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What is known about the Hodge Conjecture?

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In this talk we review the present state of our knowledge about the Hodge Conjecture, one of the central problems in complex algebraic geometry. In his 1950 Congress address [12], Hodge reported on the topological and differential-geometric methods in studying algebraic varieties and complex manifolds which had been initiated by Lefschetz and developped by Hodge himself. He raised there many problems, and most of them were settled in 1950's by extensive works due to Kodaira and others. One notable exception to this is the so-called *Hodge Conjecture* which, if true, will give a characterization of cohomology classes of algebraic cycles on a nonsingular projective variety, generalizing the Lefschetz criterion for the case of divisors. This conjecture has an arithmetic flavour, as is common to most problems concerning algebraic cycles, which makes the problem interesting and difficult at the same time.

In § 1, we recall the formulation of Hodge's general conjecture—the original one due to Hodge and modified version due to Grothendieck [8]. In § 2, we review the various cases where this conjecture has been verified. Although the examples are still not so many, the reader might notice substantial increase of the known cases as compared to those listed in [8] (1969). In § 3, we mention further examples some of which are new. A few remarks will be given in § 4.

§ 1. The formulation of the problem

Let V denote a nonsingular projective variety over $\boldsymbol{C}$, and let d be an integer such that $0<d<n=\dim V$. We use the same letter V to denote the compact complex manifold attached to V. As is well known (cf. [3]), an irreducible subvariety W of V (of dimension $r=n-d$) defines a topological $2r$-cycle on V so that its homology class $h(W)\in H_{2r}(V,\boldsymbol{Z})$ and the dual cohomology class $c(W)\in H^{2d}(V,\boldsymbol{Z})$ are well defined. The cohomology class of an algebraic cycle $Z=\sum n_i W_i$ ($n_i\in\boldsymbol{Z}$) is defined by linearity: $c(Z)=\sum n_i c(W_i)$. Let

$$\mathscr{C}^d(V)_{\boldsymbol{Z}}\subset H^{2d}(V,\boldsymbol{Z})$$

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denote the subgroup generated by $c(Z)$'s. The basic problem is this: *how can one characterize* $\mathscr{C}^d(V)_Z$ *in* $H^{2d}(V, \boldsymbol{Z})$?

For $d=1$ (i.e. for divisors), the answer is given by the Lefschetz-Hodge theorem: an element $\xi \in H^2(V, \boldsymbol{Z})$ belongs to $\mathscr{C}^1(V)_Z$ if and only if the image of ξ in $H^2(V, \boldsymbol{C})$ is of Hodge type (1,1). If $\dim V=2$, this is equivalent to the Lefschetz criterion: a topological 2-cycle Γ on V is algebraic (i.e. is homologous to a divisor on V) if and only if all the algebraic double integrals of the first kind on V have period zero on Γ, i.e. $\int_\Gamma \omega=0$ for all holomorphic 2-forms ω on V (cf. [22]). The Lefschetz-Hodge theorem can be easily proven by using the exponential sheaf sequence (cf. [17], [18]).

Now, saying that "it is clearly a matter of great importance to extend Lefschetz's condition for a 2-cycle to be algebraic," Hodge [12] proposed the following generalization. Given a topological p-cycle Γ on V, it is said to be *of rank* k if (i) there exists a Γ', homologous to Γ, such that the support of Γ' is contained in an algebraic subset of dimension $p-k$ of V; and (ii) k is the largest integer with the above property (i). By repeated use of Lefschetz theorem on hyperplane sections, we see that if $p \leq n$ then $k \geq 0$. Then it is easy to see

$$\left[\frac{p}{2}\right] \geq k \geq \max(0, p-n).$$

By definition, a topological p-cycle Γ is of rank $k=p/2$ if and only if Γ is homologous to an algebraic cycle.

Proposition. *If* Γ *is a topological p-cycle of rank* $\geq k$, *then the cohomology class* $c(\Gamma)_Q$ *(with rational coefficients) dual to the homology class* $h(\Gamma)$ *of* Γ *belongs to* $H^{2n-p}(V, \boldsymbol{Q}) \cap F^{n-p+k}H^{2n-p}(V, \boldsymbol{C})$, *where* $F^{\cdot}H^*(V, \boldsymbol{C})$ *denotes the "Hodge filtration" on* $H^*(V, \boldsymbol{C})$:

$$F^r H^i(V, \boldsymbol{C}) = \bigoplus_{\substack{p'+q=i \\ p' \geq r}} H^{p',q},$$

$H^{p',q}$ *being the space of cohomology classes of type* (p', q).

