Advanced Studies in Pure Mathematics 3, 1984
Geometry of Geodesics and Related Topics
pp. 193–211

On a Minimal Helical Immersion into a Unit Sphere

Kunio Sakamoto

§ 0. Introduction

Let $f: M \to \bar{M}$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold $\bar{M}$. If the image curve of each geodesic of M has universal constant curvatures and its osculating order is d, then f is said to be a helical immersion of order d. In the present paper we shall study minimal helical immersions into a unit sphere.

A compact Riemannian manifold admits a minimal helical immersion into a unit sphere if and only if it is a strongly harmonic manifold (see [3], [6]). Therefore the classification of compact Riemannian manifolds which admit a minimal helical immersion into a unit sphere is equivalent with that of strongly harmonic manifolds. So taking account of [4], [6], [7] and [8], we may conjecture that if the order d of a minimal helical immersion $f: M \to S(1)$ into a unit sphere $S(1)$ is odd, then M is isometric to a sphere of constant curvature, if the order is even, then M is isometric to a compact rank one symmetric space and f is equivalent to a standard minimal immersion (for the definition, see [8]). To answer this conjecture, we must pay attension to the fact that M is a Blaschke manifold (cf. [1], [6]). In fact, if a Riemannian manifold is diffeomorphic to a sphere or real projective space and has a Blaschke structure, then it is isometric to one of the above spaces with canonical Riemannian structure (cf. [3]). On the other hand, in [6] we obtained an explicit expression of the helical immersion $f: M \to S(1)$ by a geodesic polar coordinate around a fixed point x in M, which contains the second fundamental form and their higher order covariant derivatives at x. Thus we can compute Jacobi fields along a geodesic issuing from x. Therefore we are interested in the relation between the Blaschke structure of M and the second fundamental form of f. The study of this relation is the main purpose of this paper.

Well we give the organization of this paper. We in Section 1 shall explain the results obtained in [6] and moreover study the induced metric

Received December 28, 1982.
Revised March 8, 1983.

on the geodesic sphere in M. The property of this metric is due to that the immersion is minimal. In Section 2, we compute Jacobi fields along a geodesic by using the explicit expression of the helical immersion explained in Section 1. Section 3 is devoted to prepare lemmas for Section 4 which are derived from the basic property of Blaschke structure. In Section 4, we shall characterize the tangent space of the cut-locus and its orthogonal complement as eigenspaces of the Weingarten map corresponding to some normal vector. This characterization is stated in Theorem 4.1. Furthermore we apply these Corollaries to the minimal helical immersion of order 3 and give a different proof of a Theorem stated in [4]. For the study of Blaschke or harmonic manifolds, it is desirable to show that a cut-locus is a totally geodesic submanifold in M. It is a difficult problem. We in this paper represent the second fundamental tensor of the cut-locus in M by that of the immersion f.

The author wishes to express his hearty thanks to Professor S. Ishihara for his constant encouragement and valuable suggestions.

§ 1. Preliminaries

In this paper, the differentiability of all geometric objects will be C^∞. Let M be a complete Riemannian manifold and $f: M \to \bar{M}$ be an isometric immersion into a Riemannian manifold $\bar{M}$. If for each geodesic γ in M, the curve $\sigma = f \circ \gamma$ in $\bar{M}$ is of order d and has constant curvatures which do not depend on γ, then $f: M \to \bar{M}$ is called a *helical immersion of order d.*

In the sequel, $f: M \to S(1)$ *denotes a helical immersion of order d into a unit sphere.* Let $\iota: S(1) \to E$ be the canonical inclusion (E is a Euclidean space whose origin is the center of $S(1)$). Then in [6] we showed that $\tilde{f} = \iota \circ f: M \to E$ is a helical immersion of order d^* where $d^* = d$ if d is even and $d^* = d+1$ if d is odd. Let H be the second fundamental form of $f: M \to S(1)$ and D the van der Waerden-Bortolotti covariant differentiation. Then we have (cf. [6])

Theorem 1.1. *Let* $\gamma: \boldsymbol{R} \to M$ *be a unit speed geodesic such that* $\gamma(0) = x$ *and* $\dot{\gamma}(0) = X$. *Then Frenet frame* $\tau^{(j)}(X), j = 1, \cdots, d$ *at* x *of the cruve* $\tau = \tilde{f} \circ \gamma$ *is given by*

$$\tau^{(j)}(X) = (\lambda_1 \cdots \lambda_{j-1})^{-1}[-b_{j2}\tilde{f}(x) + \sum b_{ji}(D^{i-2}H)(X^i)] \text{ if } j \text{ is even,}$$
$$\tau^{(j)}(X) = (\lambda_1 \cdots \lambda_{j-1})^{-1} \sum b_{ji}(D^{i-2}H)(X^i) \quad \text{if } j \text{ is odd,}$$

where the index in the summation runs over the range $\{2, 4, \cdots, j\}$ *and* $\{3, 5, \cdots, j\}$ *respectively,* $\lambda_1, \cdots, \lambda_{d^*-1}$ *are curvatures of* τ, b_{ji}*'s are polynomials of* $\lambda_1^2, \cdots, \lambda_{j-2}^2$ *and* $(D^{i-2}H)(X^i)$ *denotes* $(D^{i-2}H)(X, \cdots, X)$ *evaluated by* X. *Moreover only in the case d is odd we have*

$$\tau^{(d+1)}(X)=(\lambda_1\cdots\lambda_d)^{-1}[-b_{d+1\,2}\tilde{f}(x)+\sum(b_{d+1\,k}-a_{d+1\,k})(D^{k-2}H)(X^k)]$$

where the index k in the summation runs over the range $\{2, 4, \cdots, d-1\}$ *and* $a_{d+1\,k}$ *is a polynomial of curvatures* $\kappa_1^2, \cdots, \kappa_{d-1}^2$ *of* σ.

