CHAPTER 27

Numerical analysis of a corrector results in short term sand transport problem, *by B. K. Thiam, M. M. T. Baldé, I. Faye and D. Seck*

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Abstract. (Short Abstract) The chapter deals with numerical simulation for short term dynamical of dunes related to the sand transport problems submitted to the tide near the seabed recently presented in the literature.
(See page 560 for the full abstract).

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Full Abstract. This paper is devoted to numerical simulations of sand transport problems submitted to the tide near the seabed. We characterized at first a corrector result from two-scale convergence. We aim also to do numerical simulation for short term dynamical of dunes presented in Faye *et al.*(2011) and Thiam *et al.*(2018). We consider a two-scale numerical approach based on finite element method. The stability of the scheme is established and finally, we present some numerical results

1. Introduction and results

In this paper we aim to characterized the corrector result associated to two scale parabolic problem studied in Faye *et al.* (2015). In this work, we consider a domain Ω of boundary $\partial\Omega$ of class $\mathcal{C}^1$. For $T > 0$ and $\epsilon > 0$, we consider the following problems:

$$(1.1) \qquad \begin{cases} \frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon}\Big(\mathcal{A}(t, \frac{t}{\epsilon}, x)\nabla z^\epsilon\Big) = \frac{1}{\epsilon}\nabla\cdot\mathcal{C}(t, \frac{t}{\epsilon}, x), \text{ in } [0,T)\times\Omega \\ z^\epsilon(x,0) = z_0(x) \text{ in } \Omega \\ \dfrac{\partial z^\epsilon}{\partial n} = g \text{ on } [0,T)\times\Omega \end{cases}$$

where $z_0 \in H^1(\Omega)$ and $g \in H^{-1}(\Omega)$. The coefficients $\mathcal{A}^\epsilon(t,x)$ and $\mathcal{C}^\epsilon(t,x)$ are regular and given by

$$\mathcal{A}^\epsilon(t,x) = \mathcal{A}(t, \frac{t}{\epsilon}, x) = a(1 - b\epsilon\mathbf{m})g_a(|\mathcal{U}(t, \frac{t}{\epsilon}, x)|) \text{ and}$$

$$(1.2) \qquad \mathcal{C}^\epsilon(t,x) = \mathcal{C}(t, \frac{t}{\epsilon}, x) = c(1 - b\epsilon\mathbf{m})g_c(|\mathcal{U}(t, \frac{t}{\epsilon}, x)|)\frac{\mathcal{U}(t, \frac{t}{\epsilon}, x)}{|\mathcal{U}|}$$

with a, b and c being constants and g_a and g_c being a regular function satisfying the following hypotheses

$$
\begin{cases} g_a \geq g_c \geq 0, \quad , g_c(0) = g_c'(0) = 0, \\ \begin{cases} \exists\, d \geq 0,\ \sup_{u\in\mathbb{R}^+} |g_a(u)| + \sup_{u\in\mathbb{R}^+} |g_a'(u)| \leq d, \\ \sup_{u\in\mathbb{R}^+} |g_c(u)| + \sup_{u\in\mathbb{R}^+} |g_c'(u)| \leq d, \end{cases} \\ \exists\, U_{thr} \geq 0, \quad \exists G_{thr} > 0, \ \text{such that} \ u \geq U_{thr} \Longrightarrow g_a(u) \geq G_{thr}. \end{cases} \tag{1.3}
$$

The functions $\theta \to \mathcal{U}(t,\theta,x)$ and $\mathcal{M}(t,\theta,x)$ are periodic with period 1. Equations (1.1) is a relevant model for short and mean term dynamical of dunes on a seabed of a coastal ocean where the tide is too strong. In theses equations, $z^\epsilon(t,x)$, where x is the position and t the time is the seabed altitude.

The first homogenization methods were set out by Engineers in the middle of the 1970s and then formalize by Mechanical Scientists. The homogenized problem can be formally obtained by the method of two-scale asymptotic expansion as explained in Allaire (1992); Nguetseng (1989). They are based on asymptotic expansion. Following the ideas of the engineers, the solution $z^\epsilon(t,x)$ of (1.1) is assuming to be given by the following series

$$
z^\epsilon(t,x) = Z^0(t,\frac{t}{\epsilon},x) + \epsilon Z^1(t,\frac{t}{\epsilon},x) + \epsilon^2 Z^2(t,\frac{t}{\epsilon},x) + \dots, \tag{1.4}
$$

where the functions $Z^i(t,\theta,x), i = 0,1,2,\dots,$ are periodic of period 1 with respect to θ. The function Z^0 is the homogenized profile while Z^1 is the first corrector and $Z^i, i = 2,\dots$ is the i-th profile corrector. Taking $\mathcal{A}^\epsilon$ and $\mathcal{C}^\epsilon$ is in the form of z^ϵ given in (1.4) we have

$$
\mathcal{A}^\epsilon(t,x) = \mathcal{A}^0(t,\frac{t}{\epsilon},x) + \epsilon\mathcal{A}^1(t,\frac{t}{\epsilon},x) + \epsilon^2\mathcal{A}^2(t,\frac{t}{\epsilon},x) + \dots, \tag{1.5}
$$

and

$$
\mathcal{C}^\epsilon(t,x) = \mathcal{C}^0(t,\frac{t}{\epsilon},x) + \epsilon\mathcal{C}^1(t,\frac{t}{\epsilon},x) + \epsilon^2\mathcal{C}^2(t,\frac{t}{\epsilon},x) + \dots, \tag{1.6}
$$