Proof. If ω is a closed C^∞-differential form of degree p, we have

$$\int_\Gamma \omega = \int_V \eta \wedge \omega$$

where η is a closed form of degree $2n-p$ representing $c(\Gamma)_Q$. The assertion follows immediately from this formula.

Hodge's original problem: Is the converse of the above proposition true?

In the special, but important, case where p is even and $k=p/2$, this problem reduces to the following question: write $k=n-d$, and assume that $c(\Gamma)_Q \in H^{2d}(V, \boldsymbol{Q}) \cap H^{d,d}$. Is Γ homologous to an algebraic cycle of codimension d?

As we have seen above, this is true for $d=1$ by the Lefschetz-Hodge theorem. For $d>1$, however, there is an example, due to Atiyah-Hirzebruch [1], with the property that $H^{2d}(V, \boldsymbol{Z})_{\mathrm{tor}}$ is not contained in $\mathscr{C}^d(V)_{\boldsymbol{Z}}$. Thus it is necessary to slightly modify the question by taking $\boldsymbol{Q}$-coefficients. Let $\mathscr{C}^d(V)_{\boldsymbol{Q}}=\mathscr{C}^d(V)_{\boldsymbol{Z}} \otimes \boldsymbol{Q} \subset H^{2d}(V, \boldsymbol{Q})$. Then *the* (*usual*) *Hodge Conjecture* is stated as follows:

Hodge (V, d): $\mathscr{C}^d(V)_{\boldsymbol{Q}}=H^{2d}(V, \boldsymbol{Q}) \cap H^{d,d}$?

"This conjecture is plausible enough, and (as long as it is not disproved!) should certainly be regarded as the deepest conjecture in the "analytic" theory of algebraic varieties", as Grothendieck says in [8].

As for the general case of Hodge's original problem, let us introduce the following notation. Let $F'^r H^i(V, \boldsymbol{Q})$ be the space of cohomology classes $\xi \in H^i(V, \boldsymbol{Q})$ such that ξ vanishes on $V-W$ for some algebraic subset W of codimension $\geq r$. In other words, it is the subspace of $H^i(V, \boldsymbol{Q})$ spanned by $\{c(\Gamma)_{\boldsymbol{Q}} \mid \Gamma$: topological p-cycle of rank $\geq k\}$ where $p=2n-i$ and $k=r-(n-p)$. By the previous proposition, we have

$$F'^r H^i(V, \boldsymbol{Q}) \subset H^i(V, \boldsymbol{Q}) \cap F^r H^i(V, \boldsymbol{C}),$$

and the general problem of Hodge can be restated as follows:

Hodge $(V, F^r H^i)$: $F'^r H^i(V, \boldsymbol{Q}) = H^i(V, \boldsymbol{Q}) \cap F^r H^i(V, \boldsymbol{C})$?

Now Grothendieck [8] pointed out that, while $F'^r H^i(V, \boldsymbol{Q})$ is always a sub-Hodge structure of $H^i(V, \boldsymbol{Q})$ (i.e. stable under the Hodge decomposition), $H^i(V, \boldsymbol{Q}) \cap F^r H^i(V, \boldsymbol{C})$ is not necessarily one. After giving certain 3-fold product of an elliptic curve with itself as a counterexample to Hodge $(V, F^r H^i)$ $(r=1, i=3)$, Grothendieck proposed the following modified version:

G-Hodge $(V, F^r H^i)$: Is $F'^r H^i(V, \boldsymbol{Q})$ the largest sub-Hodge structure contained in $H^i(V, \boldsymbol{Q}) \cap F^r H^i(V, \boldsymbol{C})$?

In this form, no counterexample has been known so far. Note that no modification is necessary for Hodge $(V, F^d H^{2d})$, which is nothing but Hodge (V, d), because $H^{2d}(V, \boldsymbol{Q}) \cap H^{d,d}$ is itself a sub-Hodge structure of pure type (d, d).

§ 2. Review of known cases

First we consider the usual Hodge Conjecture. To simplify the notation, we write

$$\begin{cases}\mathscr{C}^d(V)=\mathscr{C}^d(V)_Q\otimes C\\ \mathscr{B}^d(V)=(H^{2d}(V,Q)\cap H^{d,d})\otimes C.\end{cases}$$

The elements of $\mathscr{B}^d(V)$ will be called *Hodge cycles* (or Hodge classes) of codimension d. The conjecture Hodge(V, d) is equivalent to asserting $\mathscr{B}^d(V)=\mathscr{C}^d(V)$.

(1) Hodge $(V, 1)$ is always true (Lefschetz-Hodge theorem).

