqq \measuredangle_p(\phi_n^{-1} p, x) \to 0$, we obtain $\phi_n y \to z$. From the assumption $L(G) = H(\infty)$, we find that A contains no elliptic elements other than the identity. Hence $L(A)$ is not empty. Take a point $x \in L(A)$. Next we observe that x can be joined only with points of $L(A)$. Let $\{\phi_n\} \subset A$ be a sequence such that $\phi_n p \to x$. We may assume that $\{\phi_n^{-1} p\}$ converges to a point of $L(A)$, say y. Take a point $z \in H(\infty)$ which can be joined with y. For a point p on a complete geodesic joining z and y, $\measuredangle_p(\phi_n p, \phi_n z_p) = \measuredangle_{\phi_n^{-1} p}(p, z) \leqq \measuredangle_p(y, \phi_n^{-1} p) \to 0$. Then $\phi_n z$ converges to x. Since $L(A)$ is invariant under G, z can be joined only with points of $\overline{Gy} \subset L(A)$. Put $y_n = \gamma_{p\phi_n z}(-\infty)$. Then $y_n \in L(A)$ and $y_n \to \gamma_{px}(-\infty) \in L(A)$. This implies x can be joined with a point of $L(A)$, hence only with points of $L(A)$. Let $y \in L(A)$ be a point joined with x and γ be a complete geodesic joining x and y. Take a point p on γ and $\{\phi_n\} \subset A$ such that $\phi_n p \to x$. Since A acts trivially on $L(A)$, $\measuredangle_p(\phi_n^{-1} p, y) = \measuredangle_p(\phi_n^{-1} p, \phi_n^{-1} y) = \measuredangle_{\phi_n p}(p, y) \leqq \measuredangle_p(\phi_n p, y) \to 0$. Summarizing, we conclude that a point $x \in L(A)$ can be joined with a unique point $y \in H(\infty)$ belonging to $L(A)$.

If a point $x \in H(\infty)$ can be joined with only one point $y \in H(\infty)$, we say y is an antipodal point of x. In our case, x and y are antipodal. Then integral curves of the vector field $V(p, x)$ (for definition, cf. 2.4) are biasymptotic to each other. By the flat strip theorem, $V(p, x)$ is a parallel vector field and the flow $\{\phi_t\}$ of $V(p, x)$ are Clifford translations. By Wolf's splitting theorem, H splits into $H_0 \times H_1$, where H_0 is the nontrivial flat de Rham factor. Since H_1 contains no nontrivial flat de Rham factor, $A \subset I(H_0) \times \{1\}$. Noticing that $I(H_0)$ contains no parabolic elements, we can see that every element of A is a Clifford translation. This completes the sketch of the proof.

From this proposition, we obtain:

Theorem 30 (Chen and Eberlein [21]). *Assume that $I(H)$ satisfies the duality condition and that H contains no nontrivial flat de Rham factor. Then $I(H)$ is discrete or $I_0(H)$ is semi-simple Lie group without center and without compact factors.*

This can be seen as follows. We assume that $I_0(H)$ is not trivial We first show that $I_0(H)$ is not compact. If $I_0(H)$ is compact, it has a fixed point p by Cartan's fixed point theorem. Since $I_0(H)$ is normal in $I(H)$, every element of $I_0(H)$ fixes $I(H)p$ pointwisely. Since $L(I(H)) = H(\infty)$, we conclude that $I_0(H) = \{1\}$. Next we assume $I_0(H)$ is not semi-simple. From Proposition 29, we find that H has a nontrivial flat de Rham factor which is a contradiction. Repeating the argument in the first part, we can also show that $I_0(H)$ has no nontrivial compact factors.

5.3. Splitting theorem

In this section, we shall give a splitting theorem of a Hadamard manifold under the assumption that the isometry group satisfies the duality condition.

First we prepare the following:

Proposition 31 (Eberlein [29]). *Let B be a closed totally geodesic submanifold of a Hadamard manifold H and G be a subgroup of $I(H)$ defined by $G = \{\phi \in I(H) : \phi(B(\infty)) = B(\infty)\}$. Assume that $L(G) = H(\infty)$. Then there is a totally geodesic submanifold B' such that H splits into $B \times B'$.*

The splitting in this theorem is obtained in the following manner. From the assumption $L(G) = H(\infty)$, we find that for any point $p \in H$, there exists a totally geodesic submanifold B_p such that $B_p(\infty) = B(\infty)$. Let N be the distribution on H defined by $N_p = (B_p)_p$. We can show that N and $N^\perp$ are parallel distributions by a method similar to that used in the proof of Wolf's splitting theorem, replacing the flat strip theorem by the sandwich lemma. Hence H splits into $B \times B'$ where B' is a leaf of $N^\perp$.

Theorem 32 (Eberlein [29]). *Let H be a Hadamard manifold without* *flat de Rham factor and $G \neq \{1\}$ be a closed connected subgroup of $I(H)$. Assume that the normalizer $N_{I(H)}(G)$ of G in $I(H)$ satisfies the duality condition. Then H splits into $H_1 \times H_2$, where*

(1) *H_1 is a symmetric space of noncompact type, and*

(2) $H_2(\infty) = \mathrm{Fix}(G) \cap H(\infty)$.

We shall give a sketch of the proof. Since the normalizer of G satisfies the duality condition, every element of the center of G is a Clifford translation. Hence G is without center, for H has no flat de Rham factor. By arguments similar to those in the proof of Theorem 30, we can show that G is a semi-simple Lie group without compact factors. Let K be a maximal compact subgroup of G. Then it follows that $\mathrm{Fix}(K) \cap H(\infty) = \mathrm{Fix}(G) \cap H(\infty)$. Hence there exists a totally geodesic submanifold H_1 of H such that $H_1(\infty) = \mathrm{Fix}(G) \cap H(\infty)$. Since $N_{I(H)}(G)$ satisfies the duality condition, there is a totally geodesic submanifold H_2 such that $H = H_1 \times H_2$ by Proposition 31. This is the desired splitting.

