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Moduli of Stable Sheaves — Generalities and the Curves of Jumping Lines of Vector Bundles on $\mathbf{P}^2$

Masaki Maruyama

Introduction

For the last two decades great progress has been made in the problem of moduli of algebraic vector bundles, especially in the framework of the geometric invariant theory by D. Mumford. After fruitful works on curves by D. Mumford, M. S. Narasimhan, P. E. Newstead, S. Ramanan and C. S. Seshadri, etc., F. Takemoto started his attempt to generalize the results to higher dimensional cases. And then we succeeded in constructing moduli spaces of algebraic vector bundles. We know, however, only a few on the structure of each moduli space. The author hopes that it is useful at this moment to publish an expository account on the field.

This note consists of two parts. Part I will treat generalities of the moduli spaces of algebraic vector bundles. Except for a few, almost all are well-known to the experts of this field. In § 1 the meaning of the moduli spaces is explained and we shall show that we can not construct the moduli spaces without restricting ourselves to a suitable subfamily of algebraic vector bundles. § 2 is devoted to showing the results on curves as a good prototype. It should be pointed out that in this case there are a lot of beautiful works other than those which are mentioned in this article. The aim of § 3 is to show the present stage of the problem of moduli of algebraic vector bundles. Unfortunately, for lack of his ability and space, the author could not cover all the results. Especially he excluded those on vector bundles on $\mathbf{P}^3$ by R. Hartshorne, W. Barth, G. Ellingsrud and S. A. Strømme, etc.

Part II deals, contrary to the general viewpoint of Part I, with the curve of jumping lines of a stable vector bundle of rank 2 on $\mathbf{P}^2$ with the first Chern class 0, in connection with the moduli spaces of such vector bundles. The curve of jumping lines is defined in § 1. The complete proof of the assertions in this section will be given in [20]. The purpose of § 2 and § 3 is to generalize Barth's work on the θ-characteristic associated with a vector bundle of rank 2 on $\mathbf{P}^2$ to relative cases. We can

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carry this out by following, with slight modifications, the work of Barth. An application of the generalization to the moduli problem will be described in § 4.

Part I. **Moduli of Stable Sheaves—Generalities**

§ 1. What is the moduli?

Let X be a non-singular projective variety over an algebraically closed field k and $\mathscr{V}\mathscr{B}(X)$ be the set of the isomorphism classes of algebraic vector bundles on X. The problem of moduli of algebraic vector bundles is intuitively to endow $\mathscr{V}\mathscr{B}(X)$ with a *natural* structure of scheme VB_X. What is natural then? We require at least the following properties of VB_X:

(1.1) Let S be an algebraic k-scheme and E a vector bundle on $X \times_k S$. Then the map $S(k) \ni s \to [E_s] = [E \otimes_{\mathcal{O}_S} k(s)] \in \mathscr{V}\mathscr{B}(X)$ is induced by a morphism of S to VB_X, where [ ] denotes the isomorphism class.

Fix an ample line bundle $\mathcal{O}_X(1)$ on X (As for the language in this section, we shall follow that of [15] and [22]). By the invariance of the Hilbert polynomial under flat deformations, VB_X must be a direct sum of VB_X^H, where H is a numerical polynomial of degree $=\dim X$ and each k-rational point of VB_X^H corresponds to a vector bundle E with $\chi(E(m)) = H(m)$. Pick two vector bundles E_1, E_2 on X with $\chi(E_1(m)) = \chi(E_2(m)) = H(m)$. If m is a sufficiently large integer, then $E_i(m)$ has the following property:

(1.2) $E_i(m)$ is generated by its global sections and $H^q(X, E_i(m)) = 0$ for all $q \geq 1$.

By replacing E_i by $E_i(m)$, we may assume that E_i itself has the property. Let us take a look at the quot-scheme $Q = \mathrm{Quot}^H (V \otimes_k \mathcal{O}_X / X/k)$, where V is a k-vector space of dimension $N = H(0) = \dim H^0(X, E_i)$. If $V \otimes_k \mathcal{O}_{X \times Q} \xrightarrow{\theta} F \to 0$ is the universal quotient sheaf on $X \times_k Q$, there exists an open set U of Q such that $U(k) = \{y \in Q(k) \mid F_y = F \otimes_k k(y)$ has the properties (a), (b) and (c)$\}$, (a) F_y is locally free, (b) θ_y: $V \otimes_k \mathcal{O}_X \to F_y$ induces an isomorphism $\Gamma(\theta_y)$: $V = H^0(X, V \otimes_k \mathcal{O}_X) \to H^0(X, F_y)$ and (c) $H^q(X, F_y) = 0$ for all $q \geq 1$. By abuse of notation we denote the restriction of the universal quotient sheaf to $X \times_k U$ by $\theta : \mathcal{O}_{X \times U}^{\oplus N} \to F \to 0$. Through the natural action on V, $G = GL(V)$ acts on Q and U is a G-invariant open subscheme. It is easy to see that the center G_m of G acts trivially on Q and hence $\bar{G} = G/G_m$ acts on U. For a point $y \in U(k)$, an automorphism σ of F_y induces an element σ' of G by the property (b):

$$\begin{array}{ccc} V & \xrightarrow{\Gamma(\theta_y)} & H^0(X, F_y) \\ {\scriptstyle\sigma'}\downarrow & & \downarrow{\scriptstyle\Gamma(\sigma)} \\ V & \xrightarrow{\Gamma(\theta_y)} & H^0(X, F_y) \end{array}$$

Obviously, σ' is contained in the stabilizer group $\mathrm{St}_G(y)$ of G at y. Conversely, if τ is an element of $\mathrm{St}_G(y)$, then τ gives rise to an automorphism of F_y because τ determines a transformation among the global sections which generate F_y. Thus we see that $\mathrm{St}_G(y) \xrightarrow{\sim} \mathrm{Aut}(F_y)$ for every $y \in U(k)$. The image of the center G_m of G by this isomorphism is the multiplications by elements of $k^\times$ on F_y, that is,

$$\mathrm{St}_{\bar{G}}(y) \xrightarrow{\sim} \mathrm{Aut}(F_y)/k^\times. \tag{1.3}$$

An observation similar to the above shows that

(1.4) for $y_1, y_2 \in U(k)$, both y_1 and y_2 belong to the same orbit of $\bar{G}$ if and only if $F_{y_1} \cong F_{y_2}$.

Now, by (1.2) for E_i $(m=0)$, we have a surjective $\eta_i\colon V \otimes_k \mathcal{O}_X \to E_i$ such that $\Gamma(\theta_i)\colon V \to H^0(X, E_i)$ is bijective. The universal property of Q provides us with points z_1, z_2 of $Q(k)$ such that $F_{z_i} \cong E_i$ and $\theta_{z_i} = \eta_i$. Both z_1 and z_2 are points of $U(k)$ because of (1.2).

Assume that VB_X^H exists. Then, by the property (1.1) for VB_X, we obtain a morphism $f\colon U \to VB_X^H$. For each k-rational point x of VB_X^H, $f^{-1}(x)$ is a $\bar{G}$-orbit $o(y)$ of a $y \in U(k)$ by (1.4). On one hand, $\dim_y f^{-1}(f(y))$ is upper semi-continuous on U by a theorem of Chevalley. On the other hand, $\dim_y f^{-1}(f(y)) = \dim o(y) = \dim \bar{G} - \dim \mathrm{St}_{\bar{G}}(y) = N^2 - 1 - \dim \mathrm{St}_{\bar{G}}(y)$ is lower semi-continuous. Thus $\dim \mathrm{St}_{\bar{G}}(y)$ is constant over each connected component of U. This and (1.3) imply that $\dim \mathrm{Aut}(F_y) = \dim_k \mathrm{End}_{\mathcal{O}_X}(F_y)$ depends only on the connected component containing y. We see therefore

(1.5) if $\dim_k \mathrm{End}_{\mathcal{O}_X}(E_1) \neq \dim_k \mathrm{End}_{\mathcal{O}_X}(E_2)$, then z_1 and z_2 belong to different connected components.

Example 1.6. Let T be a non-singular subvariety of codimension 1 in X and D a very ample divisor on T. Assume

$$H^0(T, \mathcal{O}_X(T) \otimes \mathcal{O}_T(-D)) = 0, \tag{1.6.1}$$

$$\dim H^0(T, \mathcal{O}_T(D)) \geq \dim X + 1 = n + 1. \tag{1.6.2}$$

Choose an $(n+1)$-ple $(s_0, \cdots, s_n)$ of elements of $H^0(T, \mathcal{O}_T(D))$ so generally that $s_0, \cdots, s_n$ are independent over k and $\bigcap_{i=1}^{n-1} (s_i)_0 = \phi$. Set $Z = X \times_k \mathbf{A}^1$ and $S = T \times_k \mathbf{A}^1$. On S we have a homomorphism $\alpha\colon \mathcal{O}_S^{\oplus(n+1)} \to p^*(\mathcal{O}_T(D))$ such that on $S_u = T \times u$, $u \in \mathbf{A}^1$, α_u is defined by $\Gamma(\alpha_u)(e_i) = s_i$ $(0 \leq i \leq n-1)$ and $\Gamma(\alpha_u)(e_n) = us_n$, where $p\colon S \to T$ is the projection and $\{e_0, \cdots, e_n\}$ is a fixed free basis of $\mathcal{O}_S^{\oplus(n+1)}$. By the fact that $\bigcap_{i=0}^{n-1} (s_i)_0 = \phi$ we see that α_u is surjective for all $u \in \mathbf{A}^1$ and hence α is surjective by Nakayama's lemma. Put $K = \ker(\alpha)$, then K is a vector bundle of rank n on S. Dualizing $K \to \mathcal{O}_S^{\oplus(n+1)}$ and composing it with the natural $\mathcal{O}_Z^{\oplus(n+1)} \to \mathcal{O}_S^{\oplus(n+1)}$, we have a surjective homomorphism $\beta\colon \mathcal{O}_Z^{\oplus(n+1)} \to \mathcal{O}_S^{\oplus(n+1)} \to K^\vee$. It is well-known that $E = \ker(\beta)$ is a vector bundle of rank $n+1$ on Z; E is the elementary transform of $\mathcal{O}_Z^{\oplus(n+1)}$ along $K^\vee$. If $\mathbf{A}^1 \ni u$ is not zero, $s_0, \cdots, s_{n-1}, us_n$ are independent and then (1.6.1) implies that $\mathrm{End}_{\mathcal{O}_X}(E_u) \cong k$ (see [12] Theorem 3.4). At the origin $0 \in \mathbf{A}^1$, $E_0 = E' \oplus \mathcal{O}_X$, whence $\dim_k \mathrm{End}_{\mathcal{O}_X}(E_0) \geq 2$ (see [12] loc. cit.).

