CHAPTER 28

Numerical simulations of sand transport problems, by C. Diédhiou, B. K. Thiam and I. Faye

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Abstract. This paper is devoted to numerical simulations of sand transport problem submitted to the tide near the seabed. We consider a two-scale numerical approach based on finite element method. The stability of the scheme is solved and finally, we present some numerical results.

Keywords. Short term dynamical of dune; finite element method; PDE; modeling; PDE; homogenization; two scale convergence.

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1. Introduction and Results

The understanding of sand transportation near the seabed is a challenge for scientists as all supernatural phenomenon. Many mathematical models are done by scientists. The challenge is to use a sand transport equation Balde (2017); Faye et al. (2011); Idier (2002) and an equation described the movement of the fluid (Navier Stokes equation or Shallow water). The objective of the this paper is to built a Two-Scale numerical method to simulate the sand dune in tidal area. The model considered in this paper is built and studied in Faye et al. (2011).

The concept of two-scale convergence was introduced by Nguetseng (1989) and Allaire (1992). Numerical method based on two-scale convergence was used successfully by many authors. In Aillot et al. (2002), such a method is use to manage the tide oscillation for long term drift forcast of objects in coastal ocean water. Frénod et al. (2007) made simulations of the 1D Euler equation using a Two-scale Numerical Method. In Frénod et al. (2009), such a method is used to simulate a charge particle beam in a periodic focusing channel. Mouton (2009) developed a Two-scale Semi-Lagradian Method for a beam and plasma application. In Faye et al. (2015), such a method is use to simulate the evolution of sand transport equation by using Fourier approach.

In this paper, we consider the following model presented in Faye et al. (2011); Thiam (2018). The system is modeled as follows

\[
\begin{align*}
\frac{\partial z^\epsilon(t,x)}{\partial t} - \frac{1}{\epsilon} \nabla \cdot (A^\epsilon \nabla z^\epsilon) &= \frac{1}{\epsilon} \nabla \cdot C^\epsilon \quad \text{in } ]0,T[ \times \Omega \\
\frac{\partial z^\epsilon(t,x)}{\partial n} &= g \quad \text{in } [0,T) \times \partial \Omega \\
z^\epsilon(0,x) &= z_0(x) \quad \text{in } \Omega
\end{align*}
\]

(1.1)

where \( z^\epsilon(t,x) \) is the dimensionless seabed altitude, \( t \in [0,T) \), for a given \( T \) and \( x \in \Omega \), \( \Omega \) being a two dimensional domain of class \( C^2 \)of \( \mathbb{R}^2 \). \( A^\epsilon \) and \( C^\epsilon \) are given by

\[
A^\epsilon = \begin{cases} 1 & \text{if } x \in \Omega_1 \\
0 & \text{elsewhere}
\end{cases}
\]

\[
C^\epsilon = \begin{cases} 1 & \text{if } x \in \Omega_2 \\
0 & \text{elsewhere}
\end{cases}
\]
\( A'(t, x) = a(1 - b\epsilon M(t, \frac{t}{\epsilon}, x))|U(t, \frac{t}{\epsilon}, x)|^3 \)

(1.3) \( C'(t, x) = c(1 - b\epsilon M(t, \frac{t}{\epsilon}, x))|U|^3 \cdot \frac{U(t, \frac{t}{\epsilon}, x)}{|U(t, \frac{t}{\epsilon}, x)|} \)

for \( a, b \) and \( c \) are three constants positives and \( M \) and \( U \) are respectively the water variation and velocity. \( z_0 \in L^2(\Omega) \) and \( g \in L^2([0, T], L^2(\Omega)) \) are given functions. One can justify the boundary condition of (1.1) by the fact that if we consider a big domain \( \Omega \) in which the sand does not go out, what is translated by the fact that the flux \( q \) is zero on \( \partial\Omega \), i.e. \( q \cdot n = 0 \) on \( \partial\Omega \), where \( n \) is the normal exterior vector and \( q \) is given by

(1.4) \( q = q_f - |q_f|\lambda \nabla z, \)

where \( q_f \) and \( \lambda \) are respectively the water velocity induced sand flow on a flat seabed and the inverse value of the maximum slope of the sediment surface when the water velocity is 0. From this equation we have, assuming that \( q_f \neq 0 \) on \( \partial\Omega \),

(1.5) \( \frac{\partial z(t, x)}{\partial n} = \nabla z \cdot n = \frac{q_f \cdot n}{|q_f|\lambda} = g \) on \( \partial\Omega \).

The small parameter \( \epsilon \) involved in the model is the ratio between the main tide period \( \frac{1}{\omega} = 13 \) hours and and observation time which is about three months i.e. \( \epsilon = \frac{1}{13} = \frac{1}{200} \). In Faye et al. (2015), the authors used equation (1.1) in a domain without boundary: the two dimensional \( \mathbb{T}^2 \). In this paper, we suppose that the domain \( \mathbb{T}^2 \subset \Omega \), which is bounded with boundary \( \partial\Omega \) and functions \( U \) and \( M \) are regular and satisfy the following hypotheses.
\[ \theta \mapsto (U, M) \text{ is periodic of period 1} \]

\[
\begin{align*}
|U|, & \quad |\frac{\partial U}{\partial t}|, & \quad |\frac{\partial U}{\partial \theta}|, & \quad |\nabla \cdot U| \\
|M|, & \quad |\frac{\partial M}{\partial t}|, & \quad |\frac{\partial M}{\partial \theta}|, & \quad |\nabla M| \quad \text{are bounded by } d,
\end{align*}
\]

(1.6) \[ \exists U_{thr} \text{ such that } \forall (t, \theta, x) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega, |U(t, \theta, x)| \leq U_{thr} \quad \implies \]

\[ \begin{align*}
\frac{\partial U}{\partial \theta}(t, \theta, x) & = 0, \quad \nabla \cdot U(t, \theta, x) = 0 \\
\frac{\partial M}{\partial t}(t, \theta, x) & = 0, \quad \nabla M(t, \theta, x) = 0 \end{align*} \]

\[ \exists \theta_\alpha < \theta_\omega \in [0, 1] \text{ such that } \forall \theta \in [\theta_\alpha, \theta_\omega] \implies |U(t, \theta, x)| \geq U_{thr} \]

The precise aim of this paper is to develop a two-scale numerical method based on finite element method to solve equation (1.1). It is known that in Faye et al. (2011) and Thiam (2018), if \( z_0 \in H^1(\Omega) \), for any \( \epsilon > 0 \) and any \( T \in [0, T) \), the system (1.1) admit a unique solution \( z^\epsilon \in L^\infty([0, T), H^1(\Omega)) \). In addition, the sequence of solutions to (1.1) is bounded in \( L^\infty([0, T), H^1(\Omega)) \).