(2) Hodge $(V, n-1)$ $(n=\dim V)$ is true. More generally, if Hodge (V, d) is true for $d<n/2$, then Hodge $(V, n-d)$ is also true. Indeed, taking an ample class $L\in H^2(V,Q)$, consider the map

$$L^{n-2d}\colon H^{2d}(V,Q)\longrightarrow H^{2(n-d)}(V,Q)$$

defined by multiplication by L^{n-2d}. By the strong Lefschetz theorem, this map is an isomorphism. Further it sends $\mathscr{C}^d(V)_Q$ into $\mathscr{C}^{n-d}(V)_Q$, and $\mathscr{B}^d(V)$ onto $\mathscr{B}^{n-d}(V)$ (note that L is of type (1, 1)). This shows the assertion (cf. [16]).

It follows from (1), (2) that Hodge (V, d) is always true if $n=\dim V\leq 3$. For $n\geq 4$ and $1<d<n-1$, all the known results are restricted to some special type of varieties, for which the cohomological structure is well understood.

(3) If V is a flag variety G/P (e.g. projective space, Grassmannian, etc.), then it is known that $H^*(V,Z)=\mathscr{C}^*(V)_Z$. Hence Hodge (V, d) is trivially true for all d.

(4) Let $V\subset P^{n+1}$ be a nonsingular hypersurface of degree m. From the structure of $H^*(V,Q)$, Hodge (V, d) is non-trivial only if $n=\dim V$ is even and $d=n/2$. When $n=2d$, the conjecture has been verified for the following cases:

a) $V=$any 4-fold of degree $m\leq 5$. The verification of Hodge $(V, 2)$ has been made by various people: the case $m\leq 2$ is easy, the case $m=3$ (cubic 4-folds) by Griffiths and Zucker [43] by the method of normal functions, and by Murre [27] by geometric argument which also works for unirational 4-folds; the case $m=4$ by Bloch and Murre, and the case $m=5$ by Conte and Murre [4]. The method of Murre and others is to find a dominant rational map of a product $T\times P^1$ (T: 3-fold) to the given V; then, if we let $U\to T\times P^1$ be a composite of blowing-ups (with nonsingular centers of dimension ≤ 2) which gives a dominant morphism $U\to V$, Hodge $(V, 2)$ readily follows from Hodge $(U, 2)$ which in turn is a consequence of the easy fact Hodge $(T\times P^1, 2)$.

b) $V=X_m^n$, the Fermat variety of dimension n and degree m under certain arithmetic conditions on m and n (Ran [30], Shioda [34]). In particular, Hodge$(X_m^n, n/2)$ is known to be true if (i) $m=$prime, any n (Ran), (ii) $m\leq 20$, any n (Shioda) or (iii) $m=2\cdot$(prime), any n (Shioda, Miyawaki). For the Fermat 4-fold X_m^4, we can verify Hodge $(X_m^4, 2)$ for all $m\leq 24$, but the case $m=25$ cannot be settled by our inductive method (cf. [35], Appendix). More generally if $m=p^2$ (p: prime), then Hodge $(X_m^n, n/2)$ is true for $n<p-1$, but it is unknown for $n=p-1$ ($p\geqslant 5$). Further, if m is as in (i), (ii) or (iii), then Hodge (X, d) is true for arbitrary product $X=\prod_{i=1}^k X_m^{n_i}$ and for any codimension d ([34, Th. IV]).

(5) Before discussing the case of abelian varieties, we insert a remark. We set

$$\mathscr{B}^*(V)=\sum_{d=0}^{n}\mathscr{B}^d(V), \qquad \mathscr{C}^*(V)=\sum_{d=0}^{n}\mathscr{C}^d(V);$$

these are graded subrings of $H^{2*}(V, \boldsymbol{C})$, called respectively the *Hodge ring* of V and the *algebraic cycle ring* of V. Let $\mathscr{D}^*(V)=\sum \mathscr{D}^d(V)$ denote the subring of $\mathscr{B}^*(V)$ generated by $\mathscr{B}^1(V)$. Since $\mathscr{B}^1(V)=\mathscr{C}^1(V)$ by (1), we have

$$\mathscr{B}^d(V)\supset\mathscr{C}^d(V)\supset\mathscr{D}^d(V) \qquad \text{(any } d).$$

Thus, if $\mathscr{B}^d(V)=\mathscr{D}^d(V)$ holds, then (i) Hodge(V, d) is true and (ii) any algebraic cycle of codimension d is homologous over $\boldsymbol{Q}$ to a linear combination of intersections of d divisors, and conversely. In particular, if $\mathscr{B}^*(V)=\mathscr{D}^*(V)$, the usual Hodge Conjecture is true for V in all codimensions.