Let us consider the E in the above example. If m is sufficiently large, for $E(m) = E \otimes_{\mathcal{O}_X} \mathcal{O}_X(m)$ and for all u in $\mathbf{A}^1(k)$, $E(m)_u$ is generated by its global sections and $H^i(X, E(m)_u) = 0$ for all $i > 0$. By virtue of the base change theorem of cohomologies, $q_*(E(m))$ is locally free, the natural map $\psi\colon q^*q_*(E(m)) \to E(m)$ is surjective and $k^{\oplus N} \cong H^0(X, \mathcal{O}_X^{\oplus N}) \cong H^0(X, (q^*q_*(E(m)))_u) \to H^0(X, E(m)_u)$ is bijective, where q is the projection of Z to $\mathbf{A}^1$. Since every vector bundle on $\mathbf{A}^1$ is trivial, $q_*(E(m)) \cong V \otimes_k \mathcal{O}_{\mathbf{A}^1}$ with a k-vector space V. We obtain therefore a surjective $\psi\colon V \otimes_k \mathcal{O}_Z \to E(m)$. What we have shown implies that there is a morphism $g\colon A^1 \to \mathrm{Quot}^H(V \times \mathcal{O}_X/X/k)$ such that $(1_X \times g)^*(\theta)$ is isomorphic to $\psi\colon V \otimes_k \mathcal{O}_Z \to E(m)$, where H is the Hilbert polynomial of an $E(m)_u$. $g(\mathbf{A}^1)$ is contained in the open set U given in the above for $\mathrm{Quot}^H(V \otimes \mathcal{O}_X/X/k)$. Set $E_1 = E_u$ with $u \neq 0$, $E_2 = E_0$. Then $z_1 = g(u)$ and $z_2 = g(0)$ belong to the same connected component of U. This violates (1.5) because $\dim_k \mathrm{End}_{\mathcal{O}_X}(E_1) = 1 < 2 \leq \dim_k \mathrm{End}_{\mathcal{O}_X}(E_2)$. For every couple (X, T) with $\dim X \geq 2$, there are D's which satisfy the conditions (1.6.1) and (1.6.2). We see therefore

(1.7) There does not exists VB_X.

(When $\dim X = 1$, the construction of the family of vector bundles on X as in Example 1.6 is much easier.)

What we have seen so far shows that to get moduli spaces of algebraic vector bundles we have to restrict ourselves to a suitable *subfamily* $\mathcal{F}$ of $\mathcal{V}\mathcal{B}(X)$. What kind of subfamily should be taken? The answer depends on, of course, how good properties we require of the moduli space to be obtained. At least, however, the above discussion tells us that $\dim_k \mathrm{End}_{\mathcal{O}_X}(E)$ must be constant over $E \in \mathcal{F}$.

Definition 1.8. A vector bundle E on X is simple if $\mathrm{End}_{\mathcal{O}_X}(E)=$ {multiplications by constants} $=k$.

Theorem 1.9 (A. B. Altman and S. L. Kleiman [1]). *$\mathcal{V}\mathcal{B}^0(X)=\{E\in\mathcal{V}\mathcal{B}(X)\,|\,E$ is simple$\}$ carries a natural structure of algebraic space which is not necessarily separated.*

We demand

(1.10) The moduli space of $\mathcal{F}$ is a *scheme* and each of its connected components is *quasi-projective*.

§ 2. On curves

First let us consider a special but very good case; $\dim X=1$. In this case we have really successful story thanks to D. Mumford, M. S. Narasimhan, P. E. Newstead, S. Ramanan and C. S. Seshadri etc. Throughout this section X is a non-singular projective curve of genus g.

Definition 2.1 (Mumford [21]). A vector bundle E on X is stable (or, semi-stable) if for all subbundle F of E with $0\neq F\neq E$, $d(F)/r(F)<$ (or, $\leq$, resp.) $d(E)/r(E)$, where for a vector bundle G on X, $r(G)$ denotes the rank of G and $d(G)$ does the degree of the first Chern class of G.

Remark 2.2. (1) If a vector bundle E on X is stable, then it is simple. The converse is, however, not necessarily true.

(2) In the above definition, "all subbundles F with $0\neq F\neq E$" may be replaced by "all coherent subsheaves F with $0\neq r(F)\neq r(E)$". In fact, if F is a coherent subsheaf of E, then there is a subbundle F' of E such that $F'\supset F$, $r(F')=r(F)$ and $d(F')\geq d(F)$. Moreover, $d(F)=d(F')$ if and only if $F=F'$.

For a couple of integers r, d, let $S(r, d)$ be the set of the isomorphism classes of stable vector bundles E on X of rank r with $d(E)=d$.

Theorem 2.3 (Mumford [21], Narasimhan, Seshadri [27], [32] [33]). *$S(r, d)$ carries a natural structure of non-singular quasi-projective variety. We have* $\dim S(r, d)=r^2(g-1)+1$. *Moreover, if $(r, d)=1$, then $S(r, d)$ is projective.*

The map $c\colon S(r, d)\ni E\to c_1(E)\in \mathrm{Pic}^d(X)$ is a surjective morphism, where $\mathrm{Pic}^d(X)$ is the connected component of the Picard scheme $\mathrm{Pic}(X)$ of X which consists of line bundles of degree d. It is easy to see that c is smooth and $c^{-1}(x)$ is isomorphic to $c^{-1}(y)$ for all x, $y\in\mathrm{Pic}^d(X)(k)$ and $\dim c^{-1}(x)=(r^2-1)(g-1)$. Let us mention the following question:

Problem 2.4. Assume $(r, d)=1$. Is $c^{-1}(x)$ rational?

P. E. Newstead answered this problem affirmatively in some special cases; for example, if $d \equiv \pm 1 \bmod r$, then $c^{-1}(x)$ is rational ([28] and [29]).

To state the results of Seshadri on a compactification of $S(r, d)$, we need

Proposition 2.5. *Let E be a semi-stable vector bundle on X. Then there exists a filtration $0=E_0 \subset E_1 \subset \cdots \subset E_\alpha = E$ by subbundles with the following properties;*

(a) *E_i/E_{i-1} is stable, $1 \leq {}^\forall i \leq \alpha$,*

(b) *$d(E_i)/r(E_i) = d(E)/r(E)$, $1 \leq {}^\forall i \leq \alpha$.*

Moreover, if $0=E'_0 \subset E'_1 \subset \cdots \subset E'_\beta = E$ is another filtration with the properties (a) *and* (b), *then $\alpha = \beta$ and there is a permutation σ of $\{1, 2, \cdots, \alpha\}$ such that $E_i/E_{i-1} \cong E'_{\sigma(i)}/E'_{\sigma(i)-1}$.*

The proposition shows that the isomorphism class of $\mathrm{gr}(E) = \bigoplus_{i=1}^{\alpha} E_i/E_{i-1}$ is independent of the choice of the filtration. Thus we come to

Definition 2.6. Two semi-stable vector bundles E and E' are S-equivalent to each other if $\mathrm{gr}(E) \cong \mathrm{gr}(E')$ and then we denote the equivalence by $E \underset{S}{\sim} E'$.

By the definition of S-equivalence the set $\bar{S}(r, d) = \{E \mid E$ is semi-stable, $r(E)=r$ and $d(E)=d\}/\underset{S}{\sim}$ contains $S(r, d)$ injectively.

Theorem 2.7 (Seshadri [31], [32] Narasimhan and Ramanan [25]). *$\bar{S}(r, d)$ carries a natural structure of normal, projective variety. $S(r, d)$ is an open subscheme of $\bar{S}(r, d)$. The singular locus of $\bar{S}(r, d)$ is $\bar{S}(r, d) - S(r, d)$ except in the case $g=2$, $r=2$ and d even where $\bar{S}(2, d)$ is non-singular.*

In connection with this theorem, interesting and important is

Problem 2.8. Does there exist a *canonical* resolution of singularities of $\bar{S}(r, d)$?

When $r=2$, Narasimhan and Ramanan [26] and Seshadri [34] have developed beautiful theories on this matter.

§ 3. Moduli of stable sheaves

The aim of this section is to show how to generalize the results on curves described in the preceding section to those on non-singular projective varieties. First of all, if we restrict ourselves to *vector bundles*, then the moduli spaces are not necessarily proper (see Proposition 3.23). It seems natural to consider our problem in the category of *torsion free sheaves*.

Definition 3.1. A coherent sheaf E on a variety X is torsion free if the natural map $E \to E \otimes_{\mathcal{O}_X} K(X)$ is injective, where $K(X)$ is the sheaf of rational functions on X.

When X is projective as is assumed always in the sequel, E is torsion free if and only if E is a subsheaf of a vector bundle. On a non-singular curve a coherent sheaf is torion free if and only if it is a vector bundle. This is peculiar to curves. In fact, for a vector bundle E on X, pick a point x of X and a surjective homomorphism $\rho: E \to k(x)$. Then $E' = \ker(\rho)$ is torsion free but not locally free if $\dim X \geq 2$.

Let us have another look at the notion of stable vector bundles on a non-singular projective curve X of genus g. Fix an ample line bundle $\mathcal{O}_X(1)$ on X with degree d. Then, for a coherent sheaf E on X with $r(E) > 0$, the Riemann-Roch theorem tells us $P_E(m) = \chi(E(m))/r(E) = dm + d(E)/r(E) - g + 1$, where $E(m) = E \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes m}$. Thus a vector bundle E on X is stable (or, semi-stable if and only if for every coherent subsheaf F of E with $0 \neq r(F) \neq r(E)$, $P_F(m) <$ (or, $\leq$, resp.) $P_E(m)$ for every large integer m (see Definition 2.1 and Remark 2.2, (2)). In order to generalize this notion, let $(X, \mathcal{O}_X(1))$ be a couple of a non-singular projective variety X over an algebraically closed field k and an ample line bundle $\mathcal{O}_X(1)$ on X. If F is a coherent sheaf on X, there is a non-empty open set U of X such that $F_{|U} \cong (\mathcal{O}_{X|U})^{\oplus r}$. The rank $r(F)$ is defined to be the r. When $r(F) \neq 0$, set $P_F(m) = \chi(F(m))/r(F)$, where $\chi(F(m))$ is the Euler characteristic of $F(m) = F \otimes_{\mathcal{O}_X} \mathcal{O}_X(1)^{\otimes m}$. The above observation taken into account, the following definition seems to be adequate.

Definition 3.2. A coherent sheaf E on $(X, \mathcal{O}_X(1))$ is said to be stable (or, semi-stable) if (1) E is torsion free and (2) for every coherent subsheaf F of E with $0 \neq r(F) \neq r(E)$, we have $P_F(m) <$ (or, $\leq$, resp.) $P_E(m)$ for all large integers m.

Stable sheaves have good properties.

Theorem 3.3. (1) *A stable sheaf E is simple, namely,* $\mathrm{End}_{\mathcal{O}_X}(E) =$ $\{$*multiplications by constants*$\} \cong k$.

(2) *The property that a coherent sheaf is stable* (*or, semi-stable*) *is open. In other words, let $f: X \to S$ be a smooth, projective, geometrically integral morphism of locally noetherian schemes, $\mathcal{O}_X(1)$ an f-ample line bundle*

on X and E an S-flat coherent sheaf on X. Then there is a open set U (or, V, resp.) of S such that for every algebraically closed field K, $U(K)=\{s \in S(K) \mid E_s=E\otimes_{\mathcal{O}_S} k(s)$ is stable on $(X_s, \mathcal{O}_X(1)\otimes_{\mathcal{O}_S} k(s))\}$ (or, $V(K)=\{s \in S(K) \mid E_s$ is semi-stable on $(X_s, \mathcal{O}_X(1)\otimes_{\mathcal{O}_S} k(s))\}$, resp.).