By injecting formulas (1.4), (1.5) and (1.6) in (1.1), identifying and taking into account the first corrector, we get easily a system of equations from Z^0, Z^1 .... The first and the second term corrector satisfy the following equations:

$$\text{(1.7)} \qquad \begin{cases} \dfrac{\partial Z^0}{\partial \theta} - \nabla\Big(\widetilde{\mathcal{A}}\nabla Z^0\Big) = \nabla \cdot \widetilde{\mathcal{C}} \text{ in } [0,T) \times \mathbb{R} \times \Omega, \\ \frac{\partial Z^0}{\partial n} = g \text{ on } [0,T) \times \mathbb{R} \times \partial\Omega, \\ Z^0(0,0,x) = z_0(x) \text{ in } \Omega, \end{cases}$$

where $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{C}}$ are given by

$$\text{(1.8)} \qquad \widetilde{\mathcal{A}} = ag_(|\mathcal{U}(t,\theta,x)|),\ \widetilde{\mathcal{C}} = cg_c(|\mathcal{U}(t,\theta,x)|)\frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}$$

and

$$\text{(1.9)} \qquad \begin{cases} \frac{\partial Z^1}{\partial \theta} - \nabla \cdot (\widetilde{\mathcal{A}}\nabla Z^1) = \nabla \cdot \widetilde{\mathcal{C}}_1 + \frac{\partial Z}{\partial t} + \nabla \cdot (\widetilde{\mathcal{A}}_1 \nabla Z^0) \text{ in } [0,T) \times \mathbb{R} \times \Omega, \\ \frac{\partial Z^1}{\partial n} = 0 \text{ on } [0,T) \times \mathbb{R} \times \partial\Omega, \\ Z^1(0,0,x) = 0 \text{ in } \Omega. \end{cases}$$

where

$$\widetilde{\mathcal{A}}_1 = -ab\mathcal{M}(t,\theta,x)g_a(|\mathcal{U}(t,\theta,x)|),$$

$$\text{(1.10)} \qquad \widetilde{\mathcal{C}}_1 = -bc\mathcal{M}(t,\theta,x)g_c(|\mathcal{U}(t,\theta,x)|)\frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}.$$

In equations (1.7) and (3.3), the variable t is only a parameter. The solution z^ϵ solution to (1.1) can be approximated by $Z^0,\ Z^0 + \epsilon Z^1, \ldots$.
We now give our first mathematical result concerning the first corrector.

THEOREM 77. *Under assumptions (1.4), if the coefficient $\mathcal{A}^\epsilon$ and $\mathcal{C}^\epsilon$ are given by (1.5) and (1.6), the first term of the expansion of z^ϵ solution to (1.1) two-scale converges to Z^0 solution to the homogenized problem (1.7), with the homogenized coefficient $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{C}}$ given by (1.8).*

Furthermore the first corrector term Z^1 is solution to (1.9) with $\widetilde{\mathcal{A}}_1$ and $\widetilde{\mathcal{C}}_1$ given by (1).

In section 2, in an other hand, we consider the equation given by the first term corrector (1.9). We are going to introduce a two scale numerical method to study equation (1.9). The convergence of this method, will allow us to get a good approximation of z^ϵ solution to the reference problem (1.1) in the sense that $z^\epsilon \sim Z^0 + \epsilon Z^1$ for a too small choice of ϵ. In Diedhiou *et al.* (2018); Faye *et al.* (2015), the authors used a two scale numerical method to study the equation satisfied by Z^0 by a Fourier and finite element method. This methods will permit us to get a very good approximation of z^ϵ.

Denoting by $\Omega_h \subset \Omega$ the union of triangulation T_h and h the maximal length of the sight, let's define the following matrix: A a matrix of $N \times N$ where the coefficients are given by

$$A_{ij} = \int_{\Omega_h} \widetilde{\mathcal{A}} \nabla \omega^i \cdot \nabla \omega^j dx,$$

A' a matrix of size $N \times N$ where the coefficients are given by

$$A'_{ij} = \int_{\Omega_h} \widetilde{\mathcal{A}}_1 \nabla \omega^i \cdot \nabla \omega^j dx,$$

M a matrix of size $N \times N$ where the coefficients are given by

$$M_{ij} = \int_{\Omega_h} \omega^i \omega^j dx$$

and C is a vector given by

$$C_j = \int_{\Omega_h} \nabla \cdot \widetilde{\mathcal{C}}_1 \omega^j dx + \int_{\partial\Omega_h} \widetilde{\mathcal{A}}_1 g \omega^j d\sigma.$$

where $w_j, j = 1, \ldots, N$ is the standard basis of V_h the set all continuous piecewise linear functions defined on T_h.We have the following theorem of stability

THEOREM 78. *Let I be the identity Matrix and $\left\|\left(I + \Delta\theta M^{-1} A\right)^{-1}\right\|$ be the spectral norm of the matrix $\left(I + \Delta\theta M^{-1} A\right)^{-1}$. Then, $\forall \Delta\theta > 0$ and $h > 0$, if*

$$\left\|\left(I + \Delta\theta M^{-1} A\right)^{-1}\right\| \leq 1,$$

we have the stability of the scheme. In addition, we get

$$\left\|Z^1_{h,n+1}\right\|_{L^2(\Omega_h)} \leq \left\|\left(I + \Delta\theta M^{-1} A\right)^{-1}\right\|^n \left\|Z^1_0\right\|_{L^2(\Omega_h)} +$$

$$\Delta\theta\Big\|M^{-1}\Big\|\sum_{k=1}^{n}\Big\|\Big(I+\Delta\theta M^{-1}A\Big)^{-1}\Big\|^{k}\Big(\sup_{0\leq n\leq N}\Big\|B\Big\|^{n}\Big)$$

The paper is organized as follows. In section 2, we present the mathematical convergence and homogenization. The section 3 is devoted to the presentation of two-scale numerical method and the last section is devoted to the numerical illustration.

2. Mathematical convergence and Homogenization

The mathematically rigorous justification of theorem 77 concerning the two first terms Z^0 and Z^1 given in the expansion (1.4) has been done in Faye *et al.* (2015) and Thiam *et al.* (2018) by using the two-scale convergence method see Allaire (1992) and Nguetseng (1989). We shall not reproduce this argument and we content ourselves in recalling their main theorem.