The theorem is stated as follows:

**Theorem 83.** Under assumption (1.6), for any \( T \), not depending on \( \epsilon \), the sequence \( (z^\epsilon) \) of solutions to (1.1), with coefficients given by (1.2) and (1.3), Two-Scale converges to the profile \( U \in L^\infty([0, T), L^\infty(\mathbb{R}, L^2(\Omega))) \) solution to

(1.7) \[ \begin{align*}
\frac{\partial U}{\partial \theta} - \nabla \cdot (\tilde{A} \nabla U) & = \nabla \cdot \tilde{C} \text{ in } (0, T) \times \mathbb{R} \times \Omega \\
\frac{\partial U}{\partial n} & = g \text{ on } (0, T) \times \mathbb{R} \times \partial \Omega
\end{align*} \]

where \( \tilde{A} \) and \( \tilde{C} \) are given by

(1.8) \[ \tilde{A}(t, \theta, x) = a |U(t, \theta, x)|^3 \text{ and } \tilde{C}(t, \theta, x) = c |U(t, \theta, x)|^3 \frac{U(t, \theta, x)}{|U(t, \theta, x)|}. \]

Furthermore, if the supplementary assumption

(1.9) \[ U_{thr} = 0, \]

is done, we have

(1.10) \[ \tilde{A}(t, \theta, x) \geq \tilde{G}_{thr} \text{ for any } t, \theta, x \in [0, T] \times \mathbb{R} \times \Omega, \]

and, defining \( U^\epsilon = U^\epsilon(t, x) = U(t, \frac{1}{\epsilon}, x) \), the following estimate holds for \( z^\epsilon - U^\epsilon \)

(1.11) \[ \left\| \frac{z^\epsilon - U^\epsilon}{\epsilon} \right\|_{L^\infty([0,T), L^2(\Omega))} \leq \alpha, \]

where \( \alpha \) is a constant not depending on \( \epsilon \).

2. Finite element method for Two scale limit

The aim of this section is to develop a numerical method based on finite element method which allows us to resolve (1.1) in a precise way and more expensive. Because of theorem 83, we can approximate the solution \( z^\epsilon(t, x) \) of (1.1) by the solution \( U^\epsilon(t, x) = U(t, \frac{1}{\epsilon}, x) \), where \( U \) is solution to (1.7).

We first consider a uniform mesh on \([0, T]\). For the discretization of the time, we suppose that the time step \( \Delta \theta \) is constant and we use the notation \( \theta_n = n \Delta \theta \). Denoting by \( U^n \) the approximation of \( U(\cdot, \theta_n, \cdot) \), using finite differences, we can approximate \( \frac{\partial U}{\partial \theta} \) in the form

\[ \frac{\partial U}{\partial \theta}(t, \theta_n, x) \sim \frac{U(t, \theta_{n+1}, x) - U(t, \theta_n, x)}{\Delta \theta} = \frac{U^{n+1} - U^n}{\Delta \theta}. \]

Hence, system (1.7) becomes

\[
\begin{cases}
\frac{U^{n+1} - U^n}{\Delta \theta} - \nabla \cdot (\tilde{A} \nabla U^n) = \nabla \cdot \tilde{C} \text{ on } [0, T) \times \mathbb{R} \times \Omega \\
\frac{\partial U^{n+1}}{\partial n} = g \text{ on } [0, T) \times \mathbb{R} \times \partial \Omega.
\end{cases}
\]

Let

\[ V_0 = \{ w \in H^1(\Omega) : \frac{\partial w}{\partial n} = g \text{ on } \partial \Omega \}. \]

then multiplying (2.1) by \( v \in V_0 \) and integrating, we get the following variational problem: we seek for
\begin{equation}
\left\{ \begin{array}{l}
U^n \in V_0,
\forall v \in V_0, \int_\Omega \frac{U^{n+1} - U^n}{\Delta \theta} v dx + \int_\Omega \tilde{A} \nabla U^n \nabla v dx = \int_{\partial \Omega} g v d\sigma + \int_\Omega \nabla \cdot \tilde{C} v dx
\end{array} \right. \tag{2.2}
\end{equation}

Let \( \{ T_h, h \to 0 \} \) be a quasi-uniform family of admissible triangulation of \( \Omega \). We denote by \( \Omega_h \subset \Omega \), the union of triangles of \( T_h \), and \( h \) the maximal length of the sides of the triangulation \( T_h \). And let \( V_h \subset V \) be the set of all continuous piecewise linear functions defined on \( T_h \). Let \( \{ w_i \}_{j=1}^N \) be the standard basis of \( V_h \). Then, using conformal finite element with a finite element discrete space \( V_h \subset V_0 \), the discrete variational problem is to find \( U^{n+1}_h \in V_h \) such that \( \forall v_h \in V_h \):

\begin{equation}
\int_{\Omega_h} \left[ \frac{U^{n+1}_h - U^n_h}{\Delta \theta} v_h + \tilde{A} U^{n+1}_h \nabla v_h \right] dx = \int_{\partial \Omega_h} g v_h d\sigma + \int_{\Omega_h} \nabla \cdot \tilde{C} v_h dx \quad \forall v_h \in V_h. \tag{2.3}
\end{equation}

Let \( w_i, i = 1, \ldots, N \) a basis of \( V_h \), then \( \forall U^n_h \in V_h \) we have

\begin{equation}
U^n_h(x) = \sum_{i=1}^N u^n_i w^i(x) \, \forall n, \forall x \in \Omega, \tag{2.4}
\end{equation}

where \( u^n_i, i = 1, \ldots, N \) are the components of \( U^n_h \) in the base \( (w_i)_{i=1,\ldots,N} \).