Summarizing the above arguments, we obtain:

Theorem 33 (Eberlein [29]). *Let H be a Hadamard manifold such that $I(H)$ satisfies the duality condition. Then H splits into $H_0 \times H_1 \times H$ with the following properties:*

(1) *H_0 is a Euclidean space,*

(2) *H_1 is a symmetric space of noncompact type, and*

(3) *$I(H_2)$ is discrete and satisfies the duality condition.*

5.4. Lattices

If a Hadamard manifold H admits a lattice, then H splits as in Theorem 33. We now remark some results concerning lattices in a Hadamard manifold.

Let Γ be a lattice of a Hadamard manifold H. We say that Γ is reducible if there exists a finite covering of H/Γ which has a nontrivial splitting as a Riemannian manifold. If Γ is not reducible, we say Γ is irreducible. Even if H is reducible as a Riemannian manifold, a lattice in H is not necessarily reducible (cf. [82]). Eberlein [27], [29] investigated the reducibility of a lattice in a Hadamard manifold. One of his results is the following:

Theorem 34 (Eberlein [29]). *Let Γ be an irreducible lattice in a Hadamard manifold H. Assume that Γ contains no Clifford translations. Then:*

(1) *$I(H)$ is discrete and Γ is of finite index in $I(H)$ and H is irreducible, or*

(2) *$H = H_0 \times H_1$ where H_0 is a Euclidean space and H_1 is a symmetric*

space of noncompact type.

As a special case, it follows the following:

Theorem 35. *Let H be a Hadamard manifold without flat de Rham factor and $H=H_1\times H_2$ be a nontrivial splitting. Assume that $I(H_1)$ is a discrete group. Then there are no irreducible lattices in H.*

Lattices in a symmetric space have strong rigidity properties. Mostow's rigidity theorem [74] can be stated as follows: Let H, H' be symmetric spaces of noncompact type without two dimensional components. If lattices Γ in H and Γ' in H' are isomorphic as a group, then H/Γ and H'/Γ' are isometric up to normalizing constants of the metrics.

If H/Γ is a compact locally symmetric space of negative curvature, then a small perturbation of the metric yields a new compact Riemannian manifold of negative curvature. However, if H is a symmetric space of noncompact type whose rank is greater than 2, a compact quotient H/Γ has a strong rigidity. That is, if the fundamental group of a compact Riemannian manifold M^* of nonpositive curvature is isomorphic to the fundamental group of $M=H/\Gamma$, then M^* is isometric to M up to normalizing constants of the metrics, where H is a symmetric space of noncompact type whose rank is greater than 2 and Γ is an irreducible uniform lattice in H (Eberlein [28], Gromov).

§ 6. Fundamental groups of compact Riemannian manifolds of nonpositive curvature

Let M be a complete Riemannian manifold of nonpositive curvature. Then the universal covering manifold H of M is a Hadamard manifold and M can be written as H/Γ with Γ, some fixed point free properly discontinuous subgroup of $I(H)$.

Taking base points of M and H, Γ can be identified with the fundamental group of M in a canonical manner. This identification depends on the choice of base points, but it is not essential in the following arguments. We denote the fundamental group of M simply by $\pi_1(M)$, for it is not necessary to specify the base point of the fundamental group.

It should be noted that the fundamental group is the most important topological invariant in the sense that the homotopy type of M is determined completely by the fundamental group.

Now let M be a compact manifold of nonpositive curvature. From the compactness of M, if follows that $\pi_1(M)$ is finitely generated. Note that the volume of M is finite and the volume of the universal covering manifold is infinite. This implies that $\pi_1(M)$ is an infinite group.

It is known, by a theorem of Smith [85], that a finite cyclic group of isometries of a Hadamard manifold has a common fixed point. Since Γ is fixed point free, $\pi_1(M)$ does not contain a nontrivial finite cyclic group, i.e., $\pi_1(M)$ is torsion free.

To verify these properties, we only need the fact that the universal covering is diffeomorphic to the Euclidean space. So these are also true for the fundamental group of a compact manifold without conjugate points.

6.1. The fundamental groups of compact manifolds of negative curvature

First we notice a basic fact concerning the fundamental group of a compact Riemannian manifold of nonpositive curvature. Let $M=H/\Gamma$ be a compact Riemannian manifold of nonpositive curvature. Let K be a compact fundamental domain. Then for any element ϕ of Γ, the restriction of the displacement function d_ϕ to K attains the minimum value at a point of K, say p. It is easily verified that ϕ translates a complete geodesic through p and $\phi(p)$, and ϕ is an axial isometry. In consequence, every element of Γ is an axial isometry.

Now we assume that M is a compact Riemannian manifold of negative curvature. The first remarkable result was obtained by Preismann [77]:

Theorem 36. *Let M be a compact Riemannian manifold of negative curvature. Then every nontrivial abelian subgroup of $\pi_1(M)$ is infinite cyclic.*

In fact, let A be a nontrivial abelian subgroup of $\pi_1(M)\cong\Gamma$. Take ϕ, $\psi\in A$, $\phi\neq 1$ and $\psi\neq 1$. If γ is an axis of ϕ, then, since ϕ and ψ commute, ϕ also translates the geodesic $\psi\circ\gamma$. Since the curvature is negative, γ and $\psi\circ\gamma$ coinside up to parametrizations. Hence every element of A translates the same geodesic. This means that A is infinite cyclic.

Byer [15] generalized Preismann's theorem in the following way.