Remark. According to Seshadri, an analogous condition over $\boldsymbol{Z}$ appears in some other context. Namely, if $V=G/P$ is a flag variety, then the subring $\mathscr{D}^*(V)_{\boldsymbol{Z}}$ of $\mathscr{C}^*(V)_{\boldsymbol{Z}}$ generated by $\mathscr{C}^1(V)_{\boldsymbol{Z}}$ is of finite index in $\mathscr{C}^*(V)_{\boldsymbol{Z}}$, and $\mathscr{D}^*(V)_{\boldsymbol{Z}}=\mathscr{C}^*(V)_{\boldsymbol{Z}}$ holds if and only if G has the following property: any principal bundle over a nonsingular variety with structure group G which is locally isotrivial is locally trivial (see Grothendieck [9]).

(6) Let us review the case of abelian varieties. The preceding fact in (5) has been practically the only way for establishing the (usual) Hodge Conjecture for an abelian variety, except for the case f) below.

a) "*general*" *abelian varieties* (Mattuck [23]). Let $A=\boldsymbol{C}^n/(\Omega\boldsymbol{Z}^{2n})$ be a complex abelian variety with the normalized period matrix Ω:

$$\Omega=\begin{pmatrix} e_1 & & 0 & \tau_{11}\cdots\tau_{1n} \\ & \ddots & & \vdots \quad\quad \vdots \\ 0 & & e_n & \tau_{n1}\cdots\tau_{nn} \end{pmatrix} \qquad \begin{array}{l} e_i\geq 1, \quad e_i \mid e_{i+1} \\ \mathrm{Im}\,((\tau_{ij}))>0 \end{array}$$

Set $\tau_{ij}=x_{ij}+\sqrt{-1}\,y_{ij}$. If A is "general" in the sense that x_{ij}, y_{ij} $(1\leq i\leq i\leq n)$ form $n(n+1)$ algebraically independent real numbers, then Mattuck proved that $\mathscr{B}^*(A)=\mathscr{D}^*(A)$ holds and $\dim \mathscr{B}^d(A)=1$ $(0\leq d\leq n)$. More generally he verified the general Hodge Conjecture Hodge (A, F^rH^i) for all r, i for such an A, by an inductive argument.

b) *powers of an elliptic curve* (Tate [39], Murasaki [26]). If $A=E^n$ (E: an elliptic curve), then we have again $\mathscr{B}^*(A)=\mathscr{D}^*(A)$ so that the usual Hodge Conjecture is true. Further

$$\dim \mathscr{B}^d(A)=\binom{n}{d}^2-\binom{n}{d-1}\cdot\binom{n}{d+1}\delta(E) \qquad (0\leq d\leq n),$$

where $\delta(E)=0$ or 1 according to whether or not E has complex multiplication.

In each of the examples below, both cases $\mathscr{B}^*(A)=\mathscr{D}^*(A)$ and $\mathscr{B}^*(A)\neq\mathscr{D}^*(A)$ occur, but the Hodge Conjecture has been verified only in the former case, except for the recent results stated in f).

c) *generic fibre of certain families of abelian varieties* (Kuga[20]).

d) *characterization of the Hodge ring via the Hodge group* (Mumford [24], [25]). In [24], Mumford introduced the important notion of the *Hodge group*, Hg (A), of an abelian variety A. It is an algebraic subgroup of $GL\,(H_1(A, \boldsymbol{Q}))$ defined over $\boldsymbol{Q}$ with the property that the Hodge ring of $A^k=A\times\cdots\times A$ (k-times) is the ring of invariants of Hg (A) in $H^*(A^k, \boldsymbol{Q})$ under the natural action;

$$\mathscr{B}^*(A^k)_{\boldsymbol{Q}}=[H^*(A^k, \boldsymbol{Q})]^{\mathrm{Hg}(A)} \qquad (k=1, 2, \cdots).$$

Thus if one knows enough about the group Hg (A) and its representation on $H^1(A, \boldsymbol{Q})$ for a given A, one can compute $\mathscr{B}^*(A)$ and check whether $\mathscr{B}^*(A)=\mathscr{D}^*(A)$ holds. For instance, the cases a) and b) above can also be treated by this method (cf. Imai [14] for the case of a product of elliptic curves). On the other hand, it is easy to give examples of abelian varieties A such that $\mathscr{B}^*(A)\neq\mathscr{D}^*(A)$. Further results based on this method will be mentioned below.

e) *abelian varieties of CM type* (Pohlmann [29, § 2]). In this case, Pohlmann gave a combinatorial description of the Hodge ring $\mathscr{B}^*(A)$ in terms of the action of the *CM* field on the complex cohomology $H^*(A, \boldsymbol{C})$, and proved the equivalence of the Hodge Conjecture and the Tate Conjecture for this type of abelian varieties. There is given an explicit example (due to Mumford) of a 4-dimensional abelian variety of *CM* type such that $\mathscr{B}^2(A)\neq\mathscr{D}^2(A)$, for which Hodge $(A, 2)$ is still unknown.