A proof of (2) in the above is found in [14]. As for the existence of stable sheaves, we have

Theorem 3.4 ([16] Proposition A.1 and [19] Theorem 3.1). *Let $(X, \mathcal{O}_X(1))$ be as in Definition 3.2 and D a divisor on X and r, s integers with $\dim X\leq r$. Assume $\dim X\geq 2$. Then there exists a stable vector bundle E on $(X, \mathcal{O}_X(1))$ such that $r(E)=r$, $c_1(E)=D$ (rational equivalence) and the degree of $c_2(E)$ with respect to $\mathcal{O}_X(1)$ is greater than s. Moreover, if $\dim X=3$, the above holds for $r=2$.*

Let Λ be a universally Japanese ring, $f\colon X\to S$ a smooth, projective, geometrically integral morphism of Λ-schemes of finite type and $\mathcal{O}_X(1)$ an f-ample line bundle on X. To fix ideas let us introduce the following functor. Pick a numerical polynomial $H(x)$. For a locally noetherian S-scheme T, we set

$$\Sigma^H_{X/S}(T)=\left\{ E \;\middle|\; \begin{array}{l} E \text{ is a coherent sheaf on } X\times_S T \text{ with} \\ \quad\text{the following properties (a) and (b)} \end{array} \right\}\Big/\sim$$

(a) E is T-flat,

(b) for all geometric points t of T, E_t is stable on $(X_t, \mathcal{O}_X(1)\otimes_{\mathcal{O}_S} k(t))$ and $\chi(E_t\otimes\mathcal{O}_X(m))=H(m)$.

For E_1, E_2 with the properties (a) and (b), $E_1\sim E_2$ if there is a line bundle L on T such that $E_1\otimes_{\mathcal{O}_T} L\cong E_2$. It is easy to see that $\Sigma^H_{X/S}$ is, as usual, a contravariant functor of the category of locally noetherian S-schemes to that of sets.

The result which is a generalization of Theorem 2.3 is stated as follows.

Theorem 3.5 ([6] Theorem 0.3 and [15] Theorem 5.6). *$\Sigma^H_{X/S}$ has a coarse moduli scheme $M_{X/S}(H)$ (for the notion of coarse moduli schemes, see [22] Definition 5.6). $M_{X/S}(H)$ is separated and locally of finite type over S.*

As in the curve case, we have

Proposition 3.6 ([16] Proposition 1.2). *If E is a semi-stable sheaf on a non-singular projective variety $(Y, \mathcal{O}_Y(1))$, there is a filtration $0=E_0\subset E_1$*

$\subset \cdots \subset E_\alpha = E$ *by coherent subsheaves with the following properties*;

(a) E_i/E_{i-1} *is stable* $(1 \leq {}^\forall i \leq \alpha)$,

(b) $P_{E_i}(m) = P_E(m)$ $(1 \leq {}^\forall i \leq \alpha)$.

Moreover, if $0 = E'_0 \subset E'_1 \subset \cdots \subset E'_\beta = E$ *is another filtration with* (a) *and* (b), *then* $\alpha = \beta$ *and there is a permutation* σ *of* $\{1, 2, \cdots, \alpha\}$ *such that* $E_i/E_{i-1} = E'_{\sigma(i)}/E'_{\sigma(i)-1}$. *Thus the isomorphism class of* $\mathrm{gr}(E) = \bigoplus_{i=1}^{\alpha} E_i/E_{i-1}$ *is independent of the choice of the filtration.*

The S-equivalence is now defined as follows.

Definition 3.7. Semi-stable sheaves E_1 and E_2 on a non-singular projective variety $(Y, \mathcal{O}_Y(1))$ are S-equivalent to each other if $\mathrm{gr}(E_1) \cong \mathrm{gr}(E_2)$. When E_1 is S-equivalent to E_2, we denote $E_1 \underset{S}{\sim} E_2$.

Fix a numerical polynomial $H(x)$. For a geometric point s of S, set

$$\bar{\Sigma}^H_{X/S}(s) = \left\{ E \,\middle|\, \begin{array}{l} E \text{ is a semi-stable sheaf on the geometric fiber} \\ X_s \text{ and } \chi(E(m)) = H(m) \end{array} \right\} \Big/ \underset{S}{\sim}$$

Then Theorem 2.7 extends to

Theorem 3.8 ([6] Theorem 0.3 and [16] Theorem 4.11 and Corollary 5.9.1). *There exists an S-scheme* $\bar{M}_{X/S}(H)$ *with the following properties*:

(1) $\bar{M}_{X/S}(H)$ *is locally of finite type and separated over S.*

(2) $\bar{M}_{X/S}(H)$ *contains* $M_{X/S}(H)$ *as an open subscheme.*

(3) *For each geometric point s of S, there is a natural bijection* ζ_s: $\bar{\Sigma}^H_{X/S}(s) \to \bar{M}_{X/S}(H)(k(s))$.

(4) *For a locally noetherian S-scheme T, let E be a T-flat coherent sheaf on* $X \times_S T$ *such that for every geometric point s of S and k(s)-valued point t of T,* E_t *is semi-stable and the S-equivalence class* $[E_t]$ *of* E_t *is contained in* $\bar{\Sigma}^H_{X/S}(s)$. *Then we have an S-morphism* f_E *of T to* $\bar{M}_{X/S}(H)$ *such that* $f_E(t) = \zeta_s([E_t])$ *for all t in* $T(k(s))$. *Moreover, for every S-morphism g*: $T' \to T$, *we have* $f_E \cdot g = f_{(1_X \times g)^*(E)}$.

(5) *The triple* $(\bar{M}_{X/S}(H), \zeta_s, f_E)$ *is universal with respect to the properties* (3) *and* (4).

Furthermore, if $\bar{M}_{X/S}(H)$ *is of finite type over S or equivalently* $\bigcup_s \bar{\Sigma}^H_{X/S}(s)$ *is bounded, then* $\bar{M}_{X/S}(H)$ *is projective over S.*

Problem 3.9. (1) When does the closure of $M_{X/S}(H)$ in $\bar{M}_{X/S}(H)$ coincide with $\bar{M}_{X/S}(H)$?

(2) Is $\bar{M}_{X/S}(H)$ always projective over S?

Concerning (2), a partial answer is known.

Theorem 3.10 ([18]). $\bar{M}_{X/S}(H)$ *is projective over S if one of the follow-*

ing conditions is satisfied:

(1) Λ *is a field of characteristic zero.*

(2) rank≤ 3

(3) $\dim X/S \leq 2$.

To study stable sheaves and their restriction to a general hyperplane section, the notion of the Harder-Narasimhan filtration is inevitable. First let us define μ-semi-stable sheaves.

Definition 3.11. Let $(X, \mathcal{O}_X(1))$ be the same as in Definition 3.2. A coherent sheaf E on $(X, \mathcal{O}_X(1))$ is μ-semi-stable if (1) E is torsion free and (2) for every coherent subsheaf F of E with $F \neq 0$, $\mu(F) \leq \mu(E)$, where for a coherent sheaf G on $(X, \mathcal{O}_X(1))$, $d(G)$ is the degree of the first Chern class of G with respect to $\mathcal{O}_X(1)$ and where $\mu(G) = d(G)/r(G)$ if $r(G) \neq 0$.

Note that the Riemann-Roch theorem implies that a semi-stable sheaf is μ-semi-stable. The Harder-Narasimhan filtration is a notion to measure how far away from the μ-semi-stability a coherent torsion free sheaf is.

Proposition-Definition 3.12 ([17] Proposition 1.5). *Let* $(X, \mathcal{O}_X(1))$ *be the same as in Definition* 3.2. *If* E *is a coherent torsion free sheaf on* $(X, \mathcal{O}_X(1))$, *then there is a unique filtration* $0 = E_0 \subset E_1 \subset \cdots \subset E_\alpha = E$ *by coherent subsheaves such that* (1) E_i/E_{i-1} *is* μ*-semi-stable for* $1 \leq {}^\forall i \leq \alpha$ *and* (2) $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$ *for* $1 \leq {}^\forall i \leq \alpha - 1$. *The filtration is called the Harder-Narasimhan filtration.*

By using the above filtration, we can state the following which is a generalization of the theorem of Grauert-Mülich-Spindler and from which Theorem 3.10, (1) is obtained directly.

Theorem 3.13 ([5] Theorem 4.2 and [18] Theorem 4.6). *Let* X *be a non-singular projective variety over an algebraically closed field* k *of characteristic zero,* $\mathcal{O}_X(1)$ *a very ample line bundle and* L *a very ample linear subsystem of* $|\mathcal{O}_X(1)|$. *For a* μ*-semi-stable sheaf* E *on* $(X, \mathcal{O}_X(1))$, *if* $Y_1, \cdots, Y_{n'}$ $(n' < \dim X)$ *is sufficiently general member of* L, *then* $Y = Y_1 \cdots Y_{n'}$ *is a non-singular subvariety of codimension* n', $E|_Y$ *is torsion free and moreover, for the Harder-Narasimhan filtration* $0 = E_0 \subset E_1 \subset \cdots \subset E_\alpha = E|_Y$, *we have* $\mu(E_i/E_{i-1}) - \mu(E_{i+1}/E_i) \leq d$, *where* d *is the degree of* X *with respect to* $\mathcal{O}_X(1)$.

Another interesting result is

Theorem 3.14 ([17] Theorem 3.1). *Let the triple* $(X, \mathcal{O}_X(1), L)$ *be the same as in Theorem* 3.13 *except that we only assume* k *to be an algebraically closed field of arbitrary characteristic. Let* E *be a* μ*-semi-stable sheaf on* X.

Assume that $r(E)<\dim X$. For a sufficiently general member Y of L, $E|_Y$ is μ-semi-stable.

The case of rank 2 of Theorem 3.10, (2) is deduced from the above theorem and Theorem 3.10, (3) which was proved in [13] (see also [6]).

For an open subscheme U of $M_{X/S}(H)$, a coherent sheaf F on $X\times_S U$ is said to be a universal sheaf of U if (1) F is flat over U and (2) for every geometric point u of U lying over s of S, the isomorphism class of F_u is $\zeta_s^{-1}(u)$. Let $H(m)$ be a numerical polynomial. Then $H(m)$ can be written in the form

$$H(m)=\sum_{i=0}^{n} a_i\binom{m+n-i}{n-i} \qquad \text{with } n=\deg H(m) \text{ and } a_i \text{ integers.}$$

Set $\delta(H)=G.C.D.\{a_0, \cdots, a_n\}$.