Theorem 37. *Let M be a compact Riemannian manifold of negative curvature. Then every nontrivial solvable subgroup of $\pi_1(M)$ is infinite cyclic.*

Some other generalizations of Preismann's theorem has been obtained by Chen [20] and Eberlein and O'Neill [31].

In 6.2, we shall see that when M is of nonpositive curvature, abelian subgroups of $\pi_1(M)$ give some information concerning totally geodesic flat submanifolds of M.

From Preismann's theorem, we know that the product manifold M

$=M_1\times M_2$ of compact manifolds M_1 and M_2 does not admit a Riemannian metric with negative curvature. But, it is not known whether there exists a Riemannian metric with positive curvature on a product manifold of compact manifolds. This is a problem known as Hopf's problem.

Next we state a theorem of Milnor [70]. To this end, we define the growth function of a finitely generated group G. Let $S=\{g_1, \cdots, g_s\}$ be a system of generators of G. For a positive integer r, we denote by $g_S(r)$ the number of elements of the set $\{g=g_{i_1}^{n_1}g_{i_2}^{n_2}\cdots g_{i_t}^{n_t}: |n_1|+\cdots+|n_t|\leqq r\}$. We call the function $g_S(r)$ the growth function of G relative to a system of generators S. The growth function depends on the choice of a system of generators S, but the notions in the following definition does not.

Definition. Let G be a finitely generated group and S be a system of generators of G. G is said to be of polynomial growth if there exist $a>1$ and $c>0$ such that $g_S(r)\leqq cr^a$. G is said to be of exponential growth if there exists $a>1$ such that $g_S(r)\geqq a^r$.

For example, a free abelian group $G=\boldsymbol{Z}^r$ is of polynomial growth and a finitely generated nonabelian free group is of exponential growth. For more arguments about the growth function, see [51], [71], [89].

A result of Milnor is the following:

Theorem 38 (Milnor [70]). *Let M be a compact Riemannian manifold. Then*:

(1) *If Ricci curvature of M is positive semidefinite, then $\pi_1(M)$ is of polynomial growth.*

(2) *If M is of negative curvature, then $\pi_1(M)$ is of exponential growth.*

The proof is done by comparing the growth order of the volume of geodesic balls. More precisely, in the case of positive semidefinite Ricci curvature, we use a volume comparison theorem of Bishop [8], and in the case of negative curvature, we use a standard volume comparison theorem.

The second part of the theorem was generalized by Eberlein as follows:

Theorem 39 (Eberlein [26]). *Let H be a Visibility manifold and Γ a fuchsian subgroup of $I(H)$. Then there exists an infinite subset S of Γ such that S generates a nonabelian free subgroup of Γ.*

To prove this, we need the following two facts, and refer to [23] for their proof.

1. Let H be a Visibility manifold and Γ be a fixed point free properly discontinuous subgroup of $I(H)$. If the limit set $L(\Gamma)$ is an infinite set, then any pair of points x and y of $L(\Gamma)$, not necessarily distinct, are

dual relative to Γ.

2. Let H and Γ be as above. If $L(\Gamma)$ is an infinite set, then $L(\Gamma)$ is a perfect set, i.e., $L(\Gamma)=L(\Gamma)^d$, the derived set of $L(\Gamma)$.

Now we shall sketch the proof of the theorem. Take a point $z_1 \in L(\Gamma)$ and a neighbourhood U_1 of z_1 so that $\bar{U}_1 \not\supset L(\Gamma)$. Take a point $z_2 \in L(\Gamma)-\bar{U}_1$ and a neighbourhood U_2 of z_2 so that $\bar{U}_1$ and $\bar{U}_2$ are disjoint and $\bar{U}_1 \cup \bar{U}_2 \not\supset L(\Gamma)$. Inductively, we choose, for each positive integer n, a point $z_n \in L(\Gamma)$ a neighbourhood U_n of z_n so that $\bar{U}_n$, $n=1, 2, \cdots$ are mutually disjoint and $\bigcup_i \bar{U}_i \not\supset L(\Gamma)$. Since z_{2n-1} and z_{2n} are dual relative to Γ, there exists $\phi_n \in \Gamma$, such that $\phi_n(H-U_{2n-1}) \subset U_{2n}$ and $\phi_n^{-1}(H-U_{2n}) \subset U_{2n-1}$. Put $S=\{\phi_1, \phi_2, \cdots, \phi_n, \cdots\}$. Then there are no relations among elements of S. In fact, let $\phi=\phi_{i_1}^{n_1}\phi_{i_2}^{n_2} \cdots \phi_{i_r}^{n_r}$, where r is a positive integer and i_k, $1 \leqq k \leqq r$, are positive integers such that $i_k \neq i_{k+1}$ and n_k, $1 \leqq k \leqq r$, are non zero integers. Take a point $p \in H-\bigcup_j \bar{U}_{i_j}$. Then, by virtue of the choice of $\{\phi_i\}$ and $\{U_i\}$, we can show that $\phi(p) \in U_{2i_1}$ if $n_1>0$ and that $\phi(p) \in U_{2i_1-1}$ if $n_1<0$. Hence $\phi(p) \neq p$. This means that $\phi \neq 1$ and that there are no relations among elements of S.

Since a finitely generated nonabelian free group is of exponential growth, the above theorem implies the second part of Theorem 38.

The assumption of the Visibility of H in the above theorem can be weakened somewhat to the assumption that there exists a complete geodesic in H which does not bound a flat half plane (Ballmann [6], [7]).

6.2. The fundamental groups of compact manifolds of nonpositive curvature

In 6.1, we have studied the fundamental group of a compact manifold of negative curvature. In this section, we shall study the case of nonpositive curvature.