According to Mumford [25], an abelian variety A is of *CM* type in the extended sense (i.e. isogenous to a product of abelian varieties of *CM* type in the usual sense) if and only if its Hodge group $\mathrm{Hg}(A)$ is an algebraic torus. We have $\dim \mathrm{Hg}(A) \leq \dim A$, and A is called *non-degenerate* if equality holds (Kubota [19], Ribet [31]). For an abelian variety A of *CM* type, the two conditions (i) A is non-degenerate and (ii) $\mathscr{B}^*(A)=\mathscr{D}^*(A)$ seem closely related. A recent result of Ribet and Lenstra (private communication in May 1981) shows that (i) and (ii) are indeed equivalent if A is an abelian variety with the *CM* field which is an abelian extension of $\boldsymbol{Q}$. F. Hazama shows that if A is simple, then (i) implies (ii) in general.

f) *abelian varieties of Fermat type* (Shioda [35]). An abelian variety is said to be *of Fermat type* (of degree m) if it is isogenous to a product of certain "admissible" factors of the Jacobian variety of the Fermat curve $x^m+y^m+z^m=0$. The Hodge ring of such an abelian variety can be explicitly described as in the case e). On the other hand, by applying the results on Fermat varieties and their products in (4b), we can verify the Hodge Conjecture Hodge (A, d) in certain cases *even if* $\mathscr{B}^d(A)=\mathscr{D}^d(A)$ *does not hold.* As an explicit example, we mention the special case where $A=J(C_m)$ is the Jacobian variety of the curve $C_m: y^2=x^m-1$ (m: odd). In this case, Hodge $(A, 2)$ is true for any odd m, while we have $\mathscr{B}^2(A)=\mathscr{D}^2(A)$ if and only if $m \not\equiv 0 \pmod 3$, and

$$\dim (\mathscr{B}^2(A)/\mathscr{D}^2(A)) \geq m/3-1 \qquad \text{if } m \equiv 0 \pmod 3;$$

further, if m is prime, $\mathscr{B}^*(A)(=\mathscr{D}^*(A))$ is isomorphic to the exterior algebra $\bigwedge^* (\mathscr{B}^1(A))$ and Hodge (A, d) is true for all d.

g) *abelian varieties satisfying certain conditions on the endomorphism algebra.* Let $\mathfrak{g}$ denote the Lie algebra of the complex Hodge group $\mathrm{Hg}(A) \otimes_{\boldsymbol{Q}} \boldsymbol{C}$, and let $\rho: \mathfrak{g} \to \mathrm{End}(E)$ be the Lie algebra representation induced from the natural representation of $\mathrm{Hg}(A)$ on $E=H^1(A, \boldsymbol{C})$. We have

$$\dim \mathscr{B}^d(A) = \dim [\textstyle\bigwedge^{2d} E]^{\mathfrak{g}}.$$

Now if the center of the $\boldsymbol{Q}$-algebra $\mathrm{End}(A) \otimes \boldsymbol{Q}$ is a product of totally real fields, then the Hodge group $\mathrm{Hg}(A)$ is semisimple. Then $\dim [\bigwedge^{2d} E]^{\mathfrak{g}}$ can be computed by the general theory of representations of a semisimple Lie algebra. In a series of papers ([36], [37], [38]), Tankeev has made a systematic study from this viewpoint, and obtained several sufficient conditions for $\mathscr{B}^*(A)=\mathscr{D}^*(A)$. In the recent article [38], he has proved that any 5-dimensional *simple* abelian variety satisfies $\mathscr{B}^*(A)=\mathscr{D}^*(A)$ and hence the (usual) Hodge Conjecture. (The corresponding fact for a 4-dimensional simple abelian variety is false.) Further Hazama [11], and independently K. Murty, have proved by the similar method that the Jacobian varieties $J_0(N)$ and $J_1(N)$ of modular curves $X_0(N)$ or $X_1(N)$ (in the standard notation) satisfy the condition $\mathscr{B}^*(A)=\mathscr{D}^*(A)$.

h) *abelian varieties admitting complex multiplication by an imaginary quadratic field* (Weil [41]). In searching for possible counterexamples to the Hodge Conjecture, Weil [41] (cf. [32], [33]) examined families of abelian varieties admitting such multiplication as stated above, and showed among others that the generic fibre has the property that $\mathscr{B}^*(A) \neq \mathscr{D}^*(A)$. In concluding the case of abelian varieties, we quote the last paragraph from [41] where a weaker version of the Hodge Conjecture is suggested: "Thus it may happen that imposing a Hodge class upon an algebraic variety B is equivalent to imposing an algebraic class (even one of codimension 1) upon a product of two or more factors isomorphic to B. Even if this were not so in general, it might still be true for abelian varieties."