Theorem 3.15 ([16] Theorem 6.11). *If U is a quasi-compact open subscheme of $M_{X/S}(H)$ and if $\delta(H)=1$, then U has a universal sheaf.*

To explain the meaning of the theorem we need

Lemma 3.16. *Let $g\colon Y\to T$ be a proper morphism of locally noetherian schemes and E, F coherent $\mathcal{O}_Y$-modules which are faithfully flat over T. Assume that* (1) *there exist an open covering $\{U_\lambda\}$ of T and an isomorphism γ_λ: $E_{U_\lambda}\to F_{U_\lambda}$ for each λ and* (2) *for all closed points t of T, $\mathrm{Hom}_{\mathcal{O}_{Y_t}}(E_t, F_t)\cong \mathrm{End}_{\mathcal{O}_{Y_t}}(E_t)$ is canonically isomorphic to $k(t)$. Then there is a line bundle L on T such that $E\otimes_{\mathcal{O}_T}L\cong F$.*

Proof. Since E and F are faithfully flat over T, ψ_λ: $\mathcal{O}_{U_\lambda}\to (g_{U_\lambda})_*(\mathcal{H}om_{\mathcal{O}_{Y\times U_\lambda}}(E_{U_\lambda}, F_{U_\lambda}))$; $a\to a\gamma_\lambda$ is injective for all λ. If all the ψ_λ are isomorphisms, then $g_*(\mathcal{H}om_{\mathcal{O}_Y}(E, F))=L$ is an invertible sheaf on T and $g_*(\mathcal{H}om_{\mathcal{O}_Y}(E\otimes_{\mathcal{O}_T} L, F))=\mathcal{O}_T$. The global section γ of $\mathrm{Hom}_{\mathcal{O}_Y}(E\otimes_{\mathcal{O}_T} L, F)$ $=\Gamma(T, \mathcal{O}_T)$ which comes from 1 of $\mathcal{O}_T$ induces γ_λ on Y_U. The γ meets our requirement. Pick a closed point t of T and set $A=\hat{\mathcal{O}}_{T,t}$, the completion of $\mathcal{O}_{T,t}$ with respect to the maximal ideal. Since $T'=\mathrm{Spec}(A)\to T$ is flat and E is coherent, we see

$$\begin{aligned} g_*(\mathcal{H}om_{\mathcal{O}_Y}(E, F))\otimes_{\mathcal{O}_T}\mathcal{O}_{T'} &=(g_{T'})_*(\mathcal{H}om_{\mathcal{O}_Y}(E, F)\otimes_{\mathcal{O}_T}\mathcal{O}_{T'}) \\ &=(g_{T'})_*(\mathcal{H}om_{\mathcal{O}_{Y_{T'}}}(E_{T'}, F_{T'})). \end{aligned}$$

Thus we may assume that $T=T'$, that is, T is $\mathrm{Spec}(A)$ with $(A, \mathfrak{m})$ a complete local ring and that $E=F$. When A is artinian, our assertion was proved in [16] Lemma 6.1. Thus, for a $\tau\in \mathrm{End}_{\mathcal{O}_Y}(E)=\mathrm{Hom}_{\mathcal{O}_Y}(E, E)$, there is a sequence $(b_0, b_1, \cdots)$ of elements $b_n\in A/\mathfrak{m}^n$ such that $\tau_n=\tau\otimes$

$A/\mathfrak{m}^n = b_n id$. Since $A/\mathfrak{m}^n = \mathrm{End}_{\mathcal{O}_{Y_n}}(E \otimes A/\mathfrak{m}^n)$, $Y_n = Y \times_T \mathrm{Spec}\,(A/\mathfrak{m}^n)$, b_n is determined uniquely by τ. Hence the equality

$$b_{n+1} id \bmod (\mathfrak{m}^n) = \tau_{n+1} \otimes A/\mathfrak{m}^n = \tau_n = b_n id$$

shows that $b_{n+1} \bmod (\mathfrak{m}^n) = b_n$. We can find, therefore,an element b of A such that $b \bmod (\mathfrak{m}^n) = b_n$. Then $(\tau - b \cdot id) \otimes A/\mathfrak{m}^n = \tau_n - b_n id = 0$ for all n, which means that $\tau = b \cdot id$. q.e.d.

Combining Theorem 3.15 and Lemma 3.16, we have

Theorem 3.17. *If $M_{X/S}(H)$ is quasi-compact or equivalently quasi-projective and if $\delta(H)=1$, then the functor $\Sigma^H_{X/S}$ is representable, that is, $M_{X/S}(H)$ is a fine moduli scheme of $\Sigma^H_{X/S}$.*

Proof. Since $M = M_{X/S}(H)$ itself is quasi-compact and $\delta(H)=1$, we have a universal sheaf F on $X \times_S M$ by Theorem 3.15. By Theorem 3.5, we have a morphism of functors $\Phi\colon \Sigma^H_{X/S} \to h_M = \mathrm{Hom}_S(*, M)$. By the construction of the moduli space M and the universal sheaf F, we may assume that there are an open set P of $\mathrm{Quot}^H(V \otimes \mathcal{O}_X/X/S)$ and a morphism $h\colon P \to M$ such that $h\colon P \to M$ is a principal fibre bundle with group $PGL(N)$ and $(1_X \times h)^*(F) = \tilde{F} \otimes_{\mathcal{O}_P} L$ for a line bundle L on P, where V is a free $\boldsymbol{Z}$-module of rank $N = \dim H^0(X, F_y)$, $y \in M$ and $\alpha\colon V \otimes \mathcal{O}_{X_P} \to \tilde{F}$ is the universal quotient sheaf on $X_P = X \times_S P$. Moreover, we may assume also that for every closed point y of M, F_y is generated by its global sections and $H^i(X_y, F_y) = 0$, $i > 0$. Then there is an open covering $\{V_\lambda\}$ of M such that for each λ, we have a surjection $V \otimes_{\boldsymbol{Z}} \mathcal{O}_{X \times V_\lambda} \xrightarrow{\alpha_\lambda} F_{V_\lambda}$ such that α_λ induces an isomorphism of $H^0(X_y, V \otimes_{\boldsymbol{Z}} \mathcal{O}_{X_y})$ to $H^0(X_y, F_y)$ at every $y \in V$. Then we get a section $s_\lambda\colon V_\lambda \to P_V$, $h_{V_\lambda} \cdot s_\lambda = id_{V_\lambda}$ and α_λ is the pull-back of α_{V_λ} (we may assume that $L|_{s_\lambda(V_\lambda)}$ is trivial). Therefore, P_{V_λ} is a trivial bundle in the Zariski topology, $P_{V_\lambda} \cong PGL(N) \times_S V_\lambda$. Let T be a locally noetherian S-scheme and E a representative of a class $[E]$ in $\Sigma^H_{X/S}(T)$. E defines a morphism $f_E = \Phi([E])$ of T to M. As in the case of F, there is an open covering $\{U_\nu\}$ of T and a morphism $t_\nu\colon U_\nu \to P$ such that $E_{U_\nu} = (1_X \times t_\nu)^*(\tilde{F})$ and $f_E(U_\nu) \subset V_\lambda$ for a λ. The morphism $c\colon U_\nu \xrightarrow{t_\nu} P_{V_\lambda} \to PGL(N) \times_S V_\lambda \to PGL(N)$ defines a U_ν-valued point of $PGL(N)$ and it is induced from a U_ν-valued point $\tilde{c}$ of $GL(N)$. The composition $V \otimes_{\boldsymbol{Z}} \mathcal{O}_{X \times U_\nu} \xrightarrow{\tilde{c}^{-1}} V \otimes_{\boldsymbol{Z}} \mathcal{O}_{X \times U_\nu} \xrightarrow{(1_X \times t_\nu)^*(\alpha)} (1_X \times t_\nu)^*(\tilde{F}) \cong E_{U_\nu}$ provides us with another morphism $t'_\nu\colon U_\nu \to P$ such that $(1_X \times t'_\nu)^*(\tilde{F}) \cong E_{U_\nu}$ and $t'_\nu = c^{-1} t_\nu$. This means that $s_\lambda(f_E)_{U_\nu} = t'_\nu$ and hence $E_{U_\nu} \cong (1_X \times f_E)^*(F)_{U_\nu}$. Then we can apply Lemma 3.16 to our situation and get $E \cong (1_X \times f_E)^*(F) \otimes_{\mathcal{O}_T} L_E$ with L_E a line bundle on T, namely, $[E]=[(1_X\times f_E)^*(F)]$. Now given $u\in \mathrm{Hom}_S(T, M)$, $(1_X\times u)^*(F)$ defines a class in $\Sigma^H_{X/S}(T)$ and this map is functorial with respect to T. Thus a morphism of functor ψ: $h_M\to\Sigma^H_{X/S}$ is obtained. What we have done in the above is $\Psi\Phi=id$. In fact, $\Psi\Phi([E])=\Psi(f_E)=[(1_X\times f_E)^*(F)]=[E]$. On the other hand, $\Phi\Psi(u)=\Phi((1_X\times u)^*(F))=f_{(1_X\times u)^*(F)}=f_F u$, the construction of f_F and the fact that $(1_X\times h)^*(F)\cong\tilde{F}\otimes_{\mathcal{O}_P}L$ show that $f_F=1_M$. Thus $\Phi\Psi=id$. q.e.d.

When X is a curve over a field k, then $M_{X/k}(H)$ is the fine moduli if and only if $\delta(H)=1$ or equivalently $(r, d)=1$ (Narasimhan and Ramanan [24] and Ramanan [30]). When $X=\mathbf{P}^2_k$, rank is 2 and the characteristic of k is zero, $M_{X/S}(H)$ is the fine moduli if and only if $\delta(H)=1$ (Le Potier [11]).

Problem 3.18. Is $M_{X/S}(H)$ the fine moduli if and only if $\delta(H)=1$?

Remark 3.19. If $\delta(H)=1$, then $M_{X/S}(H)=\overline{M}_{X/S}(H)$ and hence $M_{X/S}(H)$ is projective over S, for example, in the case where Λ is a field of characteristic zero.

There are several examples of $(X, \mathcal{O}_X(1))$ and H such that X is a non-singular projective surface over a field k and $M_{X/k}(H)$ has singularities ([13] Example 4.14).

Proposition 3.20. ([16] Corollary 6.7.3). *Suppose that* $\dim X/S=2$. *If for a geometric point s of S, the degree of a canonical divisor of X_s with respect to $\mathcal{O}_X(1)\otimes_{\mathcal{O}_S}k(s)$ is negative, then $M_{X/S}(H)$ is smooth at every point of $M_{X/S}(H)_s$. Moreover, if $S=$* Spec (k) *with k a field, $\overline{M}_{X/S}(H)$ is normal.*

From this proposition we see the following:

(3.21.1) If X is $\mathbf{P}^2$ or a rational ruled surface over a field k, then $M_{X/S}(H)$ is smooth, quasi-projective over k and $\overline{M}_{X/S}(H)$ is normal, projective over k.

(3.21.2) If X is a ruled surface over a field k and if $\mathcal{O}_X(1)$ is suitably chosen, then for every H, $M_{X/S}(H)$ (or, $\overline{M}_{X/S}$) for $(X, \mathcal{O}_X(1))$ is smooth, quasi-projective (or, normal, projective, resp.) over k.

Let $M_{\mathbf{P}^2}(r, c_1, c_2)$ ($M_{\mathbf{P}^2}(r, c_1, c_2)_0$ or $\overline{M}_{\mathbf{P}^2}(r, c_1, c_2)$) be the moduli space of stable sheaves (stable vector bundles or semi-stable sheaves, resp.) E on $\mathbf{P}^2$ with $r(E)=r$, $c_1(E)=c_1$ and $c_2(E)=c_2$. When $r=2$, the structure of these moduli spaces is known quite well.

Theorem 3.22. (1) *$M_{\mathbf{P}^2}(r, c_1, c_2)$ is smooth, quasi-projective, $\overline{M}_{\mathbf{P}^2}(r, c_1, c_2)$ is normal, projective and* $\dim M_{\mathbf{P}^2}(r, c_1, c_2)=(1-r)c_1^2+2rc_2-r^2+1$. *If c_1 is a multiple of r and if the characteristic of the ground field is zero, then $M_{\mathbf{P}^2}(r, c_1, c_2)$ is connected (Hulek [10]).*

(2–a) $\overline{M}_{\mathbf{P}^2}(2, c_1, c_2)$ *is irreducible and rational* (*Barth* [3], *Maruyama* [16], *Hulek* [9] *and Ellingsrud, Strømme* [4]).

(2–b) $M_{\mathbf{P}^2}(2, c_1, c_2)=\overline{M}_{\mathbf{P}^2}(2, c_1, c_2)$ *if and only if* $c_1^2-4c_2\not\equiv 0 \bmod (8)$.