Taking \( v_h = w^j, j = 1, \ldots, N \) we get from (2.2) that

\begin{equation}
\int_{\Omega_h} \left[ \frac{U^{n+1}_h - U^n_h}{\Delta \theta} w^j dx + \tilde{A} U^{n+1}_h \nabla w^j \right] dx = \int_{\partial \Omega_h} g w^j d\sigma + \int_{\Omega_h} \nabla \cdot \tilde{C} w^j dx, \forall 1 \leq j \leq N. \tag{2.5}
\end{equation}

Using (2.4), we have

\begin{equation}
\sum_{i=1}^N \frac{1}{\Delta \theta} \left( u_i^{n+1} - u^n_i \right) \int_{\Omega_h} w^j w^i dx + \sum_{i=1}^N u_i^{n+1} \int_{\Omega_h} \tilde{A} \nabla w^i \nabla w^j dx = \int_{\partial \Omega_h} g w^j d\sigma + \int_{\Omega_h} \nabla \cdot \tilde{C} w^j dx \forall 1 \leq j \leq N. \tag{2.6}
\end{equation}
From this later equation, we get the following equation

\[ \sum_i \left( \frac{1}{\Delta \theta} \int_{\Omega_h} w^i w^j dx + \int_{\Omega_h} \tilde{A} \nabla w^i \nabla w^j \right) u^{n+1}_i = \sum_i \left( \frac{1}{\Delta \theta} \int_{\Omega_h} w^i w^j \right) u^n_i dx \]

(2.7) \[ + \int_{\partial \Omega_h} gw^j d\sigma + \int_{\Omega_h} \nabla \cdot \tilde{C} w^j dx, \quad \forall \ 1 \leq j \leq N. \]

This system can be written as follows

(2.8) \[ \left( \frac{1}{\Delta \theta} M + A \right) U^{n+1}_h = \frac{1}{\Delta \theta} MU^n_h + B, \]

where \( U^n = (u^n_1, \ldots, u^n_N)^t \) is the unknown vector and \( A \) a matrix of size \( N \times N \) where the coefficients are given by

\[ A_{i,j} = \int_{\Omega_h} \tilde{A} \nabla w^i \nabla w^j dx, \]

\( M \) a matrix of size \( N \times N \) where the coefficients are given by

\[ M_{i,j} = \frac{1}{\Delta \theta} \int_{\Omega_h} w^i w^j dx \]

and \( B \) is a vector given by

\[ B_j = \int_{\partial \Omega_h} gw^j d\sigma + \int_{\Omega_h} \nabla \cdot \tilde{C} w^j dx. \]

We have the following theorem of convergence.

**Theorem 84.** Let \( h \) be the biggest diameter of all the meshes of \( \Omega \), \( U \) be the solution to (1.7) and \( U^n_h = U(\theta_n, x_h) \in V_h \) the approximation function of \( U \). Then, the following estimate holds

(2.9) \[ \| U - U^n_h \|_{H^1(\Omega)} \leq C_0 h \| U \|_{H^1}. \]

We have also the following stability result.
Theorem 85. Let $I$ be the identity matrix and $\left\| \left( I + \Delta \theta M^{-1} A \right)^{-1} \right\|$ be the spectral norm of the matrix $\left( I + \delta \theta M^{-1} A \right)^{-1}$. Then, 

$$\forall \Delta \theta > 0 \text{ and } h > 0, \text{ if } \left\| \left( I + \Delta \theta M^{-1} A \right)^{-1} \right\| \leq 1,$$

we have the stability of the scheme.

$$\left\| U^n_h \right\|_{L^2(\Omega_h)} \leq \left\| \left( I + \Delta \theta M^{-1} A \right)^{-1} \right\| U^0 \left\|_{L^2(\Omega_h)} + \Delta \theta \left\| M^{-1} \right\| \sum_{k=1}^{n} \left\| \left( I + \Delta \theta M^{-1} A \right)^{-1} \right\| \left( \sup_{0 \leq n \leq N} \left\| B \right\| \right)^n \right) \tag{2.10}$$

Proof. We get from (2.8)

$$\left( M + \Delta \theta A \right)U^{n+1} = MU^n + \Delta \theta U^{n+1}$$

As the matrix $M + \Delta \theta A$ is invertible, we have

$$U^{n+1}_h = \left( M + \Delta \theta A \right)^{-1} MU^n + \left( M + \Delta \theta A \right)^{-1} \Delta \theta B^{n+1}.$$

Thus, by varying $n$, the following equalities hold:

$$U^n_h = \left( M + \Delta \theta A \right)^{-1} MU^{n-1} + \left( M + \Delta \theta A \right)^{-1} \Delta \theta B^n$$

$$U^{n-1}_h = \left( M + \Delta \theta A \right)^{-1} MU^{n-2} + \left( M + \Delta \theta A \right)^{-1} \Delta \theta B^{n-1}$$

$$U^{n-2}_h = \left( M + \Delta \theta A \right)^{-1} MU^{n-3} + \left( M + \Delta \theta A \right)^{-1} \Delta \theta B^{n-2}$$

... 

$$U^1_h = \left( M + \Delta \theta A \right)^{-1} MZ^0 + \left( M + \Delta \theta A \right)^{-1} \Delta \theta B^1.$$

This makes possible, to obtain the following generic formula for $U^n$.