To simplify the notations, we put for even j

$$\tilde{\tau}^{(j)}(X)=(\lambda_1\cdots\lambda_{j-1})^{-1}\sum b_{ji}(D^{i-2}H)(X^i), \quad (1\leq j\leq d)$$
$$\tilde{\tau}^{(d+1)}(X)=(\lambda_1\cdots\lambda_d)^{-1}\sum(b_{d+1\,k}-a_{d+1\,k})(D^{k-2}H)(X^k)$$

when d is odd. Let Λ be the Frenet matrix;

$$\Lambda=\begin{bmatrix} 0 & -\lambda_1 & & & 0 \\ \lambda_1 & 0 & -\lambda_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -\lambda_{d^*-1} \\ 0 & & & \lambda_{d^*-1} & 0 \end{bmatrix}$$

and ${}^t(f_1(s), \cdots, f_{d^*}(s))$ the first column vector of the matrix $(e^{s\Lambda}-I)\Lambda^{-1}$. Then $\tilde{f}$ can be expressed by a geodesic polar coordinate with center x as follows (cf. [6]):

Theorem 1.2. *For* $s\in \boldsymbol{R}$ *and* $X\in U_xM$ *(unit tangent sphere at* x*), we have*

$$\tilde{f}(\exp_x sX)=F(s)\tilde{f}(x)+f_1(s)X+\xi(s;X)+\zeta(s;X)$$

where $F(s)=1-(c_2f_2(s)+c_4f_4(s)+\cdots+c_{d^*}f_{d^*}(s))$,

$$c_j=(\lambda_2\lambda_4\cdots\lambda_{j-2})/(\lambda_1\lambda_3\cdots\lambda_{j-1}), \quad \xi(s;X)=\sum_{j:\text{even}}f_j(s)\tilde{\tau}^{(j)}(X) \quad \textit{and}$$

$$\zeta(s;X)=\sum_{j:\text{odd}\geqq 3}f_j(s)\tau^{(j)}(X).$$

From the above theorem we see that the inner product of the position vectors $\tilde{f}(x)$, $\tilde{f}(y)$ satisfy $\langle\tilde{f}(x), \tilde{f}(y)\rangle=F(\delta(x, y))$, where δ denotes the distance function of M. This fact implies following two theorems (cf. [6])

Theorem 1.3. *If a compact Riemannian manifold* M *admits a helical immersion* $f\colon M\rightarrow S(1)$, *then* M *is a Blaschke manifold.*

Theorem 1.4. *If a compact Riemannian manifold* M *admits a minimal helical immersion* $f\colon M\rightarrow S(1)$, *then* M *is a globally harmonic manifold. Furthermore if* M *is simply connected, then it is strongly harmonic.*

It is well-known that if $f\colon M\rightarrow S(1)$ is minimal, then the height functions are eigenfunctions of the Laplace operator corresponding to the eigenvalue $n=\dim M$. Since in Theorem 1.2, we expressed the immersion $\tilde{f}$ in terms of geodesic polar coordinates around x, in order to compute the Laplacian of the height functions, we must represent Laplace operator in terms of geodesic polar coordinates. Let g_s be the Riemannian metric on the unit tangent sphere U_xM induced by the map $X\mapsto \exp_x sX$ from the metric induced on the geodesic sphere with center x and radius s. Then the Laplace operator of M is represented by the radial derivative and the Laplace operator Δ_s with respect to g_s. Thus we have (cf. [6])

Theorem 1.5. *Let $\xi'(s;X),\cdots$ denote the derivatives with respect to s. If $f\colon M\to S(1)$ is a helical minimal immersion of order d, then we obtain*

$$(1.1)\quad f_1\Delta_s X=(f_1''+f_1'\omega+nf_1)X,\quad (X\in U_xM),$$

$$(1.2)\quad \Delta_s(\xi(s;X)+\zeta(s;X))=\xi''(s;X)+\zeta''(s;X)+\omega\cdot(\xi'(s;X)+\zeta'(s;X)) \\ +n(\xi(s;X)+\zeta(s;X)),$$

where $\omega(s)=(nF(s)-f_1'(s))/f_1(s)$.

Now, we remark that equation (1.1) shows the position vector on U_xM is an eigenfunction of Δ_s if $f_1(s)\neq 0$. This fact is interesting. In fact, making use of Theorem 1.4, we get

Proposition 1.6. *Let g be the canonical metric on the unit tangent sphere U_xM. Let $\left\{ {r \atop q\ p} \right\}_s$ $\left(\text{resp. } \left\{ {r \atop q\ p} \right\}\right)$ be Christoffel symbol with respect to g_s (resp. g). Then the tensor field $B(s)$ on U_xM defined by $B(s)^r_{qp}=\left\{ {r \atop q\ p} \right\}_s-\left\{ {r \atop q\ p} \right\}$ satisfies*

$$g_s,\ {}^{qp}B(s)^r_{qp}=0,\qquad B(s)^q_{qp}=0.$$

Moreover we have

$$g_s,\ {}^{pq}g_{pq}=(f_1''+f_1'\omega+nf_1)/f_1$$

and Δ_s is given by

$$\Delta_s=-g_s,\ {}^{pq}\mathring{\nabla}_p\mathring{\nabla}_q$$

where $\mathring{\nabla}_p$ denotes the covariant differentiation with respect to the Riemannian connection $\left\{ {r \atop p\ q} \right\}$.

Proof. Since M is harmonic, we may write $\sqrt{\det g_s}=\phi(s)\sqrt{\det g}$ with some function $\phi(s)$. So we have

$$\Delta_s = -\left(\frac{\partial}{\partial u^p} g_s,{}^{pq} + g_s,{}^{pq} \frac{\partial}{\partial u^p} \log\sqrt{\det g}\right)\frac{\partial}{\partial u^q} - g_s,{}^{pq}\frac{\partial^2}{\partial u^q \partial u^p}$$

where $\{u^2, \cdots, u^n\}$ is a local coordinate system on $U_x M$. Put $X_p = (\partial/\partial u^p)X$ and $X_{pq} = (\partial/\partial u^p)X_q$. Then we have

$$X_{pq} = \left\{ {r \atop p\ q} \right\} X_r - g_{pq} X.$$

Equation (1.1) implies

$$-f_1\left[\left(\frac{\partial}{\partial u^p} g_s,{}^{pq} + g_s,{}^{pq} \frac{\partial}{\partial u^p} \log\sqrt{\det g}\right) X_q + g_s,{}^{pq} X_{pq}\right] = (f_1'' + f_1'\omega + n f_1) X.$$

Thus making use of

$$\begin{aligned} g_s,{}^{qp}\left\{ {r \atop q\ p} \right\}_s &= -\left(\frac{\partial}{\partial u^p} g_s,{}^{rp} + g_s,{}^{pr}\left\{ {q \atop q\ p} \right\}_s\right) \\ &= -\left(\frac{\partial}{\partial u^p} g_s,{}^{pr} + g_s,{}^{pr}\frac{\partial}{\partial u^p}\log\sqrt{\det g_s}\right), \end{aligned}$$

we get

$$f_1\left[g_s,{}^{pq}\left(\left\{ {r \atop p\ q} \right\}_s - \left\{ {r \atop p\ q} \right\}\right) X_r + g_s,{}^{pq} g_{pq} X\right] = (f_1'' + \omega f_1' + n f_1) X,$$

which shows the assertions for $B(s)$ and $g_s,{}^{pq} g_{pq}$. Finally from

$$\Delta_s = g_s,{}^{pq}\left\{ {r \atop p\ q} \right\}_s \frac{\partial}{\partial u^r} - g_s,{}^{pq}\frac{\partial^2}{\partial u^p \partial u^q},$$

we have

$$\begin{aligned} \Delta_s &= g_s,{}^{pq}\left(\left\{ {r \atop p\ q} \right\} + B(s)^r_{pq}\right)\frac{\partial}{\partial u^r} - g_s,{}^{pq}\frac{\partial^2}{\partial u^p \partial u^q} \\ &= -g_s,{}^{pq}\left(\frac{\partial^2}{\partial u^p \partial u^q} - \left\{ {r \atop p\ q} \right\}\frac{\partial}{\partial u^r}\right) \\ &= -g_s,{}^{pq}\mathring{\nabla}_p \mathring{\nabla}_q. \end{aligned}$$

Q.E.D.