(2–c) $M_{\mathbf{P}^2}(2, c_1, c_2)$ *is the fine moduli if* $c_1^2-4c_2\not\equiv 0 \bmod (8)$. *If* $c_1^2-4c_2\equiv 0 \bmod (8)$ *and if the characteristic of the ground field is zero, then* $M_{\mathbf{P}^2}(2, c_1, c_2)$ *is not the fine moduli* (*Le Potier* [11]).

Let us close Part I by proving the following result.

Proposition 3.23. *Assume that* $c_1^2-4c_2\leq -7$, $\neq -8$. $B=M_{\mathbf{P}^2}(2, c_1, c_2)-M_{\mathbf{P}^2}(2, c_1, c_2)_0$ *is of codimension* 1 *in* $M_{\mathbf{P}^2}(2, c_1, c_2)$.

Proof. First of all note that if $c_1^2-4c_2=-3-4m$ or $-8-4m$ $(m\geq 1)$, then $M'=M_{\mathbf{P}^2}(2, c_1, c_2-1)_0\neq\phi$ ([16] Lemma 7.1). By the construction of the moduli space, there are an étale morphism $h\colon A\to M'$ and a vector bundle F on $\mathbf{P}^2\times A$ such that for every geometric point t of A, F_t is the stable vector bundle which corresponds to the point $s=h(t)$. On the projective bundle $p\colon P=\mathbf{P}(F)\to\mathbf{P}^2\times_k A$, we have a natural, surjective homomorphism $\alpha\colon F'=p^*(F)\to\mathcal{O}_P(1)$. P is embedded into $(\mathbf{P}^2\times_k A)\times_A P=\mathbf{P}^2\times_k P$ as the graph of p.

$$\begin{tikzcd}
P \arrow[r, hook, "i"] & \mathbf{P}^2\times_k P \arrow[r, "r"] \arrow[d, "q"'] & P \arrow[dl, "p"] \\
& \mathbf{P}^2\times_k A &
\end{tikzcd}$$

Setting $F''=q^*(F)$ and I to be the ideal of P in $\mathbf{P}^2\times_k P$, we have $F''/IF''=F'\xrightarrow{\alpha}\mathcal{O}_P(1)=r^*(\mathcal{O}_P(1))/Ir^*(\mathcal{O}_P(1))$. Composing this with $F''\to F''/IF''$, we obtain a surjective $\beta\colon F''\to\mathcal{O}_P(1)$. Since both $\mathcal{O}_P(1)$ and F'' are flat over P, so is $E=\ker(\beta)$. At a geometric point x of P, we have an exact sequence

$$0\longrightarrow E_x\xrightarrow{\gamma_x}F''_x\xrightarrow{\delta_x}k(y)\longrightarrow 0,$$

where $y=i(x)$. Since $F''_x=F_{p(x)}$, E_x is not locally free and $c_1(E_x)=c_1$, $c_2(E_x)=c_2(F_{p(x)})+1=c_2$. All the E_x are stable because for every coherent subsheaf L of E_x with $r(L)=1$, we have $\mu(L)<\mu(F_{p(x)})=\mu(E_x)$. Moreover, since $(E_x^\vee)^\vee=F_{p(x)}$, $F_{p(x)}$, γ_x and δ_x are determined uniquely by E_x. There fore, $E_{x_1}\not\cong E_{x_2}$ if $(1\times h)\cdot p(x_1)\neq(1\times h)\cdot p(x_2)$ or if $p(x_1)=p(x_2)$ but $x_1\neq x_2$. By the universal property of the moduli space, we have a morphism $g\colon P\to M_{\mathbf{P}^2}(2, c_1, c_2)$. What we have seen is that the map g is finite to one and $g(P)$ is contained in B. On the other hand, $\dim P=\dim\mathbf{P}^2\times_k A+1=\dim M'+3=-c_1^2+4c_2-4=\dim M_{\mathbf{P}^2}(2, c_1, c_2)-1$. q.e.d.

Part II. Moduli of Stable Sheaves of Rank 2 on $\mathbf{P}^2$ and the Curves of Jumping Lines

§ 1. The curve of jumping lines

Let k be an algebraically closed field. We shall consider a torsion free coherent sheaf E on $\mathbf{P}_k^2$ with the following properties:

$$c_1(E)=0,\ c_2(E)=n \quad \text{and} \quad r(E)=2. \tag{1.1}$$

$$\text{For a general line } l \text{ in } P_k^2,\ E|_l \cong \mathcal{O}_l^{\oplus 2}. \tag{1.2}$$

In the characteristic of k is zero, the second condition is equivalent to the μ-semi-stability of E by virtue of Grauert and Mülich [7]. When E is locally free, the set $|S(E)|$ of jumping lines of E is intuitively

$$|S(E)|=\{l \in P^* \,|E|_l \cong \mathcal{O}_l(a) \oplus \mathcal{O}_l(-a) \text{ with } a>0\},$$

where $P^* \cong \mathbf{P}_k^2$ is the dual space of $P=\mathbf{P}_k^2$. By the deformation theory of rational ruled surfaces, $|S(E)|$ must be a curve in P^*. We would like, however, to define the curve $S(E)$ of jumping lines of E so that $S(E)$ deforms along deformations of E or $S(E)$ depends algebraically on E. Thus we shall follow W. Barth to define the curve of jumping lines.

Suppose that E is a torsion free coherent sheaf with the properties (1.1) and (1.2). There is an exact sequence of coherent sheaves

$$0 \longrightarrow U \longrightarrow \oplus \mathcal{O}_P(m_i) \longrightarrow E(-1) \longrightarrow 0 \tag{1.3}$$

with $m_i<0$ and U locally free. Let us consider the following diagram

$$\begin{array}{ccc} & F & \\ {}^{p}\swarrow & & \searrow^{q} \\ P & & P^* \end{array} \tag{1.4}$$

F is the flag manifold which defines the incidence correspondence between P and P^*.

For general lines l in P, $H^1(l, E(-1)|_l)=H^1(l, \mathcal{O}_l(-1)^{\oplus 2})=0$ and then $L(E)=R^1q_*p^*(E(-1))$ is a torsion sheaf on P^*. It is easy to see that $q_*p^*(E(-1))=0$ and that both $N=R^1q_*p^*(U)$ and $M=R^1q_*p^*(\oplus\mathcal{O}_P(m_i))$ are locally free. Since $p^*(U) \longrightarrow p^*(\oplus\mathcal{O}_P(m_i))$ is generically injective and $p^*(U)$ is locally free, we have the exact sequence

$$0 \longrightarrow p^*(U) \longrightarrow p^*(\oplus\mathcal{O}_P(m_i)) \longrightarrow p^*(E(-1)) \longrightarrow 0.$$

Combining those obtained in the above, we get the following exact sequence

$$0 \longrightarrow N \xrightarrow{\lambda} M \longrightarrow L(E) \longrightarrow 0.$$

Since $r=r(N)=r(M)$, det (λ): $\bigwedge^r N \to \bigwedge^r M$ defines a section of the line bundle $\bigwedge^r M \otimes (\bigwedge^r N)^{\vee}$. det$(\lambda)$ determines, therefore, an effective Cartier divisor on P^*. This divisor is independent of the choice of the sequence (1.3).

Definition 1.5. The divisor defined by det(λ) is called the *curve of jumping lines* of E and denoted by $S(E)$.

The following is essentially due to W. Barth ([2] Theorem 2).

Proposition 1.6. *Let E be a torsion free coherent sheaf on $P=\mathbf{P}_k^2$ with the properties* (1.1) and (1.2).

(1) deg $S(E)=n$.

(2) *$S(E)$ depends algebraically on E, that is, if $\tilde{E}$ is a coherent sheaf on $P_S=\mathbf{P}_S^2$ with S locally noetherian such that $\tilde{E}$ is flat over S and for all geometric points s of S, $\tilde{E}_s$ is torsion free and has the properties* (1.1) *and* (1.2), *then there is a relative Cartier divisor $S(\tilde{E})$ on the dual space P_S^* of P_S such that for all geometric points s of S, we have $S(\tilde{E})_s=S(\tilde{E}_s)$.*

For a coherent sheaf E on P, the double dual $(E^{\vee})^{\vee}$ of E is locally free. If E is torsion free, then E can be canonically embedded in $(E^{\vee})^{\vee}$;

$$0 \longrightarrow E \longrightarrow (E^{\vee})^{\vee} \longrightarrow T \longrightarrow 0,$$

where T is a skyscraper sheaf and Supp (T) is the set of the pinching points of E. Set Supp $(T)=\{x_1, \cdots, x_t\}$. Then it is not difficult to prove

Proposition 1.7. *Let $a_i=$*length (T_{x_i}). *Assume that E has the properties* (1.1) *and* (1.2). *Then we have*

$S(E)=S((E^{\vee})^{\vee})+\sum a_i L_{x_i}$,

where L_{x_i} is the line in P^* consisting of all the lines in P passing through x_i.

Thanks to the above proposition, the study of the curve of jumping lines is reduced to the case of locally free sheaves. When E is locally free, it is clear that $|S(E)|=$Supp $(S(E))$. It is easy also to show the following.

Lemma 1.8. *Let E be a torsion free coherent sheaf on P with* (1.1) *and* (1.2) *and be fitted in an exact sequence*

$$0 \longrightarrow M_1 \longrightarrow E \longrightarrow M_2 \longrightarrow 0$$

such that M_i is an ideal of $\mathcal{O}_P$ with Supp $(\mathcal{O}_P/M_i)=\{x_{i1}, \cdots, x_{i\alpha_i}\}$. *Then* $S(E)=S(M_1 \oplus M_2)=\sum a_{ij} L_{x_{ij}}$, *where $a_{ij}=$*length $(\mathcal{O}_P/M_i)_{x_{ij}}$.

Assume that the characteristic of k is zero. Then, as mentioned in the above, every semi-stable sheaf on P with the property (1.1) satisfies (1.2) (cf. Comments after Definition 3.11 of Part I). The meaning of Lemma 1.8 is that $S(E)=S(E')$ if E and E' are semi-stable, $E\underset{S}{\sim}E'$ and E has the property (1.1). Therefore, we obtain a map $s(n)$ from the moduli space of semi-stable sheaves satisfying (1.1) to the projective space of curves of degree n in P^*, i.e

$$s(n)\colon \overline{M}_{\mathbf{P}^2_k}(2, 0, n)=\overline{M}(n) \ni [E] \longrightarrow S(E) \in \mathbf{P}(H^0(P^*, \mathcal{O}_{P^*}(n))^\vee).$$

By the construction of $\overline{M}(n)$ and Proposition 1.6, (2), we see

Proposition 1.9. *$s(n)$ is a morphism.*

§ 2. θ-characteristic

Let E be a locally free sheaf on $P=\mathbf{P}^2_k$ with the properties (1.1) and (1.2). Set $L(E)=R^1q_*p^*(E(-1))$, where p and q are the same as in the diagram (1.4). As we have seen in § 1, $|S(E)|=\operatorname{Supp}(S(E))=\operatorname{Supp}(L(E))$. The $\mathcal{O}_{P^*}$-module $\theta(E)=L(E)(-1)$ has the following properties.