At first, we consider compact flat manifolds. The fundamental groups of compact flat manifolds have been characterized by Bieberbach (cf. Wolf [90]). Let M be a compact flat manifold of dimension n. Then M can be written as $\boldsymbol{R}^n/\Gamma$ where Γ is a fixed point free properly discontinuous subgroup of the Euclidean transformation group.

Theorem 40 (Bieberbach). *Let $M=\boldsymbol{R}^n/\Gamma$ be a compact flat manifold. Then there exists a maximal free abelian normal subgroup Δ of rank n such that Γ/Δ is a finite group.*

Conversely, if a torsion free finitely generated group Γ has a free abelian normal subgroup of rank n which is of finite index and maximal abelian, then Γ is isomorphic to the fundamental group of some compact flat manifold of dimension n.

In this theorem, the normal abelian subgroup Δ is given in fact as

$\Gamma \cap \boldsymbol{R}^n$, where an element of $\boldsymbol{R}^n$ is identified with the translation defined by it. The above theorem says that a compact flat manifold is covered by a flat torus as a finite normal covering, and this is the geometric meaning of Bieberbach's theorem.

A group is called a Bieberbach group if it has the properties in the above theorem.

Bieberbach's theorem has been generalized by Wolf [89] and Yau [94]. Wolf showed that if the fundamental group of a compact manifold of nonpositive curvature has a nilpotent group of finite index, then M is flat. This was generalized by Yau as follows:

Theorem 41 (Yau [94]). *Let M be a compact manifold of nonpositive curvature. Then every solvable subgroup of $\pi_1(M)$ is a Bieberbach group. In particular, if $\pi_1(M)$ is solvable, then M is a flat manifold.*

Now we will investigate abelian subgroups and the center of the fundamental group. Let $M=H/\Gamma$ be a compact Riemannian manifold of nonpositive curvature. We assume that $\pi_1(M)=\Gamma$ contains a free abelian subgroup A of rank k. Let $\{\phi_1, \cdots, \phi_k\}$ be a system of generators of A. Then the minimum set C_{ϕ_1} of the displacement function d_{ϕ_1} of ϕ_1 splits into $D_1 \times \boldsymbol{R}$ such that $\{d\} \times \boldsymbol{R}$ is a ϕ_1-invariant geodesic, i.e., an axis of ϕ_1, where D_1 is a totally convex subset of H. If γ is a ϕ_1-invariant complete geodesic, $\phi_i \circ \gamma$ is also ϕ_1-invariant, since ϕ_1 and ϕ_i commute. Then ϕ_2 acts on C_{ϕ_1}. Hence we obtain that $C_{\phi_1} \cap C_{\phi_2}$ is not empty and $C_{\phi_1} \cap C_{\phi_2}$ splits into $D_2 \times \boldsymbol{R}^2$ such that $\{d\} \times \boldsymbol{R}^2$ is spanned by axes of ϕ_1 and ϕ_2. We continue this argument and finally obtain that $C_{\phi_1} \cap \cdots \cap C_{\phi_k} = D_k \times \boldsymbol{R}^k$ and that $\boldsymbol{R}^k$ is spanned by axes of ϕ_i, $1 \leqq i \leqq k$. Hence $\boldsymbol{R}^k/\{\phi_1, \cdots, \phi_k\}$ is a flat torus of dimension k which is immersed totally geodesically in M.

Theorem 42 (Gromoll and Wolf [50], Lawson and Yau [69]). *Let M be a compact Riemannian manifold of nonpositive curvature. If the fundamental group $\pi_1(M)$ contains a free abelian subgroup of rank k, then there exists a totally geodesically immersed flat torus of dimension k.*

If we assume that $\pi_1(M)$ has nontrivial center, we can assert more. Namely, if the fundamental group $\pi_1(M)$ of a compact Riemannian manifold of nonpositive curvature has nontrivial center Z, then Z is a free abelian group of rank k, $k \leqq n = \dim M$, and M is foliated by k-dimensional totally geodesically immersed flat tori (Lawson and Yau [69], O'Sullivan [75]).

Furthermore, Eberlein [30] determined the structure of a compact Riemannian manifold of nonpositive curvature whose fundamental group has nontrivial center. We begin by describing a general procedure for constructing examples of such manifolds. For a given integer k, let Z be a lattice in $\boldsymbol{R}^k$ and denote by T^k the torus $\boldsymbol{R}^k/Z$. Let H_2 be a Hadamard manifold and Γ be a discrete subgroup of $I(H_2)$ with trivial center, possibly containing elliptic elements, such that the quotient H_2/Γ is compact. Let $\rho\colon \Gamma \to T^k$ be a homomorphism whose kernel contains no elliptic elements. We define an action of Γ to $T^k \times H_2$ by $\phi(\xi, h) = (\rho(\phi)\xi, \phi(h))$ for each $\phi \in \Gamma$, $\xi \in T^k$and $h \in H_2$. Then the quotient manifold $M = T^k \times H_2/\Gamma$ is a compact manifold whose fundamental group has nontrivial center Z. For convenience, we call a manifold constructed as above a canonical manifold with nontrivial center.

Theorem 43 (Eberlein [30]). *Let M be a compact Riemannian manifold of nonpositive curvature whose fundamental group has nontrivial center. Then*:

(1) *M is a canonical manifold with nontrivial center, and*

(2) *there exists a finite covering M^* of M such that any finite covering M^{**} of M^* is diffeomorphic to the product $T^k \times \tilde{M}$ where $\tilde{M}$ is a compact manifold which admits a metric of nonpositive curvature and whose fundamental group has trivial center.*

The first part can be seen as follows. Since every element of center Z is a Clifford translation, H splits into $\boldsymbol{R}^k \times H_2$, where H is the universal covering of M. The action of Γ preserves the splitting. We denote by $p_i\colon \Gamma \to I(H_i)$ the projection, where $H_1 = \boldsymbol{R}^k$. Then $\Gamma_1 = p_1(\Gamma)$ act as a translations and $\Gamma_2 = p_2(\Gamma)$ is a discrete subgroup of $I(H_2)$ (cf. [27]). Let $P\colon \boldsymbol{R}^k \to T^k = \boldsymbol{R}^k/Z$ be the projection. We define $\rho\colon \Gamma_2 \to T^k$ by $\rho(p_2(\phi)) = P(p_1(\phi))$. Then we can show that M is isometric to $T^k \times H_2/\Gamma$.