$$U^n_h = \left[ \left( I + \Delta \theta M^{-1} A \right)^{-1} \right]^n U^0 + \Delta \theta M^{-1} \sum_{k=1}^{n} \left[ \left( I + \Delta \theta M^{-1} A \right)^{-1} \right]^k B^{n-k+1}$$
Taking the norm of $U^n$, we get
\[
\|U^n_h\|_{L^2(\Omega_h)} \leq \left\| (I + \Delta \theta M^{-1} A)^{-1} \right\| \left\| U^0 \right\|_{L^2(\Omega_h)} \\
+ \Delta \theta \left\| M^{-1} \right\| \sum_{k=1}^{n} \left\| (I + \Delta \theta M^{-1} A)^{-1} \right\| \left\| B \right\|^{n-k+1},
\]
giving the desired result. ■

Let us focus on Numerical method:

In this section, we develop a two scale numerical method based on finite element method in order two approximate the solution $z^\epsilon$ of (1.1).

**3.1. Finite element method for reference solution.** We proceed in a same way as in the previous section. Considering a time discretization with time step $\Delta t$ and $t_n = n\Delta t, \ t \in [0,T]$, we obtain from (1.1) the following time discretization problem

\[
\begin{cases}
\frac{z_{n+1}^\epsilon - z_n^\epsilon}{\Delta t} - \frac{1}{\epsilon} \nabla \cdot (A^\epsilon \nabla z_{n+1}^\epsilon) = \frac{1}{\epsilon} \nabla \cdot C^\epsilon \quad \text{in} \quad ]0,T[ \times \Omega \\
\end{cases}
\]

(3.1)

\[
z^\epsilon(0, x) = z_0(x) \quad \text{in} \quad \Omega
\]

\[
\frac{\partial z_{n+1}^\epsilon}{\partial n} = g \quad \text{on} \quad [0,T) \times \partial \Omega,
\]

where $z^\epsilon(t_n, x) = z_n^\epsilon$.

Multiplying (3.1) by a smooth test function $v$ and then integrating over $\Omega$ we get:

\[
\frac{1}{\Delta t} \int_\Omega (z_{n+1}^\epsilon - z_n^\epsilon) v dx + \frac{1}{\epsilon} \int_\Omega A^\epsilon \nabla z_{n+1}^\epsilon \cdot \nabla v(x) dx
\]

(3.2)

\[
- \frac{1}{\epsilon} \int_{\partial \Omega} A^\epsilon \nabla z_{n+1}^\epsilon \cdot n \ v(x) dx = \frac{1}{\epsilon} \int_\Omega \nabla \cdot C^\epsilon v(x) dx
\]

Now, due to the boundary condition (3.1), it can be rewritten as follows

\[
\frac{1}{\Delta t} \int_\Omega (z_{n+1}^\epsilon - z_n^\epsilon) v dx + \frac{1}{\epsilon} \int_\Omega A^\epsilon(x) \nabla z_{n+1}^\epsilon \cdot \nabla v(x) dx =
\]

(3.3)

\[
\frac{1}{\epsilon} \int_{\partial \Omega} A^\epsilon g v(x) dx + \frac{1}{\epsilon} \int_\Omega \nabla \cdot C^\epsilon v(x) dx.
\]

Multiplying (3.1) by $\epsilon$, we have

\[
\frac{\epsilon}{\Delta t} \int_\Omega (z_{n+1}^\epsilon - z_n^\epsilon) v dx + \int_\Omega A^\epsilon(x) \nabla z_{n+1}^\epsilon \cdot \nabla v(x) dx =
\]
(3.4) \[ \int_{\partial \Omega} A^\varepsilon g v(x)dx + \int_{\Omega} \nabla \cdot C^\varepsilon v(x)dx. \]

Using the same discretization of the domain \( \Omega \) and denoting by \( z_{n,h}^\varepsilon = z_{\varepsilon}(t_n, x_h), \ x_h \in \Omega_h \), we have the following finite element problem: find \( z_{n,h}^\varepsilon \in V_h \) such that

\[ \begin{aligned}
\frac{\varepsilon}{\Delta t} \int_{\Omega_h} (z_{h,n+1}^\varepsilon - z_{h,n}^\varepsilon) v_h dx + \int_{\Omega_h} A^\varepsilon \nabla z_{h,n+1}^\varepsilon \cdot \nabla v_h dx = \\
\int_{\partial \Omega_h} A^\varepsilon g v_h d\sigma + \int_{\Omega_h} \nabla \cdot C^\varepsilon v_h dx. 
\end{aligned} \]

(3.5)

For any \( \varepsilon, \ z_{n,h}^\varepsilon \in V_h \),

then there exists \( (z_1^n, \ldots, z_N^n) \) such that

\[ z_{n,h}^\varepsilon(t, x) = \sum_{j=1}^N z_j^i w_i(x) \]

(3.6)

then from (3.1), we have the following system

\[ \begin{aligned}
\sum_{i=1}^N \frac{\varepsilon}{\Delta t} (z_i^{n+1} - z_i^n) \int_{\Omega_h} w^i w^j dx + \sum_{i=1}^N z_i^{n+1} \int_{\Omega_h} A^\varepsilon \nabla w^i \nabla w^j dx = \\
\int_{\partial \Omega_h} g w^j d\sigma + \int_{\Omega_h} \nabla \cdot C^\varepsilon w^j dx, \ \forall 1 \leq j \leq N.
\end{aligned} \]

(3.7)

From this later equation, we get the following equation

\[ \begin{aligned}
\sum_{i=1}^N \left( \frac{\varepsilon}{\Delta t} \int_{\Omega_h} w^i w^j + \int_{\Omega_h} A^\varepsilon \nabla w^i \nabla w^j \right) z_i^{n+1} = \sum_{i=1}^N \left( \frac{\varepsilon}{\Delta t} \int_{\Omega_h} w^i w^j \right) z_i^n \\
+ \int_{\partial \Omega_h} g w^j d\sigma + \int_{\Omega_h} \nabla \cdot C^\varepsilon w^j dx, \ \forall 1 \leq j \leq N.
\end{aligned} \]