THEOREM 79. *For any $T > 0$, and any $\epsilon > 0$, under assumptions (1.3), if $\mathcal{A}^\epsilon$ and $\mathcal{C}^\epsilon$ is given by (1.5) and (1.6), there exists a unique sequence $z^\epsilon \in L^\infty([0,T), L^2(\Omega))$ solution to (1.1). Moreover, this solution satisfies the following estimate.*

$$\|z^\epsilon\|_{L^\infty([0,T),L^1(\Omega))} \leq \gamma \tag{2.1}$$

for a constant γ not depending on ϵ.

Furthermore, the sequence z^ϵ two-scale converges to $Z^0 \in L^\infty([0,T], L^\infty(\mathbb{R}, L^2(\Omega)))$ which is the unique solution to the following boundary value problem

$$\begin{cases} \dfrac{\partial Z^0}{\partial\theta} - \nabla(\widetilde{\mathcal{A}}\nabla Z^0) = \nabla\cdot\widetilde{\mathcal{C}} \text{ in } [0,T)\times\mathbb{R}\times\Omega, \\ \qquad\qquad \dfrac{\partial Z^0}{\partial n} = g \text{ on } [0,T)\times\mathbb{R}\times\partial\Omega \end{cases} \tag{2.2}$$

Having this theorem on hand, we are going to characterize the first order corrector Z^1 which prove the best approximation by homogenization or two-scale convergence. To obtain this approximation, we look for the equation satisfied by $z^\epsilon(t,x) - Z^0(t,\frac{t}{\epsilon},x)$. After, we look for uniform estimates of them. Then, we have the following theorem.

THEOREM 80. *Under assumptions (1.5) and (1.6), considering z^ϵ the solution to (1.1) with coefficients given by (1.5) and (1.6) and $Z^\epsilon = Z^\epsilon(t,x) =$*

$Z^0(t,\frac{t}{\epsilon},x)$ where Z^0 is the solution to (1.7), for any T not depending on ϵ, the following estimate holds for $z^\epsilon - Z^\epsilon$

$$\Big\|\frac{z^\epsilon - Z^\epsilon}{\epsilon}\Big\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \alpha, \tag{2.3}$$

where α is a constant not depending on ϵ.

Furthermore, sequence $\left(\frac{z^\epsilon - Z^\epsilon}{\epsilon}\right)$ two-scale converges to a profile $Z^1 \in L^\infty([0,T], L^\infty_\#(\mathbb{R}, L^2(\mathbb{R}^2)))$ which is the unique solution to

$$\begin{cases} \frac{\partial Z^1}{\partial \theta} - \nabla\cdot\left(\widetilde{\mathcal{A}}\nabla Z^1\right) = \nabla\cdot\widetilde{\mathcal{C}}_1 + \frac{\partial Z^0}{\partial t} + \nabla\cdot(\widetilde{\mathcal{A}}_1\nabla Z^0) \text{ in } [0,T)\times\mathbb{R}\times\Omega \\ \dfrac{\partial Z^1}{\partial n} = 0 \text{ on } [0,T)\times\mathbb{R}\times\partial\Omega. \end{cases} \tag{2.4}$$

Proof of Theorem 80. By using (1.5) and (1.6), Equations (1.1) becomes

$$\frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot(\widetilde{\mathcal{A}}^\epsilon\nabla z^\epsilon) = \frac{1}{\epsilon}\nabla\cdot\widetilde{\mathcal{C}}^\epsilon + \nabla\cdot(\widetilde{\mathcal{A}}_1^\epsilon\nabla z^\epsilon) + \nabla\cdot\widetilde{\mathcal{C}}_1^\epsilon. \tag{2.5}$$

For Z^ϵ, we have

$$\frac{\partial Z^\epsilon}{\partial t} = \left(\frac{\partial Z^0}{\partial t}\right)^\epsilon + \frac{1}{\epsilon}\left(\frac{\partial Z^0}{\partial \theta}\right)^\epsilon, \tag{2.6}$$

where

$$\left(\frac{\partial Z^0}{\partial t}\right)^\epsilon(t,x) = \frac{\partial Z^0}{\partial t}(t,\frac{t}{\epsilon},x) \text{ and } \left(\frac{\partial Z^0}{\partial \theta}\right)^\epsilon(t,x) = \frac{\partial Z^0}{\partial \theta}(t,\frac{t}{\epsilon},x). \tag{2.7}$$

By using (1.7), we see that Z^ϵ is solution to

$$\frac{\partial Z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\left(\widetilde{\mathcal{A}}^\epsilon\nabla Z^\epsilon\right) = \frac{1}{\epsilon}\nabla\cdot\widetilde{\mathcal{C}}^\epsilon + \left(\frac{\partial Z^0}{\partial t}\right)^\epsilon. \tag{2.8}$$

Formulas (2.5) and (2.8) give

$$\frac{\partial (z^\epsilon - Z^\epsilon)}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\left(\widetilde{\mathcal{A}}^\epsilon\nabla(z^\epsilon - Z^\epsilon)\right) = \nabla\cdot\widetilde{\mathcal{C}}_1^\epsilon + \left(\frac{\partial Z^0}{\partial t}\right)^\epsilon + \nabla\cdot(\widetilde{\mathcal{A}}_1^\epsilon\nabla z^\epsilon). \tag{2.9}$$

Multiplying equation (2.9) by $\frac{1}{\epsilon}$ and using the fact that $z^\epsilon = z^\epsilon - Z^\epsilon + Z^\epsilon$ in the right hand side of equation (2.9), $\frac{z^\epsilon - Z^\epsilon}{\epsilon}$ is solution to:

$$\frac{\partial\left(\dfrac{z^\epsilon - Z^\epsilon}{\epsilon}\right)}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\left((\widetilde{\mathcal{A}}^\epsilon + \epsilon\widetilde{\mathcal{A}}_1^\epsilon)\nabla(\frac{z^\epsilon - Z^\epsilon}{\epsilon})\right) = \frac{1}{\epsilon}\left(\nabla\cdot\widetilde{\mathcal{C}}_1^\epsilon + (\frac{\partial Z^0}{\partial t})^\epsilon + \nabla\cdot(\widetilde{\mathcal{A}}_1^\epsilon\nabla Z^\epsilon)\right). \tag{2.10}$$