For the proof of the second part of the theorem, we need deeper observations and must refer to Eberlein [30].

Finally, we state a splitting theorem.

Theorem 44 (Gromoll and Wolf [50], Lawson and Yau [69]). *Let M be a compact manifold of nonpositive curvature. If the fundamental group $\pi_1(M)$ is without center and splits into $\pi_1(M) = G_1 \times G_2$ as a group, then M has a splitting $M = M_1 \times M_2$ so that $\pi_1(M_1) = G_1$ and $\pi_1(M_2) = G_2$.*

§ 7. Kähler Hadamard manifolds and geometric function theory

We shall consider, in this section, Kähler Hadamard manifolds, i.e., simply connected complete Kähler manifolds of nonpositive curvature. Typical examples of Kähler Hadamard manifolds are the complex Euclidean space $\boldsymbol{C}^n$ with the standard flat metric and an open ball in $\boldsymbol{C}^n$ with the Bergman metric. It is well-known that the holomorphic sectional curvature of an open ball is negative constant and its sectional curvature is strictly negative.

It might be reasonable to ask how the metric structure of a Kähler manifold restricts the function theoretic properties of the underlying complex manifold. The problem that we will consider here is that under what geometrical conditions a Kähler manifold posseses properties similar to those of the above model spaces. We expect that if the curvature of a Kähler manifold is sufficiently negative, then the manifold has properties similar to those of a bounded domain of $\boldsymbol{C}^n$, and if the curvature is not so negative, then the manifold has properties similar to those of $\boldsymbol{C}^n$.

Several surveys concerning this topic have been given already (Greene [40], [41], Greene and Wu [45], Wu [93]).

7.1. Steinness

A complex manifold is called a Stein manifold if it is biholomorphic to a closed submanifold of some complex Euclidean space. It is well-known that every noncompact Riemann surface is Stein, while this is false for higher dimensional noncompact complex manifolds. Aomoto [3] conjectured that a Kähler Hadamard manifold is Stein, and this is proved by Wu [91] and Sasaki and Suzuki [80].

By Grauert's theorem (cf. [22]), in order to prove that a complex manifold M is Stein, it is sufficient to find a smooth strictly plurisubharmonic exhaustion function on M.

Let H be a Kähler Hadamard manifold. We have seen in 1.4, that the function $d(\cdot, p)^2$, for any point p of H, is smooth and strictly convex. So we are going to show that a strictly convex function on a Kähler manifold is strictly plurisubharmonic. Now we recall some definitions. Let M be a complex manifold and f a real valued C^2-function on M. The Levi form of f is defined to be the hermitian form $Lf=\sum (\partial^2 f/\partial z_i \partial \bar{z}_j) dz_i d\bar{z}_j$, where $(z_1, \cdots, z_n)$ is a local coordinate system. We say a C^2-function f is plurisubharmonic (strictly plurisubharmonic) if the Levi form Lf is positive semi-definite (positive definite) everywhere. A real valued function f on M is called an exhaustion function if the set $\{x \in M : f(x) \leqq c\}$ is compact for any $c < \sup f$. The relation of the convexity and the plurisubharmonicity is given by the following.

Proposition 45 (cf. [44]). *Let M be a Kähler manifold and J be the complex structure of M. For a real valued smooth function f on M, it holds that*

$$Lf(X, X) = \nabla^2 f(X, X) + \nabla^2 f(JX, JX).$$

As a corollary, we obtain:

Theorem 46 ([80], [91], cf. [46] also). *A Kähler Hadamard manifold is a Stein Manifold.*

Proposition 45 can be shown using the existence of the normal coordinate systems on a Kähler manifold. We remark that the assertion in the proposition is false for a hermitian manifold. In fact, the convexity of a function does not mean the plurisubharmonicity in general (cf. Klembeck [62]).

A general method of estimating the hessian of a function on a Riemannian manifold is to use, so called, the comparison argument. We refer to Siu and Yau [84], Greene and Wu [46] and Kasue [60] for the hessian comparison theorem.

It is also known that a complete noncompact Kähler manifold of positive curvature is Stein (Wu [92]).

7.2. The 1-dimensional case and model spaces

Let us first discuss the 1-dimensional case. This case is very special in the sense that a 1-dimensional simply connected noncompact complex manifold is biholomorphic to the unit disc or the complex plane $\boldsymbol{C}$. Among characterizations of these space, the following, due to Milnor, is a most typical characterization in connection with arguments in higher dimensional manifolds.

Theorem 47 (Milnor [72]). *Let M be a simply connected* 1*-dimensional Kähler manifold. Suppose that the metric is rotationally symmetric at some point o of M. Let r denote the distance function relative to o. Then*:

(1) *If the Gaussian curvature is $\geqq -1/r^2 \log r$ for large r, then M is biholomorphic to $\boldsymbol{C}$.*

(2) *If the Gaussian curvature is nonpositive and is $\leqq -(1+\varepsilon)/r^2 \log r$ for large r, then M is biholomorphic to the unit disc.*

We shall sketch the proof of the theorem. Since the metric is assumed to be rotationally symmetric at o, it can be represented as $dr^2 + f(r)^2 d\theta^2$, where (r, θ) being the geodesic polar coordinate at o. Then the Gaussian curvature is a function of r and which is denoted by $K(r)$. It is well-known that the function $f(r)$ satisfies the Jacobi equation, $f''(r) + K(r)f(r)=0$, $f(0)=0$ and $f'(0)=1$. Define a mapping $\Phi\colon M \to \boldsymbol{C}$ by $\Phi(r, \theta) = \left(\exp\left\{\int_1^r 1/f\right\}, \theta\right)$. We can see by direct calculation that Φ is conformal, i.e., is holomorphic. Hence, if $\int_1^\infty 1/f(r) < \infty$, then M is biholomorphic to the unit disc, and if $\int_0^\infty 1/f(r) = \infty$, then M is biholomorphic to $\boldsymbol{C}$.