§ 2. Jacobi fields

In the sequel, M will be compact, so that it is a Blaschke manifold. *Moreover f: $M \to S(1)$ will be an embedding.* This assumption is always admitted because of Corollary 6.3 in [6].

Let γ be a unit speed geodesic such that $\gamma(0)=x \in M$, $\dot{\gamma}(0)=X \in U_xM$, which is of course a simply closed geodesic loop. Making use of Theorem 1.2, we shall compute Jacobi fields along γ. Let $V \in U_xM$ be a unit tangent vector orthogonal to X. We in Theorem 1.2 exchange X for $\cos t\, X + \sin t\, V$. Then we have a variation of geodesics. Furthermore take a curve $\beta(t)$ in M such that $\beta(0)=x$, $\dot{\beta}(0)=V$. Let $X^*(t)$ be a vector field parallel to X along β. Then, drawing a geodesic which issues from $\beta(t)$ and is tangent to $X^*(t)$, we have another variation of geodesics. These variations yield all Jacobi fields along γ. Before stating a theorem, we must introduce notations. If by suitable numbers of $\langle X, X\rangle$ we multiply each term in the defining equations of $\tilde{\tau}^{(j)}(X)$ (j; even) and $\tau^{(j)}(X)$ (j; odd), then we can understand $\tilde{\tau}^{(j)}$ (or $\tau^{(j)}$) as a multilinear map $T_xM\times\cdots\times T_xM\to N_xM$ (normal space at x) of degree j, so that $\tilde{\tau}^{(j)}$ is a mixed tensor field. For instance, $\tilde{\tau}^{(4)}$ is regarded as ($X_1, \cdots, X_4 \in TM$)

$$\tilde{\tau}^{(4)}(X_1, X_2, X_3, X_4)=(\lambda_1\lambda_2\lambda_3)^{-1}[b_{4\,2}H(X_1, X_2)\langle X_3, X_4\rangle + b_{4\,4}(D^2H)(X_1, X_2, X_3, X_4)].$$

In similar way, we can regard $\xi(s;\,\cdot)$ (resp. $\zeta(s;\,\cdot)$) as a mixed tensor field of covariant degree d (resp. $d-1$) if d is even and $d-1$ (resp. d) if d is odd for each s. With this understanding, we put

$$\begin{aligned}
\xi_X(s; V)&=\xi(s; V, X, \cdots, X)+\xi(s; X, V, X, \cdots, X)\\
&\quad+\cdots+\xi(s; X, \cdots, X, V),\\
\zeta_X(s; V)&=\zeta(s; V, X, \cdots, X)+\zeta(s; X, V, X, \cdots, X)\\
&\quad+\cdots+\zeta(s; X, \cdots, X, V),\\
(D\xi)(s; V; X)&=(D_V\xi(s;\,\cdot))(X, \cdots, X),\\
(D\zeta)(s; V; X)&=(D_V\zeta(s;\,\cdot))(X, \cdots, X).
\end{aligned}$$

Under this notation, we have

Theorem 2.1. *Jacobi fields J_V and J_V^* satisfying $J_V(0)=0$, $\nabla_{\dot{\gamma}}J_V(0)=V$ and $J_V^*(0)=V$, $\nabla_{\dot{\gamma}}J_V^*(0)=0$ respectively are given by*

$$J_V(s)=f_1(s)V+\xi_X(s; V)+\zeta_X(s; V), \tag{2.1}$$

$$J_V^*(s)=F(s)V-A_{\xi(s;X)}V-A_{\zeta(s;X)}V + f_1(s)H(V, X)+(D\xi)(s; V; X)+(D\zeta)(s; V; X), \tag{2.2}$$

where $A_{\xi(s;X)}$ (resp. $A_{\zeta(s;X)}$) denotes Weingarten map corresponding to the normal vector $\xi(s; X)$ (resp. $\zeta(s; X)$).

Proof. Consider the variation $(s, t)\mapsto \exp_x s(\cos tX+\sin tV)$. Then we have

$$J_V(s)=\frac{d}{dt}[F(s)\tilde{f}(x)+f_1(s)(\cos tX+\sin tV)$$
$$+\xi(s;\cos tX+\sin tV)+\zeta(s;\cos tX+\sin tV)]|_{t=0}$$
$$=f_1(s)V+\xi_X(s;V)+\zeta_X(s;V)$$

which shows (2.1). Next let $\beta(t)$ and $X^*(t)$ be as above. We then consider the variation $(s, t)\mapsto \exp_{\beta(t)} sX^*(t)$. Also from Theorem 1.2, we have

$$J_V^*(s)=\frac{d}{dt}[F(s)\tilde{f}(\beta(t))+f_1(s)X^*(t)+\xi(s;X^*(t))+\zeta(s;X^*(t))]|_{t=0}$$
$$=F(s)V+f_1(s)H(V,X)-A_{\xi(s;X)}V+(D\xi)(s;V;X)$$
$$-A_{\zeta(s;X)}V+(D\zeta)(s;V;X),$$

which shows (2.2). Q.E.D.

Here we remark the following: Firstly, in the above equations (2.1) and (2.2), vectors J_V, J_V^*, V, $A_{\xi(s;X)}V$ and $A_{\zeta(s;X)}V$ are images of them by the differential map f_*. Secondly, the tangent vector V in the initial conditions need not be a unit vector.

Now, we shall combine Theorem 2.1 with the fact that the Jacobi fields J_V's ($V\in\{X\}^\perp$) span the subspace $\{\dot{\gamma}\}^\perp$ in $T_\gamma M$ at each point on γ except the conjugate point of x. Let L be the period of γ, so that the injectivity radius is $L/2$. For $s\in(0, L/2)$ we define a linear transformation $S(s)$ on the subspace $\{X\}^\perp$ in T_xM by

$$J_V^*(s)=J_{S(s)V}(s). \tag{2.3}$$

Then we have

Lemma 2.2. *The linear transformation $S(s)$ on $\{X\}^\perp$ is given by*

$$S(s)=f_1(s)^{-1}(F(s)I-A_{\xi(s;X)}-A_{\zeta(s;X)})$$

where I is the identity transformation.