REMARK 30. *Concerning notations, we have to pay attention to the fact that*

$$\widetilde{\mathcal{A}}^\epsilon \neq \widetilde{\mathcal{A}}_\epsilon \text{ and } \widetilde{\mathcal{C}}^\epsilon \neq \widetilde{\mathcal{C}}_\epsilon. \tag{2.11}$$

Here, our aim is to prove that $\frac{z^\epsilon - Z^\epsilon}{\epsilon}$ is bounded by a constant α not depending on ϵ. For this let us use that $\widetilde{\mathcal{A}}^\epsilon$, $\widetilde{\mathcal{A}}_1^\epsilon$, $\widetilde{\mathcal{C}}^\epsilon$ and $\widetilde{\mathcal{C}}_1^\epsilon$ are regular and bounded coefficients and that $\widetilde{\mathcal{A}}^\epsilon \geq G_{thr}$. Thus, $\nabla\cdot\widetilde{\mathcal{C}}_1^\epsilon$ is bounded, $\nabla\cdot(\widetilde{\mathcal{A}}_1^\epsilon\nabla Z^\epsilon)$ is also bounded. Since Z^ϵ is solution to (2.8), $\frac{\partial Z^0}{\partial t}$ satisfies the following equation

$$\frac{\partial\left(\dfrac{\partial Z^0}{\partial t}\right)}{\partial\theta} - \nabla\cdot\left(\widetilde{\mathcal{A}}\nabla\frac{\partial Z^0}{\partial t}\right) = \frac{\partial\nabla\cdot\widetilde{\mathcal{C}}}{\partial t} + \nabla\cdot\left(\frac{\partial\widetilde{\mathcal{A}}}{\partial t}\nabla Z^0\right). \tag{2.12}$$

Equation (2.12) is linear with regular and bounded coefficients. Then using a result of Ladyzenskaja *et al.* (1968), $\frac{\partial Z^0}{\partial t}$ is regular and bounded . Then the coefficients of equations (2.10) are regular and bounded.Then, using the same arguments as in the proof of theorem 3.16 in Faye *et al.* (2015), we obtain that $\left(\frac{z^\epsilon - Z^\epsilon}{\epsilon}\right)$ is bounded, that it two-scale converges to a profile $Z^1 \in L^\infty([0,T], L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$ and that this profile Z^1 satisfies equation (2.4). $\square$

3. Two scale Numerical Method s

The aim of this section is to built a two-scale numerical method based on finite element to resolve the equation satisfied by the first corrector of the sand problem. Because of the following, we will consider in this section, the following problem.

$$(3.1)\qquad \begin{cases} \frac{\partial Z^1}{\partial \theta} - \nabla\cdot(\widetilde{\mathcal{A}}\nabla Z^1) = \nabla\cdot\widetilde{\mathcal{C}}_1 + \frac{\partial Z}{\partial t} + \nabla\cdot(\widetilde{\mathcal{A}}_1\nabla Z) \text{ in } [0,T)\times\mathbb{R}\times\Omega, \\ \frac{\partial Z^1}{\partial n} = 0 \text{ on } [0,T)\times\mathbb{R}\times\partial\Omega, \\ Z^1(0,0,x) = Z_0^1(x) \text{ in } \Omega. \end{cases}$$

$$\widetilde{\mathcal{A}}_1 = -ab\mathcal{M}(t,\theta,x)g_a(|\mathcal{U}(t,\theta,x)|),$$

$$\widetilde{\mathcal{C}}_1 = -bc\mathcal{M}(t,\theta,x)g_c(|\mathcal{U}(t,\theta,x)|)\frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}$$

and Z is the solution to

$$(3.2)\qquad \begin{cases} \frac{\partial Z}{\partial \theta} - \nabla\cdot\left(\widetilde{\mathcal{A}}\nabla Z\right) = \nabla\cdot\widetilde{\mathcal{C}} \text{ in } [0,T)\times\mathbb{R}\times\Omega \\ \frac{\partial Z}{\partial n} = g \text{ on } [0,T)\times\mathbb{R}\times\partial\Omega \end{cases}$$

with

$$\widetilde{\mathcal{A}}(t,\theta,x) = ag_a(|\mathcal{U}(t,\theta,x)|)$$

and

$$\widetilde{\mathcal{C}}(t,\theta,x) = cg_c(|\mathcal{U}(t,\theta,x)|)\times\frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}.$$

In the following, we will discretize Z^1 in time at first and space in second. We will obtain a numerical scheme of the system (3.1) based on the backward of Euler method.

3.1. Discretization in time. For the Discretization in time, let $\Delta\theta$ be a time step and we use the notation $\theta_n = n\Delta\theta$. The total partial derivatives are discretized according to the method of finite differences:

$$\frac{\partial Z^1}{\partial \theta}(t,\theta_n,x) \sim \frac{Z^1(t,\theta_{n+1},x) - Z^1(t,\theta_n,x)}{\Delta\theta}.$$

Let us denote by Z_n^1 the approximation of $Z^1(\cdot,\theta_n,\cdot)$, then we can approximate $\frac{\partial Z^1}{\partial\theta}(t,\theta_n,x)$ by the form

$$\frac{\partial Z^1}{\partial \theta}(t,\theta_n,x) = \frac{Z_{n+1}^1 - Z_n^1}{\Delta\theta},$$

Hence, using an implicit Euler scheme, (3.1) becomes

$$(3.3)\qquad \begin{cases} \frac{Z^1_{n+1}-Z^1_n}{\Delta\theta} - \nabla\cdot(\widetilde{\mathcal{A}}\nabla Z^1_{n+1}) = \nabla\cdot\widetilde{\mathcal{C}}_1 + \frac{\partial Z_{n+1}}{\partial t} + \nabla\cdot(\widetilde{\mathcal{A}}_1\nabla Z_{n+1}) \\ \text{in } [0,T)\times\mathbb{R}\times\Omega, \\ \frac{\partial Z^1_{n+1}}{\partial n} = 0 \text{ on } [0,T)\times\mathbb{R}\times\partial\Omega, \\ Z^1(0,0,x) = Z^1_0(x) \text{ in } \Omega. \end{cases}$$

We have to pay attention that Z, solution to (1.9), appear in the second term of the first equation of (3.1). Then for the approximation of Z, we will use the same time and space discretization as for Z^1.