(7) total spaces of certain families of abelian varieties (Hall-Kuga [10], Kuga [21]). Kuga and his school have extended the method of (6c) to the total space V of families, and verified the condition $\mathscr{B}^d(V) = \mathscr{D}^d(V)$ in certain cases.

As for the *general Hodge Conjecture* Hodge (V, F^rH^i), the following cases have been checked.

(8) Hodge (V, F^1H^3) is true if V is a 3-fold in a projective space such that the general hyperplane section is a surface of geometric genus 0 (Hodge [13]). In particular, Hodge (V, F^1H^3) is true for a cubic 3-fold V in $\boldsymbol{P}^4$ (cf. the comments in [8]).

(9) An inductive approach to Hodge (V, F^rH^n) $(n = \dim V)$ generalizing (8) is stated in Grothendieck [8], § 3, b).

(10) The case of the "general" abelian varieties is due to Mattuck, as mentioned in (6b).

§ 3. Further examples

In this section, we discuss a few new examples. First we formulate a general fact:

Lemma. *Let H_Q be a rational Hodge structure of weight i, and let $H_C = H_Q \otimes \boldsymbol{C} = \bigoplus_{p+q=i} H^{p,q}$ (cf. [5]). Assume that there is a finite Galois extension K of $\boldsymbol{Q}$ $(K \subset \boldsymbol{C})$ such that $H_K = H_Q \otimes K$ has a decomposition into K-subspaces $W(\alpha)$ $(\alpha \in I)$ with the following three properties:*

(i) $\dim_K W(\alpha) = 1$ *for all* $\alpha \in I$;

(ii) *for any* $\alpha \in I$ *and any* $\sigma \in \mathrm{Gal}(K/\boldsymbol{Q})$, *there exists* $\beta \in I$ *such that* $W(\alpha)^\sigma = W(\beta)$ (*write* $\beta = \alpha^\sigma$);

(iii) *for any* $\alpha \in I$, $W(\alpha) \subset H^{p,q}$ *for some* (p, q). *Let* $F^rH_C = \bigoplus_{p \geq r} H^{p,q}$, *and let* $I^{p,q} = \{\alpha \in I \mid W(\alpha) \subset H^{p,q}\}$. *Then, for any r, $H_Q \cap F^rH_C$ is a sub-Hodge structure of H_Q, and we have*

$$[H_Q \cap F^rH_C] \otimes_Q \boldsymbol{C} = \bigoplus_{\alpha \in I^r} V(\alpha) \qquad (V(\alpha) = W(\alpha) \otimes_K \boldsymbol{C})$$

where $I^r=\{\alpha\in I\,|\,\alpha^\sigma\in\bigcup_{p,q\geq r}I^{p,q}$ *for all* $\sigma\in\mathrm{Gal}\,(K/Q)\}$. *Furthermore* $\bigoplus_{\alpha\in S}V(\alpha)$ *is a sub-Hodge structure of* H_Q *for any subset* S *of* I^r *which is stable under* $\mathrm{Gal}\,(K/Q)$.

Proof. With the notation above, we have $H_C=\bigoplus_{\alpha\in I}V(\alpha)$ and $H^{p,q}=\bigoplus_{\alpha\in I^{p,q}}V(\alpha)$. Hence it is easy to see that $H_Q\cap F^rH_C$ is contained in $\bigoplus_{\alpha\in I^r}V(\alpha)$. Conversely, let $\alpha\in I^r$ and take $\omega\in W(\alpha)$, $\omega\neq 0$. Then, for any σ, we have $\omega^\sigma\in V(\alpha^\sigma)\subset F^rH_C$. Thus, if $a\in K$ and $\eta=\sum_\sigma a^\sigma\omega^\sigma$, then $\eta\in H_Q\cap F^rH_C$. Let $a_1,\cdots,a_m$ be a basis of K over Q and set $\eta_i=\sum_\sigma a_i^\sigma\omega^\sigma$ $(1\leq i\leq m)$. Then each ω^σ is a K-linear combination of $\eta_1,\cdots,\eta_m$, because $\det(a_i^\sigma)\neq 0$ by Galois theory. In particular, we see that $W(\alpha)\subset[H_Q\cap F^rH_C]\otimes C$ for any $\alpha\in I^r$. The second assertion is proved in the same way. q.e.d.