Theorem 2.1 (Barth [3]). *Assume that E is stable. Then we have:*
(1) *$\theta(E)$ is an $\mathcal{O}_{S(E)}$-module.*
(2) $\theta(E)=\mathscr{E}xt^1_{\mathcal{O}_{P^*}}(\theta(E), \mathcal{O}_{P^*}(-3))$.
(3) $H^0(S(E), \theta(E))=0$.
(4) *$\theta(E)\cong\theta(E')$ if and only if $E\cong E'$.*

The property (2) in the above theorem implies

(2.2) $\mathscr{H}om_{\mathcal{O}_{S(E)}}(\theta(E), \omega_{S(E)})\cong\theta(E)$, where $\omega_{S(E)}$ is the canonical sheaf of $S(E)$.

When $\theta(E)$ is an $\mathcal{O}_{S(E)}$-invertible sheaf, (2.2) is synonymous with $\theta(E)^{\otimes 2}\cong\omega_{S(E)}$ and then, by Theorem 2.1, (3), $\theta(E)$ is an even θ-characteristic on $S(E)$ (see Mumford [23]).

Definition 2.3. The $\mathcal{O}_{S(E)}$-module $\theta(E)$ is called the θ-characteristic associated with a stable E.

The aim of this and the next sections is to generalize the above theorem to relative cases. The proof of Theorem 2.1 by Barth works well in the general case, too, with slight modifications. Let S be a locally noetherian scheme and F be a locally free sheaf of rank 2 on $P=\mathbf{P}^2_S$ such that for all geometric points s of S, F_s is *stable* and has the properties (1.1) and (1.2). We shall fix a free $\mathbf{Z}$-module V of rank 3 and identify the $\mathbf{P}^2$-bundle $f\colon P$

$\to S$ with Proj $(S(V\otimes_Z \mathcal{O}_S))$, where $S(V\otimes_Z \mathcal{O}_S)$ is the symmetric algebra of $V\otimes_Z \mathcal{O}_S$ over $\mathcal{O}_S$. Then we get a line bundle $\mathcal{O}(1)$ on P such that $f_*(\mathcal{O}(1)) = V\otimes_Z \mathcal{O}_S$ and on each fiber P_s of P, $\mathcal{O}(1)_s$ is the hyperplane bundle on $P_s = \mathbf{P}^2_{k(s)}$.

Since $H^i(P_s, F(k)_s)=0$ for $i\neq 1$ and $-3\leq k\leq 0$, $R(k)=R^1f_*(F(k))$ is locally free and $R^if_*(F(k))=0$, $i\neq 1$ for $-3\leq k\leq 0$. For $M=\wedge^2 F$, $M_0=f_*(M)$ is an invertible sheaf and $f^*f_*(M)\cong M$ by (1.1) and the seesaw theorem. Since the rank of F is 2, we have an isomorphism $\wedge: F\otimes M^\vee \ni x\to x^\wedge \in F^\vee$ and then $\wedge: R^1f_*(F(k))\otimes M_0^\vee \cong R^1f_*(F^\vee(k))$. By virtue of the duality theorem ([8] Ch. III, Corollary 5.2) and the fact $\omega_{P/S}\cong\mathcal{O}(-3)$, the pairing

$$\langle\ ,\ \rangle_{F(k)}\colon R^1f_*(F(k))\otimes R^1f_*(F(k+3)^\vee)\longrightarrow \mathcal{O}_S, \qquad -3\leq k\leq 0$$

is perfect; $R^1f_*(F^\vee(-k-3))\cong \mathcal{H}om_{\mathcal{O}_S}(R(k), \mathcal{O}_S)$. Thus, by composition, we have a perfect pairing $(\ ,\ )_k\colon R(k)\otimes R(-k-3)\otimes M_0^\vee\to\mathcal{O}_S$;

$$(a, b)_k=\langle a, b^\wedge\rangle_{F(k)}=-\langle a^\wedge, b\rangle_{F^\vee(k)}=\langle b, a^\wedge\rangle_{F(-k-3)}=(b, a)_{-k-3}$$

for all $a\in R(k)_s$ and $b\in R(-k-3)_s$ with $s\in S$. Since $f_*(\mathcal{O}(1))=V\otimes_Z\mathcal{O}_S$, there is a natural homomorphism

$$\psi_k\colon (V\otimes_Z\mathcal{O}_S)\otimes R(k)\longrightarrow R(k+1).$$

It is easy to see

(2.4) $(\psi_{k-1}(z\otimes a), b)_k=(a, \psi_{-k-3}(z\otimes b))_{k-1}$ for all $z\in V$, $a\in R(k)_s$ and $b\in R(-k-3)_s$ with $s\in S$.

In particular, the adjoint $\psi^*_{-2}\colon (V\otimes\mathcal{O}_S)\otimes R(-2)\otimes M_0^\vee\to R(-1)\otimes M_0^\vee$ of ψ_{-2} with respect to the pairings $(\ ,\ )$ is equal to $\psi_{-2}\otimes M_0^\vee$.

Set $H=R(-2)$ and $H^*=R(-1)$. Then $H^*\otimes M_0^\vee=\mathcal{H}om_{\mathcal{O}_S}(H, \mathcal{O}_S)$ by the pairing $(\ ,\ )_{-1}$. We denote the homomorphism $\psi_{-2}\colon V\otimes_Z H\to H^*$ by α and call it the *net of quadrics* associated with F. For each $z\in V$, $\alpha(z)$ denotes the restriction of α to $z\otimes_Z H\to H^*$. Note that for a basis z_0, z_1, z_2 of V, $\alpha(z_0)$, $\alpha(z_1)$ and $\alpha(z_2)$ determine the α completely.

Let $g\colon P^*\to S$ be the dual space of P, that is, $P^*=\text{Proj}\,(S(V^\vee\otimes_Z\mathcal{O}_S))$. Then we have a canonical injection

$$\sigma\colon \mathcal{O}_{P^*}(-1)\longrightarrow V\otimes_Z\mathcal{O}^*_{P}.$$

This and the α give rise to

$$A\colon H\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*}(-1)\xrightarrow{1\otimes\sigma} V\otimes_Z H\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*}\xrightarrow{\alpha\otimes 1} H^*\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*}.$$

Now let us recall a lemma which enables us to reduce our problems to the results of Barth.

Lemma 2.5 ([16] Lemma 6.5). *Let C be a noetherian local ring, B a noetherian C-algebra and I an ideal of C such that IB is contained in the Jacobson radical of B. Assume that an exact sequence of finite B-modules*

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

enjoys the following properties;

(1) *M is C-flat and $M'' \otimes_C C/I$ is C/I-flat,*

(2) *the map $u \otimes_C 1 \colon M' \otimes_C C/I \longrightarrow M \otimes_C C/I$ is injective.*

Then, M'' is C-flat and u is injective.

For all $s \in S$, $A \otimes k(s)$ is injective ([3] p. 69) and $V \otimes_Z H \otimes_{\mathcal{O}_S} \mathcal{O}_{P^*}$ is S-flat. From this and Lemma 2.5 we infer that A is injective and $L=$ coker (A) is S-flat. On the other hand, det (A) is a global section of the line bundle $(\bigwedge^n H)^\vee \otimes_{\mathcal{O}_S} \bigwedge^n H^* \otimes_{\mathcal{O}_S} \mathcal{O}_{P^*}(n) = g^*(N) \otimes \mathcal{O}_{P^*}(n)$ with N a suitable line bundle on S because $r(H)=r(H^*)=n$ by the Riemann-Roch theorem. Hence we have an exact sequence

$$0 \longrightarrow g^*(N^\vee) \otimes \mathcal{O}_{P^*}(-n) \xrightarrow{\det(A)} \mathcal{O}_{P^*} \longrightarrow \mathcal{O}_T \longrightarrow 0,$$

where T is the subscheme of P^* defined by $\det(A)=0$. Since $\det(A) \otimes k(s) \neq 0$ at every geometric point s of S, T is S-flat by Lemma 2.5.

Remark 2.6. For every geometric point s of S, the fibre T_s is the Cartier divisor $S(F_s)$ associated with F_s in Proposition 1.6 and $L(-1)_s = \theta(F_s)$, where $L=$ coker $(H \otimes \mathcal{O}_{P^*}(-1) \rightarrow H^* \otimes \mathcal{O}_{P^*})$.

Put $\theta(F)=L(-1)$ and $T=S(F)$, then we have

(2.7.1) $\theta(F)$ is an $\mathcal{O}_{S(F)}$-module and flat over S.

(2.7.2) $\theta(F) \cong \mathcal{E}xt^1_{\mathcal{O}_{P^*}}(\theta(F), \mathcal{O}_{P^*}(-3)) \otimes_{\mathcal{O}_S} M_0$.

(2.7.3) $g_*(\theta(F))=0$.

(2.7.4) If $\theta(F) \cong \theta(F') \otimes g^*(N)$ for a line bundle N on S, then $\alpha' \otimes 1_N = \delta \alpha \delta'^{-1}$ with $\delta \colon H^* \rightrightarrows H'^* \otimes N$ and $\delta' \colon H \rightrightarrows H' \otimes N$, where α' is the net of quadrics associated with F'.

Proof. The proof of the fact that $\theta(F)$ is an $\mathcal{O}_{S(F)}$-module is completely the same as Barth's in the case of $S=\operatorname{Spec}(k)$ with k a field, after suitable modifications by tensoring M_0. As in the proof of Barth ([3] 2.2), (2.4) implies that $A(-1)^T(-3) \otimes_{\mathcal{O}_S} M_0^\vee \colon H \otimes_{\mathcal{O}_S} \mathcal{O}_{P^*}(-2) \cong \mathcal{H}om_{\mathcal{O}_{P^*}}(H^* \otimes_{\mathcal{O}_S} \mathcal{O}_{P^*}(-1), \mathcal{O}_{P^*}(-3)) \otimes_{\mathcal{O}_S} M_0^\vee \rightarrow \mathcal{H}om_{\mathcal{O}_{P^*}}(H \otimes_{\mathcal{O}_S} \mathcal{O}_{P^*}(-2), \mathcal{O}_{P^*}(-3)) \otimes_{\mathcal{O}_S} M_0^\vee \cong$

$H^*\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*}(-1)$ is canonically equal to $A(-1)$. Since A^T is injective on every fibre P_s^* and $\mathcal{H}om_{\mathcal{O}_{P^*}}(H\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*}(-1), \mathcal{O}_{P^*})$ is S-flat, A^T itself is injective. Thus we obtain the following exact commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & H\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*}(-2) & \xrightarrow{A(-1)} & H^*\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*}(-1) & \longrightarrow & \theta(F) & \longrightarrow 0 \\ & \| & & \| & & \downarrow & \\ 0 \longrightarrow & H\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*}(-2) & \xrightarrow{A^t(-2)} & H^*\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*}(-1) & \longrightarrow & \mathcal{E}xt^1_{\mathcal{O}_{P^*}}(\theta(F), \mathcal{O}_{P^*}(-3))\otimes M_0 & \longrightarrow 0 \end{array}$$

This shows (2.7.2). (2.7.3) is directly deduced from Remark 2.6 and Theorem 2.1, (3). The assumption of (2.7.4) provides us with an isomorphism δ: $H^*\cong g_*(H^*\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*})=g_*(\theta(F)(1))\to g_*(\theta(F')(1))\otimes N\cong H'^*\otimes N$ which gives rise to the exact commutative diagram

$$\begin{array}{ccccccccc} 0\longrightarrow & H\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*}(-1) & \longrightarrow & H^*\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*} & \longrightarrow & \theta(F)(1) & \to 0 \\ & \downarrow{\scriptstyle \delta'\otimes 1} & & \downarrow{\scriptstyle \delta\otimes 1} & & \downarrow & \\ 0\to & (H'\otimes N)\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*}(-1) & \longrightarrow & (H'^*\otimes N)\otimes_{\mathcal{O}_S}\mathcal{O}_{P^*} & \longrightarrow & \theta(F')(1)\otimes g^*(N) & \to 0 \end{array}$$

Tensoring $\mathcal{O}_{P^*}(1)$ and taking the direct image by g, we obtain

$$\sum_i \delta(\alpha(z_i)(a))\otimes v_i=\sum_i (\alpha'(z_i)\otimes 1_N)(\delta'(a)) \qquad \text{for all } a\in H,$$

where $\{v_0, v_1, v_2\}$ is the dual basis of $\{z_0, z_1, z_2\}$. This shows our assertion.