(3.8)

which can be written as follows
(3.9) \[ A'Z^{n+1} = B'Z^n + C', \]

where \( A', B' \) are \( N \times N \) matrix defined respectively by

(3.10) \[ A'_{ij} = \frac{\epsilon}{\Delta t} \int_{\Omega} w^i w^j dx + \int_{\Omega_h} A' \nabla w^i \nabla w^j dx \]

(3.11) \[ B'_{ij} = \frac{\epsilon}{\Delta t} \int_{\Omega} w^i w^j dx \]

and \( C' \) is a vector defined by

(3.12) \[ C'_j = \int_{\partial \Omega_h} g w^j d\sigma + \int_{\Omega_h} \nabla \cdot C' w^j dx. \]

3.2. Convergence Result. In this section, we are going to proof the result containing in theorem 83.

Proof of theorem 83 Let \( \psi^\varepsilon(t, x) = \psi(t, \frac{t}{\varepsilon}, x) \) be a regular function with compact support on \([0, T) \times \Omega\) and periodic of period 1. Multiplying the first equation by \((1.1)\) by \( \psi^\varepsilon \) and integrating over \([0, T) \times \Omega\) we get:

(3.13) \[ \int_{\Omega} \int_0^T \frac{\partial z^\varepsilon}{\partial t} \psi^\varepsilon \, dt \, dx - \frac{1}{\varepsilon} \int_{\Omega} \int_0^T \nabla \cdot (A' \nabla z^\varepsilon) \psi^\varepsilon \, dt \, dx = \frac{1}{\varepsilon} \int_{\Omega} \int_0^T \nabla \cdot C' \psi^\varepsilon \, dt \, dx. \]

Using integration by parts over \([0, T)\) in the first term and Green formula over \( \Omega \) in the second integral, we get

(3.14) \[ -\int_{\Omega} z_0(x) \psi(0, 0, x) \, dx - \int_{\Omega} \int_0^T \frac{\partial \psi^\varepsilon}{\partial t} z^\varepsilon \, dt \, dx + \frac{1}{\varepsilon} \int_{\Omega} \int_0^T A' \nabla z^\varepsilon \nabla \psi^\varepsilon \, dt \, dx \]

But \( \frac{\partial \psi^\varepsilon}{\partial t} \) writes

(3.15) \[ \frac{\partial \psi^\varepsilon}{\partial t} = \left( \frac{\partial \psi}{\partial t} \right)^\varepsilon + \frac{1}{\varepsilon} \left( \frac{\partial \psi}{\partial \theta} \right)^\varepsilon, \]

where

\[
(\frac{\partial \psi}{\partial t})^\varepsilon (t, x) = \frac{\partial \psi}{\partial t} (t, \frac{t}{\varepsilon}, x) \quad \text{and} \quad \left( \frac{\partial \psi}{\partial \theta} \right)^\varepsilon (t, x) = \frac{\partial \psi}{\partial \theta} (t, \frac{t}{\varepsilon}, x),
\]

Thus, we get

\[
\int_\Omega \int_0^T z^\varepsilon \left( \frac{\partial \psi}{\partial t} \right)^\varepsilon + \frac{1}{\varepsilon} \left( \frac{\partial \psi}{\partial \theta} \right)^\varepsilon + \frac{1}{\varepsilon} \nabla \cdot (A^\varepsilon \nabla \psi^\varepsilon) \right) \, dt \, dx - \frac{1}{\varepsilon} \int_0^T \int_{\partial \Omega} A^\varepsilon g \psi^\varepsilon \, d\sigma
\]

\[
(3.17)
\]

Multiplying by \( \varepsilon \)

\[
\int_\Omega \int_0^T z^\varepsilon \left( \frac{\partial \psi}{\partial t} \right)^\varepsilon \, dt \, dx + \int_\Omega \int_0^T \left( \frac{\partial \psi}{\partial \theta} \right)^\varepsilon z^\varepsilon \, dt \, dx + \int_\Omega \int_0^T \nabla \cdot (A^\varepsilon \nabla \psi^\varepsilon) \right) z^\varepsilon \, dt \, dx
\]

\[
(3.18)
\]

As \( \psi^\varepsilon \) is regular with compact support on \([0, T] \times \Omega\), and \( A^\varepsilon \) is a regular function, the functions \( \left( \frac{\partial \psi}{\partial t} \right)^\varepsilon \), \( \left( \frac{\partial \psi}{\partial \theta} \right)^\varepsilon \), \( \nabla \cdot (A^\varepsilon \nabla \psi^\varepsilon) \) and \( \nabla \psi^\varepsilon \) can be considered as test functions. Then using two-scale convergence we get when \( \varepsilon \) goes to 0,

\[
\int_0^1 \int_\Omega \int_0^T \frac{\partial \psi}{\partial \theta} U \, dt \, d\theta \, dx + \int_0^1 \int_\Omega \int_0^T \nabla \cdot (\tilde{A} \nabla \psi) \right) U \, dt \, d\theta \, dx
\]

\[
(3.19)
\]

Using Green Formula, we get

\[
\int_\Omega \int_0^1 \int_0^T \left( \frac{\partial U}{\partial \theta} - \nabla \cdot (\tilde{A} \nabla U) \right) \psi \, dt \, d\theta \, dx = \int_0^1 \int_\Omega \int_0^T \nabla \cdot C \psi \, dt \, d\theta \, dx
\]

which is the weak formulation of
Let us characterize the homogenized equation for $\tilde{A}$ and $\tilde{C}$. Multiplying (1.2) by $\psi^\epsilon$ and integrating over $\Omega$ we get

$$\int_{\Omega} \int_{0}^{T} \tilde{A} \epsilon \psi dtdx = \int_{\Omega} \int_{0}^{T} a(1 - b \epsilon M(t, \theta, x))g_{a}(|U(t, \theta, x)|)\psi dtdx$$

then we have

$$\int_{\Omega} \int_{0}^{T} \int_{0}^{1} ag_{a}(|U(t, \theta, x)|)\psi dtdx = \int_{\Omega} \int_{0}^{T} \int_{0}^{1} A\psi d\theta dtdx.$$  