This lemma applies, for example, to the case $H_Q=H^i(V,Q)$ if V is a Fermat variety or an abelian variety of *CM* type (or any product of such), where the subspaces $V(\alpha)$ appear as the eigenspaces of certain automorphism group or of complex multiplications (cf. [29], [34], and (4b), (6e), (6f) of § 2). In fact, the above lemma (with its proof) is extracted from these cases for the possible application to some other cases. At any rate, it follows that, if V is a product of Fermat varieties or of abelian varieties of *CM* type, the modified version of the general Hodge Conjecture G-Hodge (V,F^rH^i) reduces to the original one Hodge (V,F^rH^i).

We give a few more examples which we have verified by making use of the above lemma.

(11) If E is an elliptic curve with complex multiplication, then, for any power $A=E^n$, Hodge (A,F^rH^i) is true for all r, i. (The proof is left to the reader as an exercise). Note that if E has no complex multiplication, then Hodge (A,F^rH^i) is not necessarily true as Grothendieck [8] showed.

(12) Let $V=\prod_{i=1}^k X_m^{n_i}$ be arbitrary product of Fermat varieties $X_m^{n_i}$ of degree m. Assume that $m\leq 4$. Then Hodge (V,F^rH^i) is true for all r, i. The proof is based on the "inductive structure" of Fermat varieties (cf. [34]), similar to the case (13) below.

(13) The case of the Fermat 3-fold X_m^3 of degree m. In this case, we have (cf. [30], [34])

$$[H^3(X_m^3,Q)\cap F^1H^3(X_m^3,C)]\otimes C=\bigoplus_{\alpha\in I}V(\alpha)$$

where $I=\{\alpha=(a_0,a_1,\cdots,a_4)\,|\,1\leq a_i\leq m-1,\ \sum_{i=0}^4\langle ta_i\rangle/m=2$ or 3 for all $t\in(Z/m)^\times\}$. ($\langle a\rangle$ is the least positive residue of a mod m). Let us call $\alpha=(a_i)\in I$ *decomposable* if $a_i+a_j\equiv 0\pmod m$ for some $i\neq j$, and *quasi-decomposable* if $a_3+a_4\not\equiv 0\pmod m$ and $(a_0,a_1,a_2,\langle a_3+a_4\rangle)\in\mathfrak{B}_m^2$ (after a permutation of a_i's), where

$$\mathfrak{B}^2_m=\{(b_0, b_1, b_2, b_3)\,|\,1\leq b_i\leq m-1, \sum_{i=0}^{3}\langle tb_i\rangle/m=2 \text{ for all } t\in(Z/m)^\times\}.$$

Now we claim that, if *every element of I is either decomposable or quasi-decomposable* (call this condition (R^3_m)), *then* Hodge (X^3_m, F^1H^3) *is true.* For the proof, consider the natural isomorphism obtained from the inductive structure of Fermat varieties (cf. [34]):

$$H^3(X^3_m, C)\simeq[H^2_{\text{prim}}(X^2_m, C)\otimes H^1(X^1_m, C)]^{\mu_m}\oplus[H^1(X^1_m, C)\otimes H^0_{\text{prim}}(X^0_m, C)].$$

Let $\alpha=(a_0, a_1, a_2, a_3, a_4)\in I$. If α is decomposable (say $a_3+a_4\equiv 0 \pmod m$), then $V(\alpha)$ corresponds to $V(\beta)\otimes V(\gamma)\subset H^1(X^1_m)\otimes H^0(X^0_m)$ where $\beta=(a_0, a_1, a_2)$ and $\gamma=(a_3, a_4)$. Hence, by the inductive structure, the support of elements of $V(\alpha)$ is contained in the P^1-join $Z=X^1_m\vee X^0_m$ of X^1_m and X^0_m which are embedded in X^3_m as

$$X^1_m=\{(x_0, x_1, x_2, 0, 0)\in X^3_m\}, \qquad X^0_m=\{(0, 0, 0, x_3, x_4)\in X^3_m\}.$$

This implies that $V(\alpha)\subset F'^1H^3(X^3_m)$ because the codimension of Z in X^3_m is 1. Next, if α is quasi-decomposable, we may assume that $\beta=(a_0, a_1, a_2, \langle a_3+a_4\rangle)\in\mathfrak{B}^2_m$; put $\gamma=(a_3, a_4, \langle -a_3-a_4\rangle)\in\mathfrak{A}^1_m$. Then $V(\alpha)$ corresponds to $V(\beta)\otimes V(\gamma)\subset[H^2_{\text{prim}}(X^2_m)\otimes H^1(X^1_m)]^{\mu_m}$. Now $V(\beta)$ is in the space of Hodge cycles on X^2_m and hence the support is contained in a (reducible) curve C on X^2_m (Lefschetz criterion). The support of $V(\alpha)$ in X^3_m is then contained in the image of $C\times X^1_m$ under the rational map $X^2_m\times X^1_m\to X^3_m$ (of the inductive diagram), which shows $V(\alpha)\subset F'^1H^3(X^3_m)$. This proves the claim. The reader will find a slightly different approach to this fact in Ran [30, § 3].