It is observed, by Greene and Wu [46], that the assumption of rotational symmetricity of the metric can be omitted using Ahlfor's criterion [1] of the hyperbolicity of Riemann surfaces.

Milnor's result mentioned above can be generalized to higher dimensional manifolds. Let M be a complete Riemannian manifold. We call a point $o \in M$ a pole if the exponential mapping at o, $\exp_o: M_o \to M$, is a diffeomorphism.

Definition. Let M be a complete Riemannian (Kähler) manifold with a pole o. M is called a model (Kählerian model) if any orthogonal (unitary) transformation of M_o can be realized as the linear isotropy of an isometry of M.

Let M be a Riemannian manifold with a pole o. Let ∂_p denote the radial vector at p, $p \neq o$, i.e., the tangent vector defined by $\partial_p = \dot{\gamma}_{op}(l)$, $l = d(o, p)$. The sectional curvature of a tangent plane containing the radial vector is called the radial curvature. If M is a Kähler manifold, the sectional curvature $K(\partial_p, J\partial_p)$ is called the holomorphic radial curvature.

Under these terminologies, we obtain:

Theorem 48 (Shiga [81]). *Let H be an n-dimensional Kählerian model. Then H is biholomorphic to either an open ball in $\boldsymbol{C}^n$ or $\boldsymbol{C}^n$ itself. Furthermore,*

(1) *if the holomorphic radial curvature is $\geqq -1/r^2 \log r$ for large r, then H is biholomorphic to $\boldsymbol{C}^n$, and*

(2) *if the holomorphic radial curvature is nonpositive and is $\leqq -(1+\varepsilon)/r^2 \log r$ for large r, then H is biholomorphic to an open ball.*

7.3. Kähler Hadamard manifolds with sufficiently negative curvature

First we recall some properties of bounded domains of $\boldsymbol{C}^n$. A simply connected bounded domain of $\boldsymbol{C}$ is biholomorphic to the unit disc by Riemann's mapping theorem. On the other hand, in higher dimensional case, we can not expect such a simple phenomenon. In fact, there exists a family of inequivalent bounded domains with a infinite dimensional parameter, by small perturbations of the unit ball (Burns, Shnider and Wells [13]).

One of the basic properties of bounded domains is the existence of the Bergman metric. Let D be a bounded domain of $\boldsymbol{C}^n$ and $\mathscr{H}$ be the set of all L^2-holomorphic functions on D. It is well-known that $\mathscr{H}$ is a separable Hilbert space. Take a complete orthonormal base $\{\phi_i\}$ of $\mathscr{H}$. Then the function $K(z, \bar{w}) = \sum \phi_i(z)\overline{\phi_i(w)}$ is called the Bergman kernel of D. We define a Kähler metric $ds^2 = \sum \partial^2/\partial z_i \partial \bar{z}_j \log K(z, \bar{z}) dz_i d\bar{z}_j$, which is called the Bergman metric. We notice that the Bergman metric is invariant under holomorphic automorphisms of D.

Let D be a bounded domain with smooth boundary ∂D. Take a smooth function f: $\boldsymbol{C}^n \to \boldsymbol{R}$ so that $D=\{x \in \boldsymbol{C}^n : f(x)<0\}$ and grad $f \neq 0$ on ∂D. We say D is strongly pseudoconvex if at any point of ∂D, $\sum \partial^2 f/\partial z_i \partial \bar{z}_j \xi_i \bar{\xi}_j > 0$ for any $(\xi_i) \neq 0$ satisfying $\sum \partial f/\partial z_i \xi_i = 0$. Let D be a strongly pseudoconvex bounded domain with smooth boundary. Fefferman [33] and Boutet de Monvel and Sjöstrand [12] determined the asymptotic behavior of the Bergman kernel at the boundary. From their result, we find that the Bergman metric is complete. The curvature of the Bergman metric near the boundary is calculated by Klembeck [63], using the asymptotic behavior of the Bergman kernel. The sectional curvature is negative near the boundary; in fact the curvature tensor converges, as the boundary is approached, to the curvature tensor of the unit ball. However in general, the sectional curvature is not necessarily negative on the whole of D.

Concerning the geometry of bounded domains, we notice here two interesting results. The first one is a Hamilton's theorem [53] (cf. [42] also): if D is a strongly pseudoconvex bounded domain in $\boldsymbol{C}^n$, then a small perturbation of the complex structure on $\bar{D}$ can be realized by another subdomain in $\boldsymbol{C}^n$ which is C^∞-closed to $\bar{D}$. The second one is due to Greene and Krantz [42], [43]: if D is a simply connected complex manifold on which, for any $\varepsilon>0$, there exists a complete Kähler metric with sectional curvature $\leqq -1+\varepsilon$ and $\geqq -4-\varepsilon$, then D is biholomorphic to the unit ball.