3.2. Full Discretization. To complete the Discretization, we use the finite element formalism for spacial discretization. Let us denote

$$V_0 = H^1(\Omega).$$

Then, multiplying (3.1) by $v \in V_0$ and integrating over Ω, the variational formulation reads: find $Z^1_n \in V_0$, $\forall v \in V_0$,

$$\int_\Omega \frac{Z^1_{n+1}-Z^1_n}{\Delta\theta} v dx + \int_\Omega \widetilde{\mathcal{A}}\nabla Z^1_{n+1}\cdot\nabla v dx = \int_\Omega \nabla\cdot\widetilde{\mathcal{C}}_1 v dx$$

$$(3.4)\qquad -\int_\Omega \frac{\partial Z_{n+1}}{\partial t} v dx - \int_\Omega \widetilde{\mathcal{A}}_1\nabla Z_{n+1}\cdot\nabla v dx + \int_{\partial\Omega}\widetilde{\mathcal{A}}_1 g v d\sigma.$$

Let $\{T_h, h\to 0\}$ be a quasi-uniform family of admissible triangulation of Ω. We denote by $\Omega_h \subset \Omega$, the union of triangles of T_h, and h the maximal length of the sides of the triangulation T_h. And let $V_h \subset V$ be the set of all continuous piecewise linear functions defined on T_h. Let $\{w_i\}_{j=1}^N$ be the standard basis of V_h. Then, using conformal finite element with a finite element discrete space $V_h \subset V_0$, the discrete variational problem is to find $Z^1_{h,n+1} \in V_h$ such that $\forall v_h \in V_h$:

$$\int_{\Omega_h} \frac{Z^1_{h,n+1}-Z^1_{h,n}}{\Delta\theta} v_h dx + \int_{\Omega_h} \widetilde{\mathcal{A}}\nabla Z^1_{h,n+1}\cdot\nabla v_h dx = \int_{\Omega_h}\nabla\cdot\widetilde{\mathcal{C}}_1 v_h dx -$$

$$\int_{\Omega_h} \frac{\partial Z_{h,n+1}}{\partial t} v_h dx - \int_{\Omega_h} \widetilde{\mathcal{A}}_1 \nabla Z_{h,n+1} \cdot \nabla v_h dx + \int_{\partial\Omega_h} \widetilde{\mathcal{A}}_1 g v_h d\sigma. \tag{3.5}$$

For any $Z^1_{h,n+1} \in V_h$, then there exists $(Z^1_1, Z^1_2, \cdots, Z^1_N)$ such that

$$Z^1_{h,n}(\cdot, \theta_n, \cdot) = \sum_{i=1}^{N} Z^1_{i,n} \omega^i(x) \tag{3.6}$$

In the same way, there exists $(Z_{1,n}, Z_{2,n}, \cdots, Z_{N,n})$ such that

$$Z_{h,n}(t, \cdot, \cdot) = \sum_{i=1}^{N} Z_{i,n} \omega^i(x) \tag{3.7}$$

Taking $v_h = \omega^j$, $j = 1, 2, \cdots, N$, we get from (3.2) that

$$\begin{aligned} &\int_{\Omega_h} \frac{Z^1_{h,n+1} - Z^1_{h,n}}{\Delta\theta} \omega^j dx + \int_{\Omega_h} \widetilde{\mathcal{A}} \nabla Z^1_{h,n+1} \cdot \nabla \omega^j dx = \int_{\Omega_h} \nabla \cdot \widetilde{\mathcal{C}}_1 \omega^j dx - \\ &\int_{\Omega_h} \frac{\partial Z_{h,n}}{\partial t} \omega^j dx - \int_{\Omega_h} \widetilde{\mathcal{A}}_1 \nabla Z_{h,n} \cdot \nabla \omega^j dx + \int_{\partial\Omega_h} \widetilde{\mathcal{A}}_1 g \omega^j d\sigma. \end{aligned} \tag{3.8}$$

From (3.6), we have: $\forall\, 1 \leq j \leq N$

$$\begin{aligned} &\sum_{i=1}^{N} \frac{Z^1_{i,n+1} - Z^1_{i,n}}{\Delta\theta} \int_{\Omega_h} \omega^i \omega^j dx + \sum_{i=1}^{N} Z^1_{i,n+1} \int_{\Omega_h} \widetilde{\mathcal{A}} \nabla \omega^i \cdot \nabla \omega^j dx = \int_{\Omega_h} \nabla \cdot \widetilde{\mathcal{C}}_1 \omega^j dx - \\ &\sum_{i=1}^{N} \frac{dZ_{i,n}}{dt} \int_{\Omega_h} \omega^i \omega^j dx - \sum_{i=1}^{N} Z_{i,n} \int_{\Omega_h} \widetilde{\mathcal{A}}_1 \nabla \omega^i \cdot \nabla \omega^j dx + \int_{\partial\Omega_h} \widetilde{\mathcal{A}}_1 g \omega^j d\sigma. \end{aligned} \tag{3.9}$$

Then, we get the following equation

$$\sum_{i=1}^{N} \Big(\frac{1}{\Delta\theta} \int_{\Omega_h} \omega^i \omega^j dx + \int_{\Omega_h} \widetilde{\mathcal{A}} \nabla \omega^i \cdot \nabla \omega^j dx \Big) Z^1_{i,n+1} = \sum_{i=1}^{N} \Big(\frac{1}{\Delta\theta} \int_{\Omega_h} \omega^i \omega^j dx \Big) Z^1_{i,n} \tag{3.10}$$