Proof. Consider T_xM-component of both hand sides in (2.3). Then the assertion is immediately derived from (2.1) and (2.2). Q.E.D.

The following proposition asserts that the derivative of $-S(s)$ is the inverse g_s^{-1} of the metric g_s.

Proposition 2.3. *We have*

$$g_s(S'(s)\, V, W) = -\langle V, W \rangle$$

where $s \in (0, L/2)$ and $V, W \in \{X\}^{\perp}$.

Proof. Let $\{V_k\}_{k=2,\dots,n}$ be an orthonormal base in $\{X\}^{\perp}$. The equation (2.3) is written by

$$J_k^* = \sum_{l=2}^{n} S_{kl} J_l, \qquad (J_k = J_{V_k}).$$

Thus we have

$$\nabla_{\dot\gamma}^2 J_k^* = \sum (S''_{kl} J_l + 2S'_{kl} \nabla_{\dot\gamma} J_l + S_{kl} \nabla_{\dot\gamma}^2 J_l).$$

Since J_k^* and J_l are Jacobi fields, we obtain

$$\sum (S''_{kl} J_l + 2S'_{kl} \nabla_{\dot\gamma} J_l) = 0,$$

so that

$$\sum [S''_{kl} \langle J_l, J_h \rangle + 2S'_{kl} \langle \nabla_{\dot\gamma} J_l, J_h \rangle] = 0.$$

Since $g_s(V_l, V_h) = \langle J_l, J_h \rangle$ and

$$\dot\gamma \cdot g_s(V_l, V_h) = \langle \nabla_{\dot\gamma} J_l, J_h \rangle + \langle J_l, \nabla_{\dot\gamma} J_h \rangle = 2\langle \nabla_{\dot\gamma} J_l, J_h \rangle,$$

we have

$$\sum [S''_{kl} g_s(V_l, V_h) + S'_{kl} g'_s(V_l, V_h)] = 0,$$

or $g_s(S''(s)V, W) + g'_s(S'(s)V, W) = 0$. It follows that $g_s(S'(s)V, W) =$ constant. We shall prove that this constant is $-\langle V, W \rangle$. From Lemma 2.2, we see that

$$S'(s) = -f'_1(s)/(f_1(s))^2 [F(s)I - A_{\xi(s;X)} - A_{\zeta(s;X)}]$$
$$- 1/f_1(s)[f_1(s)I + A_{\xi'(s;X)} + A_{\zeta'(s;X)}],$$

where we note $F'(s) = -f_1(s)$. Thus we have

$$S'(s) = -\frac{1}{s^2} I + O(1).$$

On the other hand we have

$$g_s = s^2 \langle\ ,\ \rangle + O(s^4).$$

Therefore $g_s(S'(s)V, W) = -\langle V, W\rangle$. Q.E.D.

Next we consider the N_xM-component of both hand sides in (2.3). We have

Theorem 2.4. *Let $V \in \{X\}^\perp$ be arbitrary. Then*

$$(2.4)\qquad f_1(s)(D\xi)(s; V; X) = -\xi_X(s; A_{\zeta(s;X)}V) + F(s)\zeta_X(s; V) - \zeta_X(s; A_{\xi(s;X)}V),$$

$$(2.5)\qquad (f_1(s))^2 H(V, X) + f_1(s)(D\zeta)(s; V; X) = F(s)\xi_X(s; V) - \xi_X(s; A_{\xi(s;X)}V) - \zeta_X(s; A_{\zeta(s;X)}V).$$

Proof. Substituting equations (2.1), (2.2) and that given in Lemma 2.2 into (2.3), we have

$$\begin{aligned} f_1(s)[f_1(s)H(V, X) + (D\xi)(s; V; X) + (D\zeta)(s; V; X)] \\ = \xi_X(s; F(s)V - A_{\xi(s;X)}V - A_{\zeta(s;X)}V) \\ + \zeta_X(s; F(s)V - A_{\xi(s;X)}V - A_{\zeta(s;X)}V). \end{aligned}$$

Exchange X for $-X$. Then we obtain the desired equations. Q.E.D.

§ 3. Lemmas

By Theorem 1.3 we know M is a Blaschke manifold. Thus we recall the definition of a Blaschke manifold. Let $x \in M$ and $\mathrm{Cut}(x)$ be the cut-locus of x in M. If for every $y \in \mathrm{Cut}(x)$ the link $\mathscr{L}_y(x) = \{\dot{\gamma}(y) \in U_yM: \gamma$ is a minimal geodesic from x to $y\}$ is a great sphere of U_yM, then M is said to be a Blaschke manifold at the point x. Moreover M is said to be a Blaschke manifold if it is a Blaschke manifold at every point in M (cf. [3]). It is well-known that M is a Blaschke manifold at x if and only if the cut-locus $\mathrm{Cut}(x)$ is spherical (cf. [3]) and that if M is a Blaschke manifold, then every geodesic is a simply closed geodesic whose length we denoted by L.

Let $\mathscr{H}_y(x)$ be the subspace spanned by $\mathscr{L}_y(x)$ in T_yM, so that $\mathscr{L}_y(x) = \mathscr{H}_y(x) \cap U_yM$ and $\mathscr{H}_y(x) \perp T_y\,\mathrm{Cut}(x)$. The dimension of $\mathscr{L}_y(x)$ is constant which is the index of the first conjugate point y of x along a geodesic from x to y. Let $e = \dim \mathscr{H}_y(x)$. Bott's theorem says that e is equal to 1, 2, 4, 8 or n, corresponding to which $n = m$, $2m$, $4m$, 16 or any ($m = 1, 2, \cdots$). Using Theorem 2.1, we shall characterize the subspace $\mathscr{H}_x(y)$ for $y \in \mathrm{Cut}(x)$.

Let $X \in U_xM$. If $y = \exp_x(L/2)X$, then $\mathscr{H}_x(y)$ is the "holomorphic section" determined by X when M is a compact rank one symmetric space.

Thus we shall adopt the notation $\mathscr{H}_x(X)$ instead of $\mathscr{H}_x(y)$ if $y=\exp_x(L/2)X$. Moreover we put $\mathscr{H}_x^*(X)=\{X\}^\perp$ in $\mathscr{H}_x(X)$.

Lemma 3.1. *We have*

$$\mathscr{H}_x(X)=\operatorname{span}\{Z\in U_xM\colon \xi(X)=\xi(Z)\},$$
$$\mathscr{H}_x^*(X)=\{Z\in T_xM\colon \xi_X(Z)=0,\ Z\perp X\},$$

where $\xi(X)=\xi(L/2;X)$ *and* $\xi_X(Z)=\xi_X(L/2;Z)$.