Multiplying (1.3) by $\psi^\epsilon$ and integrating over $\Omega$ we get

$$\int_{\Omega} \int_{0}^{T} \tilde{C} \epsilon \psi dtdx = \int_{\Omega} \int_{0}^{T} c(1 - b \epsilon M(t, \theta, x))g_{c}(|U(t, \theta, x)|)\frac{U(t, \theta, x)}{|U(t, \theta, x)|}\psi dtdx$$

we have

$$\int_{\Omega} \int_{0}^{T} \int_{0}^{1} cg_{c}(|U(t, \theta, x)|)\frac{U(t, \theta, x)}{|U(t, \theta, x)|}\psi dtdx = \int_{\Omega} \int_{0}^{T} \int_{0}^{1} C\psi d\theta dt dx.$$  

Then

$$A = ag_{a}(|U(t, \theta, x)|) \text{ and } C = cg_{c}(|U(t, \theta, x)|)\frac{U(t, \theta, x)}{|U(t, \theta, x)|}.$$  

Since the coefficients $A^\epsilon(t, x)$ and $C^\epsilon(t, x)$ of (1.1) two scale converges to $\tilde{A}(t, \theta, x)$ and $\tilde{C}(t, \theta, x)$, then these coefficients can be set in the form

$$A^\epsilon(t, x) = \tilde{A}^\epsilon(t, x) + \epsilon\tilde{A}_{1}^\epsilon(t, x) \text{ and } C^\epsilon(t, x) = \tilde{C}^\epsilon(t, x) + \epsilon\tilde{C}_{1}^\epsilon(t, x)$$

where

(3.23) \[ \mathcal{A}'(t,x) = \tilde{A}(t,\frac{t}{\epsilon},x), \quad C'(t,x) = \tilde{C}(t,\frac{t}{\epsilon},x) \]

and

(3.24) \[ \tilde{A}'_1(t,x) = \tilde{A}_1(t,\frac{t}{\epsilon},x), \quad \tilde{C}'_1(t,x) = \tilde{C}_1(t,\frac{t}{\epsilon},x) \]

We have also to notice that, under the same assumptions as in Theorem 83, the coefficients

(3.25) \[ \tilde{A}, \tilde{C}, \tilde{A}_1, \tilde{C}_1, \tilde{A}', \tilde{C}', \tilde{A}'_1, \text{ and } \tilde{C}'_1 \] are regular and bounded.

Because of (3.22), equation (1.1) becomes

(3.26) \[ \begin{cases} \frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon} \nabla \cdot \left( \tilde{A}' \nabla z^\epsilon \right) = \frac{1}{\epsilon} \nabla \cdot \tilde{C}' + \nabla \cdot \left( \tilde{A}'_1 \nabla z^\epsilon \right) + \nabla \cdot \left( \tilde{C}'_1 \nabla z^\epsilon \right) \\ \frac{\partial z^\epsilon}{\partial n} = g \end{cases} \]

From (1.7) and using the fact that

(3.27) \[ \frac{\partial U^\epsilon}{\partial t} = \left( \frac{\partial U}{\partial t} \right)^\epsilon + \frac{1}{\epsilon} \left( \frac{\partial U}{\partial \theta} \right)^\epsilon, \]

where

\[ \left( \frac{\partial U}{\partial t} \right)^\epsilon(t,x) = \frac{\partial U}{\partial t}(t,\frac{t}{\epsilon},x) \quad \text{and} \quad \left( \frac{\partial U}{\partial \theta} \right)^\epsilon(t,x) = \frac{\partial U}{\partial \theta}(t,\frac{t}{\epsilon},x) \]

We have that \( U^\epsilon \) is solution to

(3.28) \[ \begin{cases} \frac{\partial U^\epsilon}{\partial t} - \frac{1}{\epsilon} \nabla \cdot \left( \tilde{A}' \nabla U^\epsilon \right) = \frac{1}{\epsilon} \nabla \cdot \tilde{C}' + \left( \frac{\partial U}{\partial t} \right)^\epsilon \\ \frac{\partial U^\epsilon}{\partial n} = g. \]
From formulas (3.26) and (3.28) we deduce that \( \frac{z^\epsilon - U^\epsilon}{\epsilon} \) is solution to

\[
\begin{aligned}
\partial_t \left( \frac{z^\epsilon - U^\epsilon}{\epsilon} \right) - \frac{1}{\epsilon} \nabla \cdot \left( (\tilde{A}^\epsilon + \epsilon \tilde{A}_1^\epsilon) \nabla \left( \frac{z^\epsilon - U^\epsilon}{\epsilon} \right) \right) &= \frac{1}{\epsilon} \nabla \cdot \tilde{C}_1^\epsilon \\
\partial_t \left( \frac{z^\epsilon - U^\epsilon}{\epsilon} \right) + \nabla \cdot (\tilde{A}_1^\epsilon \nabla U^\epsilon) &\text{ in } [0, T] \times \Omega \\
\frac{\partial}{\partial n} \left( \frac{z^\epsilon - U^\epsilon}{\epsilon} \right) &= 0 \text{ on } [0, T] \times \partial \Omega.
\end{aligned}
\]

(3.29)

All the coefficients of (3.29) are regular and bounded, then existence of \( \left( \frac{z^\epsilon - U^\epsilon}{\epsilon} \right) \) is a consequence result of Ladyzenskaja et al. (1968). We have to notice that, as the boundary condition of (3.29) is homogeneous, there is no the boundary term to be considered. Then using the same argument as in Faye et al. (2011), we get that \( \left( \frac{z^\epsilon - U^\epsilon}{\epsilon} \right) \) solution to (3.29) is bounded in \( L^2([0, T], L^2(\Omega)) \), and we have

\[
\| z^\epsilon - U^\epsilon \|_{L^\infty([0, T], L^2(\Omega^2))} \leq \epsilon \| Z(0, 0, \cdot) \|_2 \gamma
\]