(3.11)
$$-\sum_{i=1}^{N}\frac{dZ_{i,n}}{dt}\int_{\Omega_h}\omega^i\omega^j dx-\sum_{i=1}^{N}Z_{i,n}\int_{\Omega_h}\widetilde{\mathcal{A}}_1\nabla\omega^i\cdot\nabla\omega^j dx\Big)+\int_{\Omega_h}\nabla\cdot\widetilde{\mathcal{C}}_1\omega^j dx+\int_{\partial\Omega_h}\widetilde{\mathcal{A}}_1 g\omega^j d\sigma.$$

which can be written as follows

$$\Big(\frac{1}{\Delta\theta}M+A\Big)Z^1_{h,n+1}=\Big(\frac{1}{\Delta\theta}M\Big)Z^1_{h,n}+\Big(M\frac{d}{dt}Z-A'Z\Big)+C \tag{3.12}$$

or

$$\Big(M+\Delta\theta A\Big)Z^1_{h,n+1}=MZ^1_{h,n}+\Delta\theta B, \tag{3.13}$$

where the vector $B=-\Big(M\frac{d}{dt}Z_{h,n}+A'Z_{h,n}\Big)+C$ and $Z^1_{h,n}=\Big(Z^1_{1,n},Z^1_{2,n},\cdots,Z^1_{N,n}\Big)$ is the unknown vector, the vector $Z_{h,n}=\Big(Z_{1,n},Z_{2,n},\cdots,Z_{N,n}\Big)$ is given, A a matrix of $N\times N$ where the coefficients are given by

$$A_{ij}=\int_{\Omega_h}\widetilde{\mathcal{A}}\nabla\omega^i\cdot\nabla\omega^j dx,$$

A' a matrix of size $N\times N$ where the coefficients are given by

$$A'_{ij}=\int_{\Omega_h}\widetilde{\mathcal{A}}_1\nabla\omega^i\cdot\nabla\omega^j dx,$$

M a matrix of size $N\times N$ where the coefficients are given by

$$M_{ij}=\int_{\Omega_h}\omega^i\omega^j dx$$

and C is a vector given by

$$C_j=\int_{\Omega_h}\nabla\cdot\widetilde{\mathcal{C}}_1\omega^j dx+\int_{\partial\Omega_h}\widetilde{\mathcal{A}}_1 g\omega^j d\sigma.$$

We have the following theorem of convergence

THEOREM 81. *Let h be the big diameter of the mesh of Ω, Z^1 be the solution to (3.1) and $Z^1_{h,n}=Z^1(\cdot,\theta_n,x_h)\in V_h$ the approximation function of Z^1. Then, the following estimate holds*

$$\|Z^1-Z^1_{h,n}\|_{H^1(\Omega)}\leq C_0 h\|Z^1\|_{H^1(\Omega)}.$$

We have also the following stability result.

THEOREM 82. *Let* $\left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|$ *be the spectral norm of the matrix* $\left(I+\Delta\theta M^{-1}A\right)^{-1}$. *Then,* $\forall\Delta\theta>0$ *and* $h>0$, *if*

$$\left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|\leq 1.$$

we have the stability of the scheme. In addition, we get

$$\begin{aligned}\left\|Z^1_{h,n+1}\right\|_{L^2(\Omega_h)}\leq\left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|^n\left\|Z^1_0\right\|_{L^2(\Omega_h)}+\\ \Delta\theta\left\|M^{-1}\right\|\sum_{k=1}^{n}\left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|^k\left(\sup_{0\leq n\leq N}\left\|B\right\|^n\right)\end{aligned}$$

Proof. We get from (3.13)

$$\left(M+\Delta\theta A\right)Z^1_{h,n+1}=MZ^1_{h,n}+\Delta\theta B_{n+1}\tag{3.14}$$

As the matrix $M+\Delta\theta A$ is invertible, we have

$$Z^1_{h,n+1}=\left(M+\Delta\theta A\right)^{-1}MZ^1_{h,n}+\left(M+\Delta\theta A\right)^{-1}\Delta\theta B_{n+1}\tag{3.15}$$

Thus, by varying n, the following equalities holds

$$Z^1_{h,n}=\left(M+\Delta\theta A\right)^{-1}MZ^1_{h,n-1}+\left(M+\Delta\theta A\right)^{-1}\Delta\theta B_{n}\tag{3.16}$$

$$Z^1_{h,n-1}=\left(M+\Delta\theta A\right)^{-1}MZ^1_{n-2}+\left(M+\Delta\theta A\right)^{-1}\Delta\theta B_{n-1}\tag{3.17}$$

$$Z^1_{h,n-2}=\left(M+\Delta\theta A\right)^{-1}MZ^1_{h,n-3}+\left(M+\Delta\theta A\right)^{-1}\Delta\theta B_{n-2}\tag{3.18}$$

$$\cdots$$
$$\cdots$$
$$\cdots$$

$$Z^1_{h,1}=\left(M+\Delta\theta A\right)^{-1}MZ^1_{0}+\left(M+\Delta\theta A\right)^{-1}\Delta\theta B_{0}\tag{3.19}$$

This makes possible to obtain the following generic for Z^1_n

$$Z^1_{h,n} = \left[\left(I+\Delta\theta M^{-1}A\right)^{-1}\right]^n Z^1_0 + \Delta\theta M^{-1}\sum_{k=1}^{n}\left[\left(I+\Delta\theta M^{-1}A\right)^{-1}\right]^k B_{n-k+1}$$

Taking the norm of Z^1_n, we get

$$\begin{aligned}\left\|Z^1_{h,n}\right\|_{L^2(\Omega_h)} \leq \left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|^n\left\|Z^1_0\right\|_{L^2(\Omega_h)}+\\ \Delta\theta\left\|M^{-1}\right\|\sum_{k=1}^{n}\left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|^k\left\|\left\|B_{n-k+1}\right\|\right\|.\end{aligned}$$

Then, we have

$$\begin{aligned}\left\|Z^1_{h,n}\right\|_{L^2(\Omega_h)} \leq \left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|^n\left\|Z^1_0\right\|_{L^2(\Omega_h)}+\\ \Delta\theta\left\|M^{-1}\right\|\sum_{k=1}^{n}\left\|\left(I+\Delta\theta M^{-1}A\right)^{-1}\right\|^k\left\|\sup_{0\leq n\leq N}\left\|B\right\|^n.\end{aligned}$$