When $\theta(F)$ is invertible, $\theta(F)^{\otimes 2}\cong\omega_{S(F)/S}\otimes_{\mathcal{O}_S}M_0$, that is, $\theta(F)$ is a relative θ-characteristic because $\mathcal{E}xt^1_{\mathcal{O}_{P^*}}(\theta(F), \mathcal{O}_{P^*}(-3))\cong\theta(F)^\vee\otimes\omega_{S(F)/S}$.

§ 3. Monadology

Our present aim is to prove the following theorem.

Theorem 3.1. *Let S be a locally noetherian scheme and let f and g be the structure morphism to S of the $\mathbf{P}^2$-bundle $P=\mathbf{P}^2_S$ and its dual $P^*\cong\mathbf{P}^2_S$. If F is a vector bundle on P such that for all geometric points s of S, F_s is stable and has the properties* (1.1) *and* (1.2), *then we have the following*:

(1) *There are a flat family $S(F)$ of curves over S in P^* and a coherent $\mathcal{O}_{S(F)}$-module $\theta(F)$ which is flat over S. For every morphism h: $S'\to S$ of locally noetherian schemes, $S(F\otimes_{\mathcal{O}_S}\mathcal{O}_{S'})=S(F)\times_S S'$ as subschemes of $P^*\times_S S'$ and $\theta(F\otimes_{\mathcal{O}_S}\mathcal{O}_{S'})=\theta(F)\otimes_{\mathcal{O}_S}\mathcal{O}_{S'}$. Moreover, for every geometric point s of S, $S(F)_s$ is the curve of jumping lines of F_s and $\theta(F_s)$ is the θ-characteristic associated with F_s.*

(2) $\theta(F)\cong\mathcal{E}xt^1_{\mathcal{O}_{P^*}}(\theta(F), \mathcal{O}_{P^*}(-3))\otimes M_0$.

(3) $g_*(\theta(F))=0$.

(4) *Let F' be another vector bundle on P which satisfies the same conditions as F. If $S(F)=S(F')$ and $\theta(F)\cong\theta(F')\otimes g^*(N)$ with N a line bundle on S, then $F\cong F'\otimes f^*(M)$ for some line bundle M on S.*

A proof of the first three assertions of the above theorem was given in the preceding section. The main part of this section is, therefore, devoted to giving a proof of (4) of the theorem. For this purpose we shall construct a monad $M(F)$ (in the sense of Horocks, see [3]) by using the net of quadrics α: $V\otimes_Z H\longrightarrow H^*$ associated with the vector bundle F on P which was defined in § 2. The key points of the proof are (i) for F and F' in (4) of Theorem 3.1, $H(F)=H(F')\otimes_{\mathcal{O}_S} N$ as we shall show in (3.6) and (ii) $H(F)$ is isomorphic to $F(1)$ locally over S, where $H(*)$ is the cohomology bundle of the monad $M(*)$. Once we know the facts, our assertion is an easy consequence of Lemma 3.16 of Part I.

Put $I=R(-3)$ and $I^*=R(0)$. Then, through $(\ ,\ )_0$, $I^*\otimes_{\mathcal{O}_S} M_0^\vee$ is isomorphic to $\mathcal{H}om_{\mathcal{O}_S}(I, \mathcal{O}_S)$. β and γ denote ψ_{-3} and ψ_{-1}, respectively; β: $(V\otimes_Z \mathcal{O}_S)\otimes I\longrightarrow H$ and γ: $(V\otimes_Z \mathcal{O}_S)\otimes H^*\longrightarrow I^*$. γ is the adjoint of β with respect to $(\ ,\ )$. For z, $z'\in V\otimes_Z \mathcal{O}_S$, it is easy to see

$$\gamma(z)\alpha(z')-\gamma(z')\alpha(z)=0. \tag{3.2}$$

Let T_P denote the relative tangent sheaf of P over S. Since $\bigwedge^2 T_P\cong \mathcal{O}_P(3)$, we have a canonical isomorphism $\bigwedge$: $T_P(-1)\to\Omega_P(2)$, where $\Omega_P=\Omega_{P/S}\cong T_P^\vee$. On the other hand, there are the exact sequences

$$0\longrightarrow\mathcal{O}_P(-1)\longrightarrow V^\vee\otimes_Z \mathcal{O}_P\xrightarrow{\ \varepsilon\ } T_P(-1)\longrightarrow 0$$

and its dual

$$0\longrightarrow\Omega_P(2)\xrightarrow{\ \tau\ } V\otimes_Z \mathcal{O}_P(1)\longrightarrow\mathcal{O}_P(2)\longrightarrow 0.$$

Composing ε, $\bigwedge$ and τ, we obtain a homomorphism λ: $V^\vee\otimes_Z \mathcal{O}_P\to V\otimes_Z \mathcal{O}_P(1)$. It is easy to write down the λ;

$$\lambda(v_i)=z_{i-1}\otimes\rho(z_{i+1})-z_{i+1}\otimes\rho(z_{i-1}) \qquad (i=0,\ 1,\ 2 \bmod (3)), \tag{3.3}$$

where ρ is the natural homomorphism $V\otimes_Z \mathcal{O}_P=f^*f_*(\mathcal{O}_P(1))\to\mathcal{O}_P(1)$.

By using these we can construct a monad

$$M(F):\quad H\otimes_{\mathcal{O}_S} \mathcal{O}_P\xrightarrow{a(F)} H^*\otimes_{\mathcal{O}_S} T_P(-1)\xrightarrow{c(F)} I^*\otimes_{\mathcal{O}_S} \mathcal{O}_P(1),$$

where we define $a(F)$ and $c(F)$ as follows;

$$a(F):\quad H\otimes_{\mathcal{O}_S}\mathcal{O}_P\xrightarrow{\alpha\otimes 1}(V^\vee\otimes_Z H^*)\otimes_{\mathcal{O}_S}\mathcal{O}_P$$
$$\cong H^*\otimes_{\mathcal{O}_S}(V^\vee\otimes_Z\mathcal{O}_P)\xrightarrow{1_{H^*}\otimes\varepsilon}H^*\otimes_{\mathcal{O}_S}T_P(-1),$$
$$c(F):\quad H^*\otimes_{\mathcal{O}_S}T_P(-1)\xrightarrow{1_{H^*}\otimes\tau^\wedge}H^*\otimes_{\mathcal{O}_S}(V\otimes_Z\mathcal{O}_P(1))$$
$$\cong(V\otimes_Z H^*)\otimes_{\mathcal{O}_S}\mathcal{O}_P(1)\xrightarrow{\gamma\otimes 1}I^*\otimes_{\mathcal{O}_S}\mathcal{O}_P(1).$$

(3.2), (3.3) and a computation similar to [3] p. 72 show that $c(F)\cdot a(F)=0$. For a geometric point s of S, the pull-back of $M(F)$ to the fibre P_s is the monad which was given in [3] Proposition 3 for F_s. $c(F)_s$ is surjective for all s. Then, by Nakayama's lemma, $c(F)$ itself is surjective. Since $a(F)_s$ is injective for all points s of S and $H^*\otimes_{\mathcal{O}_S}T_P(-1)$ is flat over S, $a(F)$ is injective and the cokernel $Q(F)$ of $a(F)$ is S-flat by virtue of Lemma 2.5. Over every geometric point s of S, $Q(F)_s$ is locally free by Barth [3] and hence $Q(F)$ is locally free. We see therefore that $M(F)$ is a monad on P. $M(F)$ is displayed as follows:

$$\text{(3.4)}\qquad\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H\otimes_{\mathcal{O}_S}\mathcal{O}_P & \longrightarrow & K(F) & \longrightarrow & H(F) & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H\otimes_{\mathcal{O}_S}\mathcal{O}_P & \xrightarrow{a(F)} & H^*\otimes_{\mathcal{O}_S}T_P(-1) & \longrightarrow & Q(F) & \longrightarrow & 0\\
 & & & & \downarrow{\scriptstyle c(F)} & & \downarrow{\scriptstyle d(F)} & & \\
 & & & & I^*\otimes_{\mathcal{O}_S}\mathcal{O}_P(1) & = & I^*\otimes_{\mathcal{O}_S}\mathcal{O}_P(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Note that

(3.5) for every geometric point s of S, $H(F)_s$ is stable and $c_1(H(F)_s)=2$.

For another vector bundle F', assume that α is isomorphic to α': $V\otimes_Z H'\to H'^*$, namely, there are isomorphisms δ: $H\to H'\otimes N$ and δ': $H^*\to H'^*\otimes N$ so that N is a line bundle on S and $(\alpha'\otimes 1_N)(1_V\otimes\delta)=\delta'\alpha$. Then $(a(F')\otimes 1_N)(\delta\otimes 1_{\mathcal{O}_P})=(\delta'\otimes 1_{T_P(-1)})\cdot a(F)$. Thus $Q(F)$ is naturally isomorphic to $Q(F')\otimes_{\mathcal{O}_S}N$. We claim