It is easy to check the condition (R^3_m) for $m\leq 10$. For $m\leq 7$, all elements of I are decomposable; for $m=8$, 9 or 10, there are indecomposable elements in I but all of them are quasi-decomposable. However, the condition (R^3_m) is not always satisfied. For example, for $m=11$, the element $\alpha=(1\ 1\ 5\ 7\ 8)$ of I is neither decomposable nor quasi-decomposable. Thus the general Hodge Conjecture for the Fermat 3-fold X^3_m is true for $m\leq 10$, but we cannot verify it for $m=11$ by the method described above. Incidentally, the above example shows that, when the intermediate Jacobian variety of the 3-fold X^3_m is decomposed into "admissible" factors $A_\alpha(\alpha\in I)$ which are abelian varieties with complex multiplication by cyclotomic fields, the "*CM* type" of some A_α does not necessarily occur among those obtained from admissible factors of the Jacobian variety of the Fermat curve X^1_m (cf. Weil [40]).

(14) It seems worthwhile to extend the notion of the Hodge group for the investigation of the general Hodge Conjecture. Group-theoreti-

cally this must be possible, at least in some cases of interest (e.g. abelian varieties, $\cdots$).

§ 4. Remarks

In the previous sections, we have reviewed all the cases, known to the author, where the (usual or general) Hodge Conjecture has been verified or at least examined.

There are a few subjects which we wanted to discuss here but were unable to do so, and we give references to these:

(i) The theory of normal functions associated with Lefschetz pencils (Griffiths [7], Zucker [42]). This is probably the most direct approach to the Hodge Conjecture, although it has led so far to the verification of the conjecture only in a few cases where more "elementary" geometric method works also.

(ii) Questions involving algebraic cycles with integral coefficients (rather than rational coefficients). As for some topological conditions imposed on the cohomology classes of algebraic cycles, see Atiyah-Hirzebruch [1] and Kawai [15]. In Barton-Clemens [2], there is a discussion in a special case about which integral multiples of a topological cycle defining an algebraic cycle with rational coefficients are integral algebraic cycles themselves.

(iii) Among important problems on algebraic cycles, there is the Tate Conjecture [39] which is closely related to the Hodge Conjecture. In many of the works reviewed in § 2, both conjectures are studied at the same time.

(iv) Deligne's theory of absolute Hodge cycles [6]. Deligne introduced the notion of absolute Hodge cycles, which lies in between that of usual Hodge cycles and algebraic cycles. He proved that, on an abelian variety, every Hodge cycle is an absolute Hodge cycle, and deduced from this some important arithmetic consequences. Further Ogus [28] studies a variant of this notion in crystalline cohomology.

Finally, after reviewing various examples, do you think that the Hodge Conjecture is true in general? Some people think that the answer is negative, and have made some efforts to find a counterexample. At present it may be just a wishful thinking to assume its truth in general. The essential difficulty for this (and other related) conjecture seems to lie in the fact that we know so little about how to produce algebraic cohomology classes of higher codimension, other than starting from divisors and forming various "elementary" operations (e.g. taking intersections, or transplanting cycles from one variety to another in some nice situation).

Added in proof.

The following results should be added to the list of § 2.

(4b)-bis. (iv) For the Fermat variety X^n_m, Hodge $(X^n_m, n/2)$ is true for any pair (m, n) such that m is not divisible by any prime number less than $n+2$. Furthermore, in this case, $\mathscr{C}^{n/2}(X^n_m)_Q$ is spanned by the cohomology classes of (finitely many) linear subspaces of P^{n+1} lying on X^n_m. This result is due to N. Aoki (in preparation), and is based on the results of Ran [30] and Shioda [34].

(v) Hodge$(X^4_m, 2)$ is true for the case $m=25$. This is the first case where the method of inductive structure [34] fails, but we can show that the Hodge classes in question are induced from Hodge classes on a certain quintic 4-fold for which the Hodge Conjecture is known to hold by Conte-Murre [4].

(6g)-bis. Tankeev and, independently, Ribet have recently proven that the Hodge Conjecture is true for any simple abelian variety of prime dimension.

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