Proof. It is easily shown that $f_1'=1-\lambda_1 f_2$, $f_j'=\lambda_{j-1}f_{j-1}-\lambda_j f_{j+1}$ $(j\geqq 2)$. Since f_1 is an odd function, we see that f_j is an odd (resp. even) function if j is odd (resp. even). Moreover f_j's are periodic functions with period L. Thus $f_j(L/2)=0$ if j is odd. Hence we have $\zeta(L/2;X)=0$, so that cut-locus $\operatorname{Cut}(x)$ of x is given by

$$\tilde{f}(\operatorname{Cut}(x))=\{\tilde{f}(y)=F(L/2)\tilde{f}(x)+\xi(X)\colon X\in U_xM\}.$$

Therefore we see that $\exp_x(L/2)X=\exp_x(L/2)Z$ if and only if $\xi(X)=\xi(Z)$. Q.E.D.

In the sequel, we shall put $a=f_1'(L/2)$, $b=F(L/2)$. Let $\gamma\colon s\mapsto\exp_x sX$ be the unit speed geodesic such that $\gamma(0)=x$, $\dot{\gamma}(0)=X$. We put $\bar{X}==\dot{\gamma}(L/2)$. Then we have

Lemma 3.2. *The Weingarten map* $A_{\xi(X)}$ *satisfies*

$$A_{\xi(X)}X=(b-a)X \tag{3.1}$$

and $\bar{X}$ *is given by*

$$\bar{X}=aX+\zeta'(L/2;X)=aX+(D\xi)(X). \tag{3.2}$$

Proof. The first equality of (3.2) is clear in virtue of Theorem 1.2. Now we note that

$$\tilde{f}(\gamma(s+L/2))=b\tilde{f}(\gamma(s))+\xi(\dot{\gamma}(s)).$$

Differentiate this equation and let $s=0$. Then we have

$$\bar{X}=bX-A_{\xi(X)}X+(D\xi)(X).$$

However $A_{\xi(X)}X$ is proportional to X because of Lemma 3.3 in [6]. Hence $A_{\xi(X)}X=(b-a)X$ and $\zeta'(L/2;X)=(D\xi)(X)$. Q.E.D.

Lemma 3.3. *The subspaces* $\mathscr{H}_y(\bar{X})$ *and* $T_y\operatorname{Cut}(x)$ *are given by*

$$\mathcal{H}_y(\bar{X})=\mathrm{span}\{aZ+(D\xi)(Z);\ Z\in\mathcal{H}_x(X),\ \|Z\|=1\},$$
$$T_y\mathrm{Cut}(x)=\{\xi_X(Y);\ Y\in T_x\ \mathrm{Cut}(y)\},$$

where $\xi_X(Y)=\xi_X(L/2;\ Y)$. *Moreover we see that* $\xi_X\colon T_x\mathrm{Cut}(y)\to T_y\mathrm{Cut}(x)$ *is a linear isomorphism.*

Proof. The first equality is clear from (3.2). Since the curve $\theta\mapsto b\tilde{f}(x)+\xi(\cos\theta X+\sin\theta Y)$ is contained in $\mathrm{Cut}(x)$, its tangent vector $\xi_X(Y)$ at $\theta=0$ is contained in $T_y\mathrm{Cut}(x)$. Thus we see from the fact $\dim T_x\mathrm{Cut}(y)=\dim T_y\mathrm{Cut}(x)$ that the second equality holds and $\xi_X\colon T_x\mathrm{Cut}(y)\to T_y\ \mathrm{Cut}(x)$ is an isomorphism. Q.E.D.

Hereafter we shall assume that $f\colon M\to S(1)$ *is minimal.*

Lemma 3.4. *We have* $a\neq 0$.

Proof. Let J_k ($k=2,\cdots,n$) be the Jacobi field used in the proof of Lemma 2.3. Let Θ be defined by

$$\Theta(s)=s^{1-n}(\det(\langle J_k(s),\ J_l(s)\rangle))^{1/2}.$$

Here we note that by virtue of Theorem 1.4 Θ is a function depending only on s. We have proved in [6]

$$f_1'+f_1\cdot\left(\frac{n-1}{s}+\frac{\Theta'}{\Theta}\right)=nF. \tag{3.3}$$

Let us assume that $a=f_1'(L/2)=0$. Since $F'=-f_1$ and $f_1(L/2)=f_1''(L/2)=0$, we obtain

$$F(s)=b+\alpha_4(s-L/2)^4+O[(s-L/2)^6]$$

where $4!\ \alpha_4=F^{(4)}(L/2)$. On the other hand, in [1] Allamigeon proved

$$\Theta(s)=\alpha(s-L/2)^{e-1}+O[(s-L/2)^e],$$

where α is a non-zero constant. It follows that

$$\frac{\Theta'}{\Theta}=(e-1)(s-L/2)^{-1}+O(1).$$

Substitute these expansions into (3.3). We have

$$\begin{aligned}&-12\alpha_4(s-L/2)^2+O[(s-L/2)^4]\\&-\{4\alpha_4(s-L/2)^3+O[(s-L/2)^5]\}\{(e-1)(s-L/2)^{-1}+O(1)\}\\&\quad=n\{b+\alpha_4(s-L/2)^4+O[(s-L/2)^6]\}\end{aligned}$$

on some interval which contains $L/2$. Therefore we have $b=0$ and $\alpha_4=0$. Repeating this argument, we see that all derivatives of F at $L/2$ vanish. Since F is analytic, $F\equiv 0$. This conclusion contradicts to $F(0)=1$.

Q.E.D.

Let us define $(D\xi)_X(V)$ by

$$(D\xi)_X(V)=(D\xi)(V, X, \cdots, X)+\cdots+(D\xi)(X, \cdots, X, V).$$

Proposition 3.5. *If $Z\in\mathscr{H}_x(X)$ is a unit tangent vector orthogonal to X, then $(D\xi)(Z)=(D\xi)_X(Z)$. Furthermore equation*

$$\langle aZ_1+(D\xi)(Z_1), aZ_2+(D\xi)(Z_2)\rangle=\langle Z_1, Z_2\rangle$$

holds for any unit vectors $Z_1, Z_2\in\mathscr{H}_x(X)$.

Proof. Since $aV+(D\xi)(V)\in\mathscr{L}_y(x)$ for every unit vector V in $\mathscr{H}_x(X)$, we have $aZ+(D\xi)_X(Z)\in\mathscr{H}_y(\bar{X})$, which implies that $aZ+(D\xi)_X(Z)=c(aW+(D\xi)(W))$ holds for some unit vector $W\in\mathscr{H}_x(X)$ and some constant c. It follows that $c=1$, $Z=W$. Thus we have $(D\xi)_X(Z)=(D\xi)(Z)$.