(3.30)

where \( \gamma \) is a constant. \( \blacksquare \)

We have also the following theorem of convergence

**Theorem 86.** Let \( \epsilon \) be a positive real, \( z^\epsilon \) be the solution to (1.1), \( U_h^\epsilon \) the approximation of \( U \) solution to (1.7) and \( U^\epsilon \) defined by \( U^\epsilon(t, x) = U(t, \frac{t}{\epsilon}, x) \). Then, under assumptions (1.6), \( z^\epsilon - U_h^\epsilon \) satisfies the following estimate:

\[
\| z^\epsilon - U_h^\epsilon \|_{L^\infty([0, T], L^2(\Omega^2))} \leq \epsilon \| Z(0, 0, \cdot) \|_2 + f(h, n).
\]

(3.31)

where \( f \) is a function not depending on \( \epsilon \) and satisfying \( \lim_{h \to 0} f(h, n) = 0 \).

**Proof.** We have

\[
\| z^\epsilon - U_h^\epsilon \|_{L^\infty([0, T], L^2(\Omega^2))} = \| z^\epsilon - U^\epsilon + U^\epsilon - U_h^\epsilon \|_{L^\infty([0, T], L^2(\Omega^2))}
\]

(3.32)

\[
\leq \| z^\epsilon - U^\epsilon \|_{L^\infty([0, T], L^2(\Omega^2))} + \| U^\epsilon - U_h^\epsilon \|_{L^\infty([0, T], L^2(\Omega^2))}.
\]

From (3.30), the first term of (3.2) is bounded by

\[
\| z^\epsilon - U^\epsilon \|_{L^\infty([0, T], L^2(\Omega^2))} \leq \epsilon \| Z(0, 0, \cdot) \|_2.
\]

(3.33)
For the second term, as $U^h_n$ is the approximation of $U^\epsilon(t,x) = U(t,\frac{t}{\epsilon},x)$ where $U$ is the solution to (1.7), then there exists a function $f(h,n)$ satisfying $\lim_{h\to0} f(h,n) = 0$ such that

$$
(3.34) \quad \| U^\epsilon - U^h_n \|_{L^\infty([0,T],L^2(\mathbb{T}^2))} \leq f(h,n)
$$

From (3.33) and (3.34) we get the desired result. □

4. Comparison Numerical Solution of Two-scale limit and reference solution

In this paragraph, we consider the two approximations: $U^h_n$ of the two scale limits $U$ and $z^\epsilon_{h,n}$ of $z^\epsilon(t,x)$. The objective here is to compare, for fixed $\epsilon$ and for a given time, the quantity $z^\epsilon_h(t,x_1,x_2) - U^\epsilon_h(t,\frac{t}{\epsilon},x)$ when the velocity $U$ and $M$ are given.

For the numerical simulations, concerning $z^\epsilon$, we take $z_0(x_1,x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$ and $z_0(x_1,x_2) = Z(0,0,x_1,x_2)$. In what concerns the water velocity field, we consider the function

$$
(4.1) \quad U(t,\theta,x_1,x_2) = \sin 2\pi x_1 \cos 2\pi x_2 \sin 2\pi \theta \mathbf{e}_1,
$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are respectively the first and the second vector of the canonical basis of $\mathbb{R}^2$ and $x_1$, $x_2$ are the first and the second components of $x$.

4.1. Numerical simulation of $U$ and $A$ when $U$ given by (4.1). Let us recall that the water velocity $U$ used in the simulations is given by (4.1). The coefficient $A$ is also given by

$$
(4.2) \quad A(t,\theta,x) = a |U(t,\theta,x)|^3,
$$

where $a$ is a constant.

In Figure 3, the $\theta$-evolution of $\tilde{A}(\theta)$ is also given in various points $(x_1,x_2) \in \mathbb{R}^2$. 
In Figure 1, we can see the space distribution of the first component of the velocity $\mathcal{U}$ for a given time $t = 1$ and for various values of $\theta = \frac{1}{4}, \frac{3}{4}$ and $\frac{1}{6}$.

In Figure 2, we see, for a fixed point $x = (x_1, x_2)$, how the water velocity $\tilde{\mathcal{U}}(\theta)$ evolves with respect to $\theta$. 

**Figure 1.** Space distribution of the first component of $\mathcal{U}(1, 1/4, (x_1, x_2)), \mathcal{U}(1, 3/4, (x_1, x_2))$ and $\mathcal{U}(1, 1/6, (x_1, x_2))$ when $\mathcal{U}$ is given by (4.1). Space distribution of the first component of $\mathcal{U}(1, 1/4, (x_1, x_2)), \mathcal{U}(1, 3/4, (x_1, x_2))$ and $\mathcal{U}(1, 1/6, (x_1, x_2))$ when $\mathcal{U}$ is given by (4.1).