$$\text{(3.6)}\qquad H(F)\cong H(F')\otimes_{\mathcal{O}_S}N.$$

Proof. We may regard $H(F')\otimes_{\mathcal{O}_S}N$ as a subbundle of $Q(F)$ through the isomorphism of $Q(F')\otimes_{\mathcal{O}_S} N$ to $Q(F)$. Let η be the composition of the embedding $H(F')\otimes_{\mathcal{O}_S} N \hookrightarrow Q(F)$ and $d(F)$. If η is zero, then $H(F')\otimes_{\mathcal{O}_S} N$ is contained in $H(F)$ and hence it is a subbundle of $H(F)$. Since they have the same rank, they coincide with each other. Thus we have only to prove $\eta=0$. Take a closed point s of S and let η_s be the base change of η by $\operatorname{Spec}(\mathcal{O}_{S,s})\hookrightarrow S$. It is enough to show that $\eta_s=0$ for all closed points s of S. Thus we may assume that $S=\operatorname{Spec}(R)$ with $(R, \mathfrak{m})$ a local ring and then I^* is a free R-module. Since $\bigcap_n (I^*\otimes_{\mathcal{O}_S} \mathcal{O}_P(1))\mathfrak{m}^n$ $=(I^*\otimes_{\mathcal{O}_S} \mathcal{O}_P(1))(\bigcap_n \mathfrak{m}^n)=0$, we have $\eta\otimes_R R/\mathfrak{m}^n\neq 0$ for some n if $\eta\neq 0$. Hence, replacing R by $R/\mathfrak{m}^n$, we may assume that R is an artinian local ring. Now we shall prove our assertion by induction on the length $l(R)$ of R. If $l(R)=1$, that is, R is field k, then $H(F')\otimes\bar{k}=(H(F')\otimes_{\mathcal{O}_S} N)\otimes\bar{k}$ is a stable vector bundle on $\mathbf{P}^2_{\bar{k}}$ with $c_1(H(F')\otimes\bar{k})=2$, where $\bar{k}$ is the algebraic closure of k. Then every homomorphism of $H(F')$ to $I^*\otimes_{\mathcal{O}_S} \mathcal{O}_P(1)$ $\cong \mathcal{O}_P(1)^{\oplus(n-2)}$ must vanish. Suppose that our assertion is true when $l(R)$ $<l$. If $l(R)=l$, there is a principal ideal tR of R such that $tR=R/\mathfrak{m}$ as R-module. Set $\bar{R}=R/tR$. By induction assumption, $\eta\otimes_R \bar{R}=0$. Hence $\eta(H(F')\otimes_{\mathcal{O}_S} N)\subset t(I^*\otimes_{\mathcal{O}_S} \mathcal{O}_P(1))=\mathcal{O}_{\mathbf{P}^2}(1)^{\oplus(n-2)}$. Since $\eta(\mathfrak{m}\otimes(H(F')\otimes_{\mathcal{O}_S} N))$ $\subset t\mathfrak{m}(I^*\otimes_{\mathcal{O}_S} \mathcal{O}_P(1))=0$, $\eta=\bar{\eta}\pi$ with a homomorphism $\bar{\eta}$: $(H(F')\otimes_{\mathcal{O}_S} N)\otimes$ $R/\mathfrak{m}\to\mathcal{O}_{\mathbf{P}^2}(1)^{\oplus(n-2)}$, where π: $H(F')\otimes_{\mathcal{O}_S} N\to(H(F')\otimes_{\mathcal{O}_S} N)\otimes R/\mathfrak{m}$ is the natural projection. For the same reason as before, $\bar{\eta}=0$ and hence $\eta=0$.

q.e.d.

Let $U=\operatorname{Spec}(A)$ be an affine open set of S such that $M_0|_U\cong\mathcal{O}_U$. We now show that there is an isomorphism ν: $F_U\xrightarrow{\sim} H(F)(-1)_U$. We see easily that $H^1(P_U, F(-2)_U)\xrightarrow{\sim}\Gamma(U, H_U)=H(U)$ and $H^1(P_U, F_U)\xrightarrow{\sim}\Gamma(U, I^*_U)$ $=I^*(U)$, where the subscript U means the restriction to U. These isomorphisms provide us with elements of

$$\begin{aligned}
&\operatorname{Hom}_A(H^1(P_U, F(-2)_U), H(U))\cong H^1(P_U, F^\vee(-1)_U)\otimes_A H(U)\\
&\quad\cong\Gamma(R^1f_{U*}(F^\vee(-1)_U))\otimes_A H(U)\cong\Gamma(R^1f_{U*}(F^\vee(-1)_U)\otimes_{\mathcal{O}_U} H_U)\\
&\quad\cong\Gamma(R^1f_{U*}(F^\vee(-1)_U\otimes f_U^*(H_U)))\cong H^1(P_U, F^\vee(-1)\otimes_{\mathcal{O}_U} H_U)\\
&\quad\cong\operatorname{Ext}^1_{\mathcal{O}_{P_U}}(F_U, H_U\otimes_{\mathcal{O}_U}\mathcal{O}_P(-1)_U)
\end{aligned}$$

and $\operatorname{Hom}_A(I^*(U), H^1(P_U, F_U))\cong\operatorname{Ext}^1_{\mathcal{O}_{P_U}}(I_U^*\otimes_{\mathcal{O}_U}\mathcal{O}_{P_U}, F_U)$, respectively. These define the following extensions:

$$0\longrightarrow H_U\otimes_{\mathcal{O}_U}\mathcal{O}_{P_U}(-1)\longrightarrow K'\longrightarrow F_U\longrightarrow 0$$
$$0\longrightarrow F_U\longrightarrow Q'\longrightarrow I_U^*\otimes_{\mathcal{O}_U}\mathcal{O}_{P_U}\longrightarrow 0.$$

Since $\operatorname{Ext}^i_{\mathcal{O}_{P_U}}(I_U^*\otimes_{\mathcal{O}_U}\mathcal{O}_{P_U}, H_U\otimes_{\mathcal{O}_U}\mathcal{O}_{P_U}(-1))\cong\Gamma(R^if_{U*}(I^{*\vee}_U\otimes_{\mathcal{O}_U} H_U\otimes_{\mathcal{O}_U}\mathcal{O}_{P_U}))$

$=0$ for all i, the above extensions give rise to the display of a monad $M'(F_U)$ (cf. [3]):

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & H_U \otimes_{\mathcal{O}_U} \mathcal{O}_{P_U}(-1) \arrow[r] \arrow[d, equal] & K' \arrow[r] \arrow[d] & F_U \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H_U \otimes_{\mathcal{O}_U} \mathcal{O}_{P_U}(-1) \arrow[r] & X \arrow[r] \arrow[d] & Q' \arrow[r] \arrow[d] & 0 \\ & & I_U^* \otimes_{\mathcal{O}_U} \mathcal{O}_{P_U} \arrow[r, equal] \arrow[d] & I_U^* \otimes_{\mathcal{O}_U} \mathcal{O}_{P_U} \arrow[d] & \\ & & 0 & 0 & \end{tikzcd}$$

Note that the above construction is compatible with base changes; for an A-algebra B, $M'(F_U) \otimes_A B = M'(h^*(F_U))$, where h: Spec $(B) \to U = $ Spec (A). Now the same argument as in [3] p. 75—p. 76 yields a homomorphism ν: $F_U \to H(F)(-1)_U$ such that for all geometric points s of U, $\nu \otimes k(s)$ is an isomorphism of F_s to $H(F)(-1)_s$. Then ν is surjective by Nakayama's lemma and hence ν is an isomorphism.

Now we can prove Theorem 3.1 which was stated at the beginning of this section.

Proof of Theorem 3.1. If one notes that the construction of $S(F)$ and $\theta(F)$ in § 2 is compatible with base changes, he knows that every assertion of (1) is proved in § 2. (2) and (3) are (2.7.2) and (2.7.3), respectively. If $\theta(F) \cong \theta(F') \otimes g^*(N)$, then α for F is isomorphic to α' for F' by (2.7.4). Then $H(F) \cong H(F') \otimes_{\mathcal{O}_S} N$ by virtue of (3.5). On the other hand, what we have proved just before the theorem is that there is an open covering $\{U_\lambda\}$ of S such that $H(F)(-1)_{U_\lambda} \cong F_{U_\lambda}$ for each U_λ. Since F_s is stable for every geometric point s of S, we can apply Lemma 3.16 of Part I to our situation and see that there is a line bundle M' on S such that $H(F)(-1) = F \otimes f^*(M')$. Combining these results, we know that $F = F' \otimes f^*(M)$ for some line bundle M on S. q.e.d.

Remark 3.7. We can globalize the above theorem as follows; if F is a vector bundle on $P = \mathbf{P}(E)$ with E a vector bundle of rank 3 on S such that for every geometric point s of S, F_s is stable on $P_s = \mathbf{P}^2_{k(s)}$ and has the properties (1.1) and (1.2), then the same results as in Theorem 3.1 hold for this F.

§ 4. Application moduli spaces

The meaning of Theorem 3.1 is that the couple $(S(F), \theta(F))$ determines not only a stable vector bundle F on $\mathbf{P}^2$ with the properties (1.1) and (1.2) but also its infinitesimal neighbourhood (more strongly, étale neighbourhood) in the moduli space of stable vector bundles. The aim of this section is to explain this fact and the result will be summarized in Proposition 4.2. Combining this and a result which will appear in [20], we see, for example, that $\overline{M}_{\mathbf{P}^2}(2, 0, 2)$ is isomorphic to $\mathbf{P}^5$ and $\overline{M}_{\mathbf{P}^2}(2, 0, 3)$ is birationally equivalent to a triple covering of an open set of $\mathbf{P}^9$ (cf. [3]).

Throughout this section we shall fix an algebraically closed field k of *characteristic zero*. Let $U \subset \mathbf{P}(H^0(P^*, \mathcal{O}_{P^*}(n))^\vee)$ be the open set of non-singular curves of degree n in $P^* \cong \mathbf{P}^2_k$ and $M(n)_e = s(n)^{-1}(U)$ (see Proposition 1.9). If E is semi-stable, of rank 2 and $E \not\cong \mathcal{O}_{\mathbf{P}^2}^{\oplus 2}$ on $\mathbf{P}^2$, then $c_2(E) = n \geq 2$. Thus Proposition 1.7 and Lemma 1.8 show that if $S(E)$ is non-singular, then E is stable and locally free, that is,

(4.1) $M(n)_e \subset M_{\mathbf{P}^2}(2, 0, n)_0$ (for the notation, see Theorem 3.22 of Part I).

Since $s(n)$ is proper, so is $s(n)_U \colon M(n)_e \to U$. We denote the universal family of smooth curves over U by C. Let J be the Picard scheme of C over U. J decomposes into a disjoint union $\coprod J^d$, where d is the degree of invertible sheaves which correspond to points of J^d. The relative canonical sheaf $\omega_{C/U}$ defines a section σ of U to J^{2g-2} with g the genus of curves. Consider the following fibre product

$$\begin{array}{ccccc} \Theta & \longrightarrow & J^{g-1} & \ni & x \\ \sigma\downarrow & & \downarrow & & \downarrow \\ U & \xrightarrow{\delta} & J^{2g-2} & \ni & 2x \end{array}$$

For a $u \in U(k)$, Θ_u is the discrete scheme whose points represent the θ-characteristics of C_u and it is clear that δ is étale.

As was shown in [3] p. 139 (see also [20]), for $E \in M(n)_e$, $\theta(E)$ is invertible. Then, (2.2) and Theorem 3.6, (1) show that $s(n)_U$ goes through Θ:

$$s(n)_U \colon M(n)_e \xrightarrow{t(n)} \Theta \xrightarrow{\delta} U.$$

Theorem 2.1, (4) means that $t(n)$ is injective. Applying Theorem 3.6, (4) to the case of $S = \mathrm{Spec}\,(A)$ with A artinian, we know that $t(n)$ is unramified. Since $s(n)_U$ is proper and δ is separated, $t(n)$ is proper, too. Combining these, we see that $t(n)$ is a closed immersion. Let $V = s(n)_U(M(n)_e)$

with reduced structure. Since $M(n)_e$ is reduced and irreducible by Theorem 3.22 of Part I, $t(n)(M(n)_e)$ is one of the irreducible components of Θ_V.

Proposition 4.2. (1) *$M(n)_e$ is isomorphic via $t(n)$ to one of the irreducible components of Θ_V and hence $s(n)_V$ is unramified.*

(2) *Let V_0 be the open set of smooth points of V. Then $s(n)_{V_0}$ is étale and finite. In particular, $\sharp s(n)^{-1}(v)$ is constant in $v \in V_0$.*

Proof. Everything other than the flatness of $s(n)_{V_0}$ has been proved before this proposition. The flatness of $s(n)_{V_0}$ is deduced directly from the smoothness of $M(n)_e$ (Theorem 3.22 of Part I) and V_0 and the finiteness of $s(n)_V$. q.e.d.

Remark 4.3. V is non-empty for all $n \geq 2$ ([3] 5.4).

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Department of Mathematics
Faculty of Science
Kyoto University
Kyoto, 606 Japan