If $Z_1=Z_2$, then the equality is clear. In general case, we take a unit vector Z orthogonal to Z_1 in the plane spanned by Z_1 and Z_2. Then Z_2 is written as $Z_2=\cos\theta\, Z_1+\sin\theta\, Z$. Let us put

$$\begin{aligned}\phi(\theta)&=\langle aZ_1+(D\xi)(Z_1), aZ_2+(D\xi)(Z_2)\rangle\\&=a^2\cos\theta+\langle(D\xi)(Z_1), (D\xi)(\cos\theta\, Z_1+\sin\theta\, Z)\rangle.\end{aligned}$$

Differentiating $\phi(\theta)$, we have

$$\begin{aligned}\phi'(\theta)&=-a^2\sin\theta+\langle(D\xi)(Z_1), (D\xi)_{Z_2}(-\sin\theta\, Z_1+\cos\theta\, Z)\rangle\\&=-a^2\sin\theta+\langle(D\xi)(Z_1), (D\xi)(-\sin\theta\, Z_1+\cos\theta\, Z)\rangle.\end{aligned}$$

Differentiate once more. Then

$$\begin{aligned}\phi''(\theta)&=-a^2\cos\theta+\langle(D\xi)(Z_1), (D\xi)_{-\sin\theta Z_1+\cos\theta Z}(-Z_2)\rangle\\&=-a^2\cos\theta-\langle(D\xi)(Z_1), (D\xi)(Z_2)\rangle\\&=-\phi(\theta).\end{aligned}$$

In the above computation we have used the fact that Z_2 is orthogonal to $-\sin\theta\, Z_1+\cos\theta\, Z$ and that $(D\xi)_V(W)=(D\xi)(W)$ if V, W are orthonormal vectors in $\mathscr{H}_x(X)$. We have already shown $\phi(0)=1$. Thus it suffices to show $\phi'(0)=0$. In order to show $\phi'(0)=0$, we have only to prove that if $Z_1\perp Z_2$, then $\langle(D\xi)(Z_1), (D\xi)(Z_2)\rangle=0$. This is proved as

$$\langle (D\xi)(Z_1), (D\xi)(Z_2)\rangle = \langle (D\xi)(Z_1), (D\xi)_{Z_1}(Z_2)\rangle = 0$$

where we have used the fact that $\|(D\xi)(V)\|^2 = 1 - a^2$ for every unit vector $V \in \mathscr{H}_x(X)$. Q.E.D.

Let $Y \in T_x\mathrm{Cut}(y)$ such that $\|Y\| = 1$. Since $\xi_X\colon T_y\mathrm{Cut}(x) \to T_x\mathrm{Cut}(y)$ is a linear isomorphism, there exists $\bar{Y} \in T_y\mathrm{Cut}(x)$ such that $\xi_{\bar{X}}(\bar{Y}) = Y$. From (2.1) we see that there is a Jacobi field $K_{\bar{Y}}$ along γ such that $K_{\bar{Y}}(L/2) = 0$, $\nabla_{\bar{X}} K_{\bar{Y}}(L/2) = \bar{Y}$ and $K_{\bar{Y}}(0) = Y$.

Lemma 3.6. *The Jacobi field $K_{\bar{Y}}$ is given by*

$$K_{\bar{Y}} = J^*_Y + J_W$$

where $W = 1/a\, A_{(D\xi)(X)} Y$.

Proof. Let $W = \nabla_X K_{\bar{Y}}(0)$. Then we have $K_{\bar{Y}} = J^*_Y + J_W$. The Jacobi field $K_{\bar{Y}}$ satisfies

$$\begin{aligned}\frac{d}{ds}K_{\bar{Y}}\Big|_{s=L/2} &= \Big[\tilde{\nabla}_{\bar{X}} K_{\bar{Y}} + \Big\langle \frac{d}{ds}K_{\bar{Y}}, \tilde{f}(\gamma(s))\Big\rangle \tilde{f}(\gamma(s))\Big]\Big|_{s=L/2} \\ &= \nabla_{\bar{X}} K_{\bar{Y}}(L/2) + H(\bar{X}, K_{\bar{Y}}(L/2)) \\ &= \bar{Y} \in T_y\mathrm{Cut}(x),\end{aligned}$$

where $\tilde{\nabla}$ denotes the covariant differentiation on $S(1)$. Thus we see from Theorem 2.1 and Lemma 3.2 that

$$\begin{aligned}&-A_{(D\xi)(X)}Y + aH(Y, X) + (D^2\xi)(Y, X, X, \cdots, X) + aW + (D\xi)_X(W) \\ &\quad = \bar{Y} \in T_y\mathrm{Cut}(x).\end{aligned}$$

So Lemma 3.3 implies $aW = A_{(D\xi)(X)}Y$. Q.E.D.

§ 4. Theorems

In the beginning, we shall characterize $\mathscr{H}_x(X)$ and $T_x\mathrm{Cut}(y)$ as eigenspaces of the Weingarten map $A_{\xi(X)}$.

Theorem 4.1. *We obtain*

$$\begin{aligned}\mathscr{H}_x(X) &= \{Z \in T_xM\colon A_{\xi(X)}Z = (b-a)Z\}, \\ T_x\mathrm{Cut}(y) &= \{Y \in T_xM\colon A_{\xi(X)}Y = bY\}.\end{aligned}$$

Proof. Let $Z \in \mathscr{H}_x(X)$. Then Lemmas 3.1 and 3.2 show

$$A_{\xi(X)}Z = A_{\xi(Z)}Z = (b-a)Z,$$

so that $\mathscr{H}_x(X)\subset\{Z\in T_xM: A_{\xi(X)}Z=(b-a)Z\}$. Next let $Y\in T_x\mathrm{Cut}(y)$. Then we have a Jacobi field K_Y as in Lemma 3.6. Since $K_Y(L/2)=0$, we find

$$bY-A_{\xi(X)}Y+(D\xi)(L/2;Y;X)+\xi_X\left(\frac{1}{a}A_{(D\xi)(X)}Y\right)=0,$$

which implies that $A_{\xi(X)}Y=bY$. Thus $T_x\mathrm{Cut}(y)$ is contained in the eigen-space corresponding to eigenvalue b of $A_{\xi(X)}$. Since $T_xM=\mathscr{H}_x(X)\oplus T_x\mathrm{Cut}(y)$, we obtain the assertions. Q.E.D.

Corollary 4.2. *The equality*

$$a=mb, \qquad (m=n/e)$$

holds.

Proof. The minimality of $f: M\to S(1)$ implies that trace $A_{\xi(X)}=0$ and hence $e(b-a)+(n-e)b=0$. Thus we have $a=mb$. Q.E.D.