This is the desired result. □

4. Numerical illustration

For the numerical simulations, we consider Z^1 solution to (1.9). In what concerns the water velocity, we take first

$$\mathcal{U}(t,\theta,x_1,x_2) = \sin(\pi x_1)\sin(2\pi\theta)e_1, \tag{4.1}$$

where e_1 is the first vector of the canonical basis of $\mathbb{R}^2$, x_1, x_2 are the first and the second components of x. Functions g_a and g_c are given by $g_a(|u|) = g_c(|u|) = |u|^3$, constants $a = c = 1$, $b = -1$. For the sake of simplicity, we take Z solution to (3.2) is used such that $\frac{\partial Z}{\partial t} = 0$. The first height variation $\mathcal{M}(t, \frac{t}{\epsilon}, x_1, x_2)$ is given by $\mathcal{M}(t, \frac{t}{\epsilon}, x_1, x_2) = 0$. The numerical algorithm is therefore given in the following

(1) T fixed, we choose a value of t, ϵ and the domain Ω.
(2) We solve the numerical scheme of Z^0 and we find Z^0
(3) We introduce Z^0 in the scheme of Z^1 and we find Z^1
(4) We calculate $z^\epsilon \sim Z^0 + \epsilon Z^1$

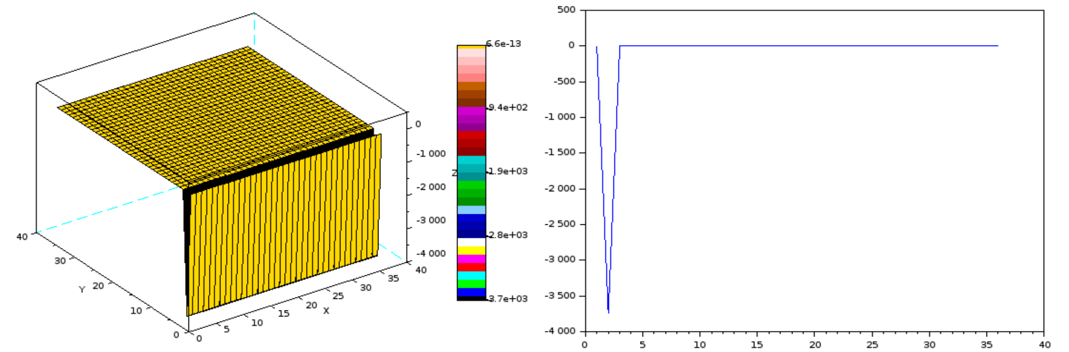

FIGURE 1.DISTRIBUTION OF THE FIRST TERM CORRECTOR $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D IN THE LEFT AND 2D IN THE RIGHT FOR $t = 1$, $\epsilon = 0.01$.. Distribution of the first term corrector $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D in the left and 2D in the right for $t = 1$, $\epsilon = 0.01$.

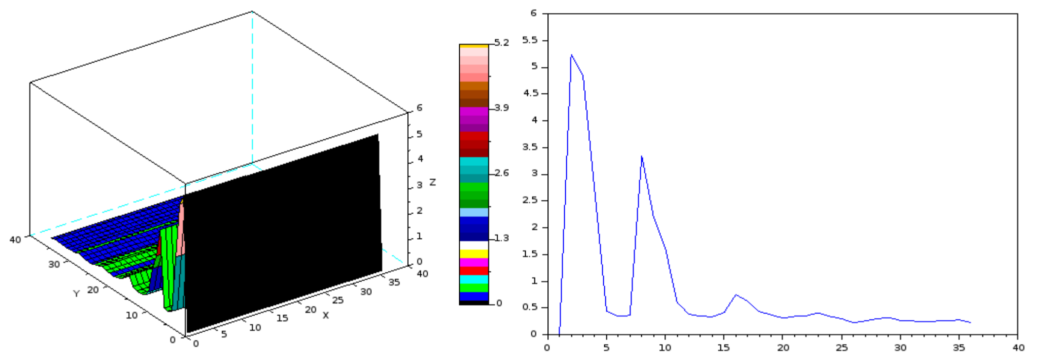

FIGURE 2.DISTRIBUTION OF THE FIRST TERM CORRECTOR $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D IN THE LEFT AND 2D IN THE RIGHT FOR $t = \frac{1}{3}$, $\epsilon = 0.01$.. Distribution of the first term corrector $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D in the left and 2D in the right for $t = \frac{1}{3}$, $\epsilon = 0.01$.

For a fixed time t, we plot $Z^1(t, \frac{t}{\epsilon}, x_1, x_2)$. The results are shown in the following figures.

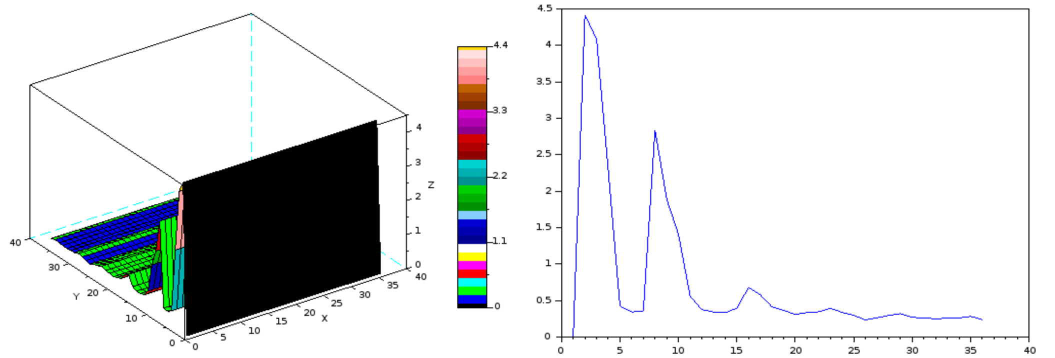

FIGURE 3.DISTRIBUTION OF THE FIRST TERM CORRECTOR $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D IN THE LEFT AND 2D IN THE RIGHT FOR $t = \frac{1}{4}$, $\epsilon = 0.01$.. Distribution of the first term corrector $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D in the left and 2D in the right for $t = \frac{1}{4}$, $\epsilon = 0.01$.