Now let us define the Bergman kernel form on a complex manifold M. Let $\mathscr{H}$ be the set of holomorphic n-form ω such that $\int_M \omega \wedge \bar{\omega} < \infty$. It is known that $\mathscr{H}$ is a separable Hilbert space. Take a complete orthonormal base $\{\omega_i\}$ of $\mathscr{H}$. The $2n$-form $K(z, \bar{w}) = \sum \omega_i(z) \wedge \overline{\omega_i(w)}$ is called the Bergman kernel form. Take a local coordinate system $(z_1, \cdots, z_n)$ and represent $K(z, \bar{w}) = (\sqrt{-1})^{n(n+1)/2} k(z, \bar{w}) dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_n$. We now assume that $k(z, \bar{z})>0$ on M. We consider the hermitian form $\sum \partial^2/\partial z_i \partial \bar{z}_j \log k(z, \bar{z}) dz_i d\bar{z}_j$. It can be seen that this hermitian form is independent of the choice of a local coordinate system. If this hermitian form is positive definite, it defines a Kähler metric on M. This metric is called the Bergman metric of M. The Bergman metric, if it exists, is invariant under holomorphic automorphisms of M.

A complex manifold M has the Bergman metric if and only if M has the following two properties (cf. Kobayashi [64]). 1) for any point $p \in M$, there exists ω such that $\omega(p) \neq 0$, and 2) for any point p there exist $\omega_i = \tilde{\omega}_i dz_1 \wedge \cdots \wedge dz_n$ such that $\omega_i(p)=0$ and $\partial \tilde{\omega}_i/\partial z_j(p) = \delta_{ij}$ for $1 \leqq i, j \leqq n$. Thus we want to find holomorphic n-forms with these properties. The most useful method to construct holomorphic n-forms on noncompact complex manifold is the, so called, L^2-estimates of $\bar{\partial}$ (cf. [40], [41], [46], [58], [84]). The version convenient for our problem is:

Theorem 49. *Let M be a Stein manifold with a complete Kähler metric ds^2. If λ_2 is a plurisubharmonic function on M and if λ_1 is a C^∞-function on M, then*:

(1) *If the Levi form $L\lambda_1$ is $\geqq c\, ds^2$ for some positive function c on M and if f is a $C^\infty(n, 1)$-form on M with $\bar{\partial} f=0$, then there exists a $C^\infty(n, 0)$-form u such that $\bar{\partial} u=f$ and*

$$\int |u|^2 e^{-\lambda_1-\lambda_2} \leqq \int |f|^2/c\, e^{-\lambda_1-\lambda_2}.$$

(2) *If $L\lambda_1+\mathrm{Ric} \geqq c\, ds^2$ for some positive continuous function c, and if f is a $C^\infty(0, 1)$-form on M with $\bar{\partial} f=0$, then there exists a C^∞-function u on M such that $\bar{\partial} u=f$ and*

$$\int |u|^2 e^{-\lambda_1-\lambda_2} \leqq \int |f|^2/c\, e^{-\lambda_1-\lambda_2}.$$

We now assume that for a point $p \in M$, it is possible to choose λ_2 so that λ_2 is continuous except at p and λ_2 tends to $-\infty$ at p with suitable order. Let ω be a holomorphic n-form defined on a neighbourhood of p, say U. Take a smooth function ρ on M which is 1 on a neighbourhood of p and whose support is contained in U. Consider the differential equation $\bar{\partial} u=\bar{\partial}(\rho\omega)$. It is immediate from the definition of ρ, that $\int c^{-1}|\bar{\partial}\rho\omega|^2 e^{-\lambda_1-\lambda_2}$ is finite. Then there is a solution u such that $\int |u|^2 e^{-\lambda_1-\lambda_2}$ is finite. Since λ_2 is singular at p, u must vanish with some order, which is determined from that of λ_2. Then $u-\rho\omega$ is a holomorphic n-form which agrees with ω at p up to some order. Hence the possibility of the choice of suitable λ_i implies the existence of the Bergman metric.

Proposition 50 (Greene and Wu [46]). *Let M be a Kähler manifold with a pole o. If the radial curvature is nonpositive and is $\leqq -(1+\varepsilon)/r^2 \log r$ for large r, then there exists a bounded exhaustion function $\varphi\colon M \to [0, 1)$ such that $\varphi^{-1}(0)=o$, $\varphi=O(r^2)$ at o and* $\log\varphi$ *is plurisubharmonic.*

From this proposition, Greene and Wu [46] obtained:

Theorem 51. *Let M be a Kähler Hadamard manifold and o be a point of M. If the sectional curvature is $\leqq -A/r^2(\log r)^{1+\varepsilon}$ for large r, then M posseses the Bergman metric.*

To see this, it is sufficient to take $\lambda_1=\varphi$ and $\lambda_1=m \log\varphi$ for some m, where φ is the function in Proposition 50. Greene and Wu also obtained a sufficient condition for the completeness of the Bergman metric in [46].

Now we turn to the hyperbolicity of bounded domains in the sense of Kobayashi [65], [67]. Let $\Delta=\{z \in C: |z|<1\}$ be the unit disc and $f: \Delta \to \Delta$ be a holomorphic mapping such that $f(0)=0$. The Schwarz lemma says $|f'(0)| \leqq 1$. This can be interpreted as follows. Let ds_Δ^2 is the Poincaré metric on Δ and $f: \Delta \to \Delta$ be a holomorphic mapping. Then $f^*ds_\Delta^2 \leqq ds_\Delta^2$. The Schwarz lemma has been generalized by Ahlfors [2], Grauert and Reckziegel [38], Kobayashi [66], Yau [95] and others. The following is one of them.

Theorem 52 (Kobayashi [66]). *Let Δ be the unit disk with the metric $ds_\Delta^2=4\, dz d\bar{z}/A(1-|z|^2)^2$ of constant negative curvature $-A$, and M be a hermitian manifold whose holomorphic sectional curvature is bounded from above by a negative constant $-B$. Then every holomorphic mapping $f: \Delta \to M$ satisfies $f^*ds_M^2 \leqq A/B\, ds_\Delta^2$.*

Kobayashi [65] defined an intrinsic pseudodistance on a complex manifold and its infinitesimal form was determined by Royden [78].