Corollary 4.3. *Let $Y\in T_x\mathrm{Cut}(y)$ and $Z\in\mathscr{H}_x(X)$ such that $\|Y\|=\|Z\|=1$, $Z\perp X$. Then equations*

$$J_Y^*(L/2)=-\frac{1}{a}\xi_X(A_{(D\xi)(X)}Y)\in T_y\mathrm{Cut}(x),$$

$$J_Z^*(L/2)=aZ+(D\xi)(Z)-\frac{1}{a}\xi_X(A_{(D\xi)(X)}Z)$$

hold.

Proof. From the proof of Theorem 4.1, we see that

$$J_Y^*(L/2)=(D\xi)(L/2;Y;X)=-\frac{1}{a}\xi_X(A_{(D\xi)(X)}Y).$$

Equation (2.2) shows that

$$J_Z^*(L/2)=aZ+(D\xi)(L/2;Z;X).$$

Differentiate (2.4) with respect to s and let $s=L/2$. Then we have

$$\begin{aligned}a(D\xi)(L/2;Z;X)&=-\xi_X(A_{(D\xi)(X)}Z)+b(D\xi)(Z)-(D\xi)_X((b-a)Z)\\&=a(\xi D)(Z)-\xi_X(A_{(D\xi)(X)}Z),\end{aligned}$$

which shows the second equation. Q.E.D.

Well, we shall apply Corollaries 4.2 and 4.3 to the case $d=3$. In this case we may exhibit

(3.4) $(D\xi)(X)=a_3'\tau^{(3)}(X), \qquad \tau^{(3)}(X)=(\lambda_1\lambda_2)^{-1}(DH)(X^3),$

(3.5) $a=1-\lambda_1 a_2, \qquad a_3'=\lambda_2 a_2-\lambda_3 a_4, \qquad b=1-c_2 a_2-c_4 a_4,$

(3.6) $(a_3')^2=1-a^2,$

(3.7) $c_2^2+c_4^2=1,$

where $a_3'=f_3'(L/2)$, $a_2=f_2(L/2)$ and $a_4=f_4(L/2)$. The last equation (3.7) is due to Lemma 5.2 in [6]. Let us assume $e\geq 2$. Codazzi structure equation shows that D_ξ is a symmetric tensor, i.e.,

$$(D\xi)(V_1, V_2, V_3)=(D\xi)(V_2, V_1, V_3)=(D\xi)(V_1, V_3, V_2).$$

Thus we have $(D\xi)(L/2; Z; X)=(D\xi)(Z, X, X)=(D\xi)(Z)/3$ for $Z \in \mathscr{H}_x(X)$ such that $\|Z\|=1$ and $Z\perp X$. It follows from Corollary 4.3 that

$$\frac{2}{3}(D\xi)(Z)=\frac{1}{a}\xi_X(A_{(D\xi)(X)}Z).$$

However Lemmas 3.1 and 3.3 show that both hand sides are orthogonal. Thus we have $(D\xi)(Z)=0$ and hence $a_3'=0$. On the other hand, by three equations in (3.5) and (3.7), we obtain $b-a=c_4\lambda_3^{-1}a_3'$, which implies $a=b$. By virtue of Corollary 4.2 we see that $n=e$, $a=-1$. Next assume that $e=1$. Let $Y \in T_xM$ such that $\|Y\|=1$ and $X\perp Y$. From Corollary 4.3, we have

$$\frac{1}{3}(D\xi)_X(Y)=-\frac{1}{a}\xi_X(A_{(D\xi)(X)}Y),$$

from which

$$\begin{aligned}\frac{1}{3}\langle(D\xi)(X), H(Y, Y)\rangle &= -\frac{1}{3}\langle(D\xi)_X(Y), H(X, Y)\rangle\\ &=\frac{1}{a}\langle\xi_X(A_{(D\xi)(X)}Y), H(X, Y)\rangle\\ &=\frac{1}{a}[-\langle\xi(X), H(A_{(D\xi)(X)}Y, Y)\rangle+\langle(b-a)A_{(D\xi)(X)}Y, Y\rangle]\\ &=-\langle(D\xi)(X), H(Y, Y)\rangle,\end{aligned}$$

where we have used Lemma 3.3 in [6]. It follows that $A_{(D\xi)(X)}=0$. So we see that $(D\xi)_X(Y)=0$ for every $Y \in T_xM$ such that $Y\perp X$. This means

$(D\xi)(X)$ is constant vector on U_xM. Since $D\xi$ is of covariant degree 3, we have $(D\xi)(X)=0$, and so $a_3'=0$. Thus we have $e=n$ which is a contradiction. In this way, we obtain

Theorem 4.4. (Nakagawa [4]). *Let f: $M\to S(1)$ be a minimal helical embedding of order* 3. *Suppose that M is compact. Then M is isometric to a sphere. Moreover f is equivalent to a standard minimal embedding of a sphere.*

Proof. Since $e=n$, we can use D.3 Corollary in [3]. Thus M is isometric to a sphere. Moreover by Wallach's linear rigidity Theorem [8], we see that f is equivalent to a standard minimal embedding of a sphere.
Q.E.D.

Thus we may conjecture that if f: $M\to S(1)$ is a minimal helical embedding of odd order, then M is isometric to a sphere. If this conjecture is true, then by virtue of Tsukada [7] we see that f is equivalent to a minimal standard embedding of a sphere. However in this paper we can only prove the converse. We need

Lemma 4.5. *If j is even, then*

$$\int_{U_xM} \tilde{\tau}^{(j)}(X)dv_g=0,$$

where dv_g denotes the volume form of the canonical metric g on U_xM.

Proof. Integrate on U_xM both hand sides of (1.2) with respect to the measure dv_{g_s}. Noting that

$$dv_{g_s}=s^{n-1}\Theta dv_g,$$

we have from Green's Theorem

$$\int_{U_xM} [\xi''(s;X)+\omega\cdot\xi'(s;X)+n\xi(s;X)]\,dv_g=0$$

for every $s\in(0, L/2)$. It follows that

$$\sum_{j:\text{even}}^{d^*} (f_j''+\omega f_j'+nf_j)\int_{U_xM} \tilde{\tau}^{(j)}(X)dv_g=0.$$

Well we note

$$f_j(s)=\frac{1}{j!}(\lambda_1\cdots\lambda_{j-1})s^j+O(s^{j+2})$$

which is easily derived from the definition of f_j. Moreover we have

$$\omega(s)=\frac{n-1}{s}+O(s).$$

Therefore for $j\geq 2$ we see

$$f_j''(s)+\omega(s)f_j'(s)+nf_j(s)=\frac{1}{(j-1)!}(\lambda_1\cdots\lambda_{j-1})(n+j-2)s^{j-2}+O(s^j).$$

It follows that we have inductively

$$\int_{U_xM}\tilde{\tau}^{(j)}(X)dv_g=0. \qquad \text{Q.E.D.}$$

Here we should remark that Lemma 4.5 means the minimality of f, in particular when $j=2$, $\tilde{\tau}^{(2)}(X)=\lambda_1^{-1}H(X, X)$, and hence

$$\int_{U_xM}H(X, X)dv_g=0.$$

This is equivalent with trace of $H=0$.