**Figure 2.** $\theta$-evolution of $\tilde{\mathcal{U}}(\theta, (1/2, 1/4))$ and $\tilde{\mathcal{U}}(\theta, (1/4, 1/4))$ when $\mathcal{U}$ is given by (4.1). $\theta$-evolution of $\tilde{\mathcal{U}}(\theta, (1/2, 1/4))$ and $\tilde{\mathcal{U}}(\theta, (1/4, 1/4))$ when $\mathcal{U}$ is given by (4.1)
4.2. Numerical result: Comparisons $z^\epsilon(t, x)$ and $U(t, \frac{1}{\epsilon}, x)$. In this paragraph, we present numerical simulations in order to validate the Two-Scale convergence presented in Theorem 83. For a given $\epsilon$, we compare $U_h^n(t, \frac{1}{\epsilon}, x)$, where $U_h^n$ is the approximation of $U(t, \frac{1}{\epsilon}, x)$, when $U$ is solution to (1.7) and $z_{h,n}$ is the approximation of the solution of $z^\epsilon$ to (1.1). For the initial condition of (1.1) we use $z_0(x) = \sin 2\pi x_1$

Before going further, let us show, what the solution $z^\epsilon$ to (1.1) converges to $U$ solution to (1.7). For this, we compare, for a given time $t_0 = 1$, $z^\epsilon(t_0, x)$ and $U(t_0, \frac{1}{\epsilon}, x)$ for $\epsilon = 0.5$, $\epsilon = 0.1$, $\epsilon = 0.05$, $\epsilon = 0.01$ and $\epsilon = 0.001$. The results is given in figure 4 and figure 5. This figure shows that if $\epsilon$ is too small, the solution $z^\epsilon$ to (1.1) is very close to $U$ solution to (1.1).

We remark that, if $\epsilon$ is too small, for a fixed time $t$, the solution $z^\epsilon$ is close to $U(t, \frac{1}{\epsilon}, x)$.

In an other hand, we will compare the two solutions, when $\epsilon$ is too small and for a given time $t$. The results show that the solution $U(t, \frac{1}{\epsilon}, x)$ is very close to $z^\epsilon(t, x)$. The results are shown in Figures 6, 7, 8 and 9.
In the Figure 10 and Figure 11, we proof also that, the reference solution is very close to his limit. The initial condition is given by $z_0(x_1, x_2) = \cos(2\pi x_1) + \cos(4\pi x_1)$ and $U(t, \theta, x) = \sin(\pi x_1) \sin(2\pi \theta) e_1$.

Besides this, by considering a value of $t$, and by making $\epsilon$ vary, we notice that the errors between $z^\epsilon(t, x)$ and $U(t, \frac{1}{\epsilon}, x)$ decrease as illustrated in the following tabular.
The results given in this table show that, at time $t = 1$, $z^\epsilon(t, x)$ is closer to $Z(t, \tfrac{t}{\epsilon}, x)$ when $\epsilon$ is very small. These results validate the results obtained in Theorem 83.

Figure 8. Comparison 3D of $z_h^\epsilon(t, x_1, x_2)$ and $U_h(t, \frac{t}{\epsilon}, x_1, x_2)$. On the left $z_h^\epsilon(t, x)$, on the right $U_h(t, \frac{t}{\epsilon}, x_1, x_2) \epsilon = 0.001, t = 10^{-2}, \epsilon = 0.01$. Comparison 3D of $z_h^\epsilon(t, x_1, x_2)$ and $U_h(t, \frac{t}{\epsilon}, x_1, x_2)$. On the left $z_h^\epsilon(t, x)$, on the right $U_h(t, \frac{t}{\epsilon}, x_1, x_2) \epsilon = 0.001, t = 10^{-2}, \epsilon = 0.01$.

Figure 9. Comparison 2D of $z_h^\epsilon(t, x_1, 0)$ and $U_h(t, \frac{t}{\epsilon}, x_1, 0)$. On the left $z_h^\epsilon(t, x)$, on the right $U_h(t, \frac{t}{\epsilon}, x_1, 0) \epsilon = 0.001, t = 1$. Comparison 2D of $z_h^\epsilon(t, x_1, 0)$ and $U_h(t, \frac{t}{\epsilon}, x_1, 0)$. On the left $z_h^\epsilon(t, x)$, on the right $U_h(t, \frac{t}{\epsilon}, x_1, 0) \epsilon = 0.001, t = 1$. 
Figure 10. Comparison of $z_h(t, x_1, x_2)$ and $U_h(t, \frac{\epsilon}{\tau}, x_1, x_2)$, $\epsilon = 0.001$, $t = 0.2$, $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$. On the left $z^*(t, x_1, x_2)$, on the right $U(t, \frac{\epsilon}{\tau}, x_1, x_2)$. Comparison of $z_h(t, x_1, x_2)$ and $U_h(t, \frac{\epsilon}{\tau}, x_1, x_2)$, $\epsilon = 0.001$, $t = 0.2$, $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$. On the left $z^*(t, x_1, x_2)$, on the right $U(t, \frac{\epsilon}{\tau}, x_1, x_2)$.

Figure 11. Comparison of $z^*(t, x_1, x_2)$ and $U(t, \frac{\epsilon}{\tau}, x_1, x_2)$, $t = 0.4$, $\epsilon = 0.001$, $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$. On the right $U(t, \frac{\epsilon}{\tau}, x_1, x_2)$, on the left $z^*(t, x_1, x_2)$. Comparison of $z^*(t, x_1, x_2)$ and $U(t, \frac{\epsilon}{\tau}, x_1, x_2)$, $t = 0.4$, $\epsilon = 0.001$, $z_0(x_1, x_2) = \cos 2\pi x_1 + \cos 4\pi x_1$. On the right $U(t, \frac{\epsilon}{\tau}, x_1, x_2)$, on the left $z^*(t, x_1, x_2)$. 

<table>
<thead>
<tr>
<th>value of $\epsilon$</th>
<th>norm $L^1$</th>
<th>norm $L^2$</th>
<th>norm $L^\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
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<td>24</td>
<td>39.47</td>
</tr>
<tr>
<td>0.01</td>
<td>0.22</td>
<td>0.30</td>
<td>0.86</td>
</tr>
<tr>
<td>0.001</td>
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<td>$8.93 \times 10^{-12}$</td>
<td>$2.79 \times 10^{-11}$</td>
</tr>
<tr>
<td>0.0001</td>
<td>$5.7 \times 10^{-12}$</td>
<td>$7.93 \times 10^{-12}$</td>
<td>$1.99 \times 10^{-11}$</td>
</tr>
</tbody>
</table>

Errors norm $U_h(t, \epsilon, x_1, x_2) - z(\epsilon, t, x_1, x_2)$, $t = 1$. 