Let M be a complex manifold. We define a seminorm on the tangent bundle TM of M by $F(X)=\inf\{|V|_{ds_\Delta^2}: V \in T\Delta$, and there is a holomorphic mapping $f: \Delta \to M$ such that $f_*(V)=X\}$. F is upper semicontinuous on TM. Given points p, q of M, we define a pseudodistance $k(p, q)$ by $k(p, q)=\inf \int_\gamma F(\dot{\gamma})$ where γ runs over all smooth curves joining p and q. This is called the Kobayashi pseudodistance. It follows from the definition that the Kobayashi pseudodistance is invariant under a biholomorphic mapping. We say a complex manifold M is hyperbolic if the Kobayashi pseudodistance is actually a distance on M. For the general theory about hyperbolic manifolds, we refer to Kobayashi [67].

From the Proposition 52, it follows that if a complex manifold M admits a hermitian metric whose holomorphic sectional curvature is bounded from above by a negative constant, then M is hyperbolic. Since every bounded domain is biholomorphic to a subdomain of the unit ball, a bounded domain is hyperbolic. Furthermore, it is known that a strongly pseudoconvex bounded domain is complete hyperbolic (Graham [37]). Modifying the proof of Theorem 51, an application of the maximal principle, Greene and Wu [46] obtained:

Theorem 53. *Let ds^2 be a hermitian metric on a complex manifold M such that its holomorphic sectional curvature is $\leqq -A/1+r^2$ for some positive number A. Then there exists a constant B depending only on A such that*

$$F(X) \geqq B/(1+r^2)^{1/2}|X|_{ds^2}.$$

In particular M is hyperbolic, and is complete hyperbolic if ds^2 is complete.

We have seen in the above argument that a Kähler Hadamard manifold whose sectional curvature is sufficiently negative has several properties similar to those of bounded domains. It is then natural to ask the existence of nonconstant bounded holomorphic functions provided that the sectional curvature of a Kähler Hadamard manifold is sufficiently negative. At the present, however, we have no answers to this question. In Riemannian category, there is an analogous question: if the sectional curvature of a Hadamard manifold is sufficiently negative, then can we assert the existence of nonconstant bounded harmonic functions. We have only some partial answers to this question (Greene and Wu [46], Kasue [61], Sasaki [79]).

7.4. Some geometric characterizations of the complex Euclidean space

Let H be a Kähler Hadamard manifold. When $\dim H=1$, we have already seen that if the Gaussian curvature is $\geqq -1/r^2 \log r$ for large r, then H is biholomorphic to $\boldsymbol{C}$. This phenomenon also holds in the higher dimensional case. Siu and Yau [84] showed:

Theorem 54. *Let H be a Kähler Hadamard manifold. If the sectional curvature is $\geqq -A/(1+r^2)^{1+\varepsilon}$ for some $A>0$ and $\varepsilon>0$, then H is biholomorphic to $\boldsymbol{C}^n$.*

They constructed global coordinate functions which constitute a biholomorphism to $\boldsymbol{C}^n$ by the method of L^2-estimates of $\bar{\partial}$. Greene and Wu [46] obtained some generalization of this theorem.

Recently, some gap phenomenon has been obtained by Mok, Siu and Yau [73] in the Kählerian case and by Greene and Wu [47], [48] in the Riemannian case.

Theorem 55 (Greene and Wu [47], [48]). *Let H be a Hadamard manifold of* $\dim H \geqq 3$, *and o be a point of H. Define k: $[0, \infty) \to \boldsymbol{R}$ by $k(s)=$* $\sup\{$*|sectional curvature at q|: $q \in H$ and $d(o, q)=s$*$\}$. *Then:*

(1) *If the dimension of H is odd and if* $\liminf_{s\to\infty} s^2 k(s)=0$, *then H is isometric to $\boldsymbol{R}^n$.*

(2) *If the dimension of H is even and* $\int sk(s)<\infty$, *then H is isometric to $\boldsymbol{R}^n$.*

This theorem is obtained by the following volume comparison theorem, which is a simple application of Rauch's comparison theorem.

Lemma 56. *Let H be a Hadamard manifold and o be a point of H. Let $v_H(r)(v_{R^n}(r))$ denote the volume of the geodesic ball of radius r in H centered at o(in $\boldsymbol{R}^n$). Then*:

(1) *For all* $r>0$, $v_H(r) \geq v_{R^n}(r)$.

(2) *If* $\liminf v_H(r)/v_{R^n}(r)=1$, *then H is isometric to* $\boldsymbol{R}^n$.

In order to show that H satisfies the condition in Lemma 55, they used the Gauss-Bonnet theorem in the odd dimensional case and some generalization of Gauss map in the even dimensional case.

For manifolds of nonnegative curvature, there also exists a gap theorem corresponding to Theorem 55 ([48]). It follows from the Theorem 55 together with a gap theorem for manifolds with positive curvature, that there exist no nontrivial examples if we assume that the curvature of a Kähler manifold is one sided and is sufficiently small at infinity. Mok, Siu and Yau [73] obtained a characterization of $\boldsymbol{C}^n$ for the case when the curvature takes both signs.

Theorem 57. *Let M be a complete Kähler manifold of dimension $n \geqq 2$ with a pole o. Suppose that the curvature of M is bounded as*

$$-A/(1+r^2)^{1+\varepsilon} \leqq \textit{sectional curvature} \leqq A/(1+r^2)^{1+\varepsilon},$$

where A is a sufficiently small constant depending on $\varepsilon>0$. Then M is biholomorphic to $\boldsymbol{C}^n$.

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