Theorem 4.6. *Let f: $M\to S(1)$ be a minimal helical embedding. Assume that M is compact and $e=n$. Then we have $\xi(X)=0$ for every $X\in UM$ and $a=b=-1$ which means geometrically that the cut point $f(\gamma(L/2))$ of $f(\gamma(0))$ is the antipodal point of $f(\gamma(0))$ in $S(1)$ for every geodesic γ. Furthermore we see that d must be odd.*

Proof. Since $e=n$, $\xi(X)$ is constant on U_xM. From the above Lemma we have

$$\int_{U_xM}\xi(X)dv_g=0,$$

so that $\xi(X)=0$ on U_xM. Since f is embedding and $\|\xi(X)\|^2=1-b^2$, we have $b=-1$. Next we shall prove that d is odd. If d is even, then $H(X, X)$, $(D^2H)(X^4)$, $\cdots$, $(D^{d-2}H)(X^d)$ are linearly independent and hence $\tilde{\tau}^{(j)}$'s are also linearly independent. Thus $f_j(L/2)=0$ for every even integer $2\leq j\leq d$, which implies $b=1$. This contradicts to the assumption that f is an embedding. Q.E.D.

Finally we shall study when the cut-locus is totally geodesic in M. As before let γ be a unit speed geodesic in M such that $\gamma(0)=x$, $\dot{\gamma}(0)=X$ and $\gamma(L/2)=y$. Then the cut-locus $\mathrm{Cut}(y)$ of y is geodesic at x in M if and only if the restriction of $A_{(D\xi)(Z)}$ to $T_x\mathrm{Cut}(y)$ vanishes for every $Z \in \mathscr{L}_x(y)$. Indeed

Theorem 4.7. *Let $f: M \to S(1)$ be a minimal helical embedding and M be compact. Let $Z \in \mathscr{L}_x(y)$. Then the Weingarten map $\bar{A}_Z$ of the submanifold* Cut(y) *in M corresponding to Z is given by*

$$\langle \bar{A}_Z Y_1, Y_2 \rangle = -\frac{1}{a} \langle A_{(D\xi)(Z)} Y_1, Y_2 \rangle$$

*for $Y_1, Y_2 \in T_x$*Cut(y). *Moreover we see that* Cut(y) *is a minimal submanifold in M.*

Proof. We may assume that $Z = X$ because of $\mathscr{H}_x(X) = \mathscr{H}_x(Z)$. Let us consider the Jacobi field K_Y as in Lemma 3.6. This is a variation vector of a geodesic variation such that each geodesic is orthogonal to Cut(y), passes through y at $s = L/2$ and has the same length $L/2$. It follows that

$$-\bar{A}_X Y - \nabla_X K_Y(0) \in \mathscr{H}_x(X),$$

(cf. [5]). Thus we have from Lemma 3.6

$$-\bar{A}_X Y - \frac{1}{a} A_{(D\xi)(X)} Y \in \mathscr{H}_x(X),$$

which implies that

$$\langle \bar{A}_X Y_1, Y_2 \rangle = -\frac{1}{a} \langle A_{(D\xi)(X)} Y_1, Y_2 \rangle$$

for $Y_1, Y_2 \in T_x$Cut(y). Next let $Z_1, Z_2 \in \mathscr{H}_x^*(X)$ which is the subspace orthogonal to X in $\mathscr{H}_x(X)$. We have

$$\begin{aligned}\langle A_{(D\xi)(X)} Z_1, Z_2 \rangle &= \langle (D\xi)(X), H(Z_1, Z_2) \rangle \\ &= -\langle (D\xi)(Z_1), H(X, Z_2) \rangle.\end{aligned}$$

If $Z_1 = Z_2$, then $\langle (D\xi)(Z_1), H(X, Z_2) \rangle = 0$ (cf. Lemma 3.3 in [6]). If $Z_1 \perp Z_2$, then

$$\langle (D\xi)(Z_1), H(X, Z_2) \rangle = -\langle (D\xi)(Z_2), H(X, Z_1) \rangle.$$

But $\langle (D\xi)(Z_1), H(X, Z_2) \rangle$ is symmetric with respect to Z_1 and Z_2, and hence $\langle A_{(D\xi)(X)} Z_1, Z_2 \rangle = 0$. Since $A_{(D\xi)(X)} X = 0$, the matrix representation of $A_{(D\xi)(X)}$ is

$$\frac{1}{a}A_{(D\xi)(X)}=\left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline 0 & 0 & Q_X \\ \hdashline 0 & {}^tQ_X & -\bar{A}_X \end{array}\right]\begin{array}{l} X \\ \}\ \mathscr{H}^*_x(X) \\ \}\ T_x\mathrm{Cut}(y) \end{array}$$

where Q_X is a $(e-1)\times(n-e)$ matrix. Therefore we have trace $\bar{A}_X=0$ since trace $A_{(D\xi)(X)}=0$. Q.E.D.

Remark. We consider the field of linear transformation $Q_{\dot{\gamma}(s)}$: $\mathscr{H}^*_{\gamma(s)}(\dot{\gamma}(s))\to T_{\gamma(s)}\,\mathrm{Cut}\,(\gamma(s+L/2))$ along γ. We easily see that $Q_{\dot{\gamma}}$ vanishes along γ if and only if $\mathscr{H}_\gamma(\dot{\gamma})$ is parallel distribution along γ.

References

[1] Allamigeon, A., Propriétés globales des espaces de Riemann harmoniques. Ann. Inst. Fourier **15** (1965), 91–132.

[2] Berger, M., Gaudechon, P., Mazet, E., Le spectre d'une variété Riemannienne., Lecture Notes in Math., **194**, Berlin, Heiderberg, New York, Springer, 1971.

[3] Besse, A., Manifolds all of whose Geodesics are Closed, Ergebnisse der Math., **93**, Springer 1978.

[4] Nakagawa, H., On a certain minimal immersion of a Riemannian manifold into a sphere. Kodai Math. J. 3, (1980), 321–340.

[5] ——, Global Riemannian Geometry. Tokyo, Kaigai Schuppan 1977 (in Japanese).

[6] Sakamoto, K., Helical immersions into a unit sphere. Math. Ann. **261** (1982), 63–80.

[7] Tsukada, K., Isotropic minimal immersions of spheres into spheres. To appear.

[8] Wallach, N. R., Symmetric spaces edited by W. M. Boothby and G. L. Weiss, New York, Marcel Dekker, 1972.

Department of Mathematics
Tokyo Institute of Technology
Meguro-ku, Tokyo 152
Japan