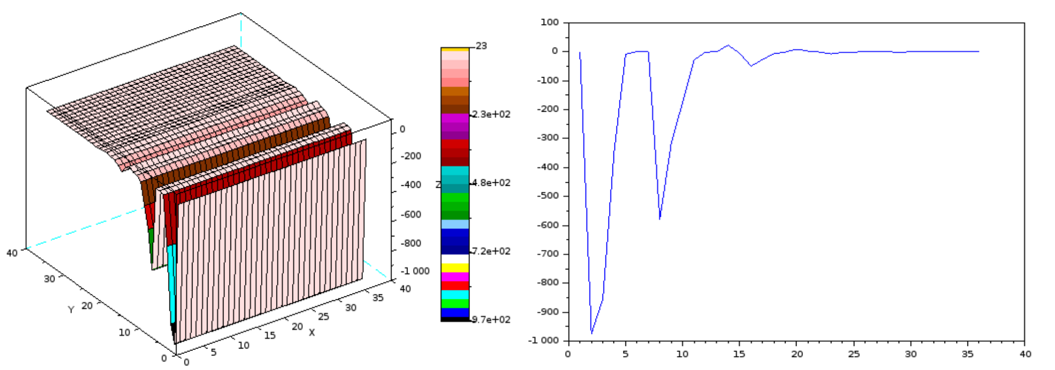

FIGURE 4.DISTRIBUTION OF THE FIRST TERM CORRECTOR $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D IN THE LEFT AND 2D IN THE RIGHT FOR $t = \frac{1}{8}$, $\epsilon = 0.01$.. Distribution of the first term corrector $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D in the left and 2D in the right for $t = \frac{1}{8}$, $\epsilon = 0.01$.

The knowledge of Z^1 makes it possible to have a better approximation of z^ϵ solution of the reference problem 1.1. The result of two-scales convergence allows to have an approximation of the type $z^\epsilon \sim Z^0 + \epsilon Z^1$. Thus knowing numerically the solution Z^0 of the problem limit (1.7), one obtains, for $\epsilon = 0.01$ and at different values of t, the following approximations of the reference problem solution. Then, we get the following results.

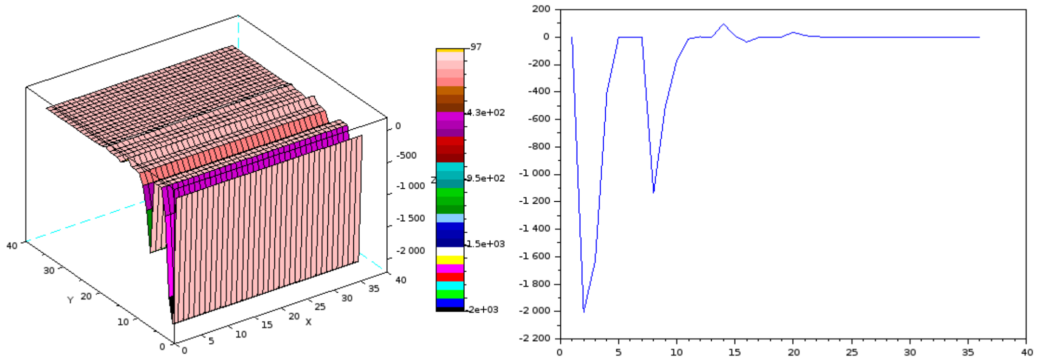

FIGURE 5.DISTRIBUTION OF THE FIRST TERM CORRECTOR $Z^1(t,\frac{t}{\epsilon},\cdot,\cdot)$ 3D IN THE LEFT AND 2D IN THE RIGHT FOR $t=\frac{1}{10}$, $\epsilon=0.01$.. Distribution of the first term corrector $Z^1(t,\frac{t}{\epsilon},\cdot,\cdot)$ 3D in the left and 2D in the right for $t=\frac{1}{10}$, $\epsilon=0.01$.

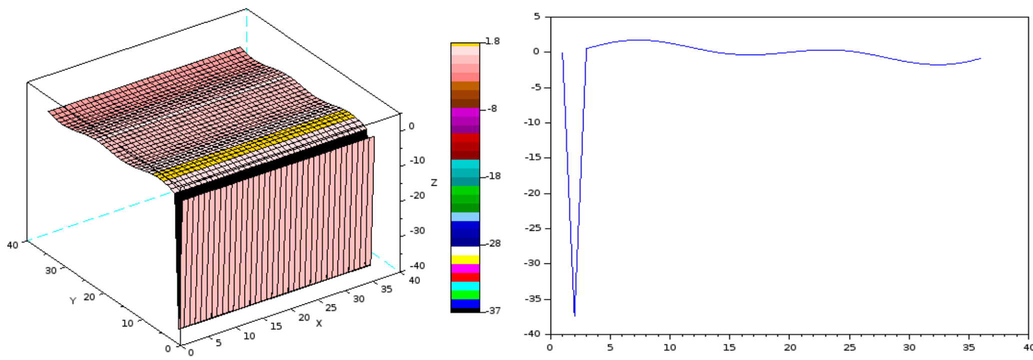

FIGURE 6.APPROXIMATION OF THE REFERENCE SOLUTION z^ϵ, FOR $\epsilon=0.01$ AT $t=1$, 3D IN THE LEFT AND 2D IN THE RIGHT, $Z^0(t,\frac{t}{\epsilon},\cdot,\cdot)+\epsilon Z^1(t,\frac{t}{\epsilon},\cdot,\cdot)$.. Approximation of the reference solution z^ϵ, for $\epsilon=0.01$ at $t=1$, 3D in the left and 2D in the right, $Z^0(t,\frac{t}{\epsilon},\cdot,\cdot)+\epsilon Z^1(t,\frac{t}{\epsilon},\cdot,\cdot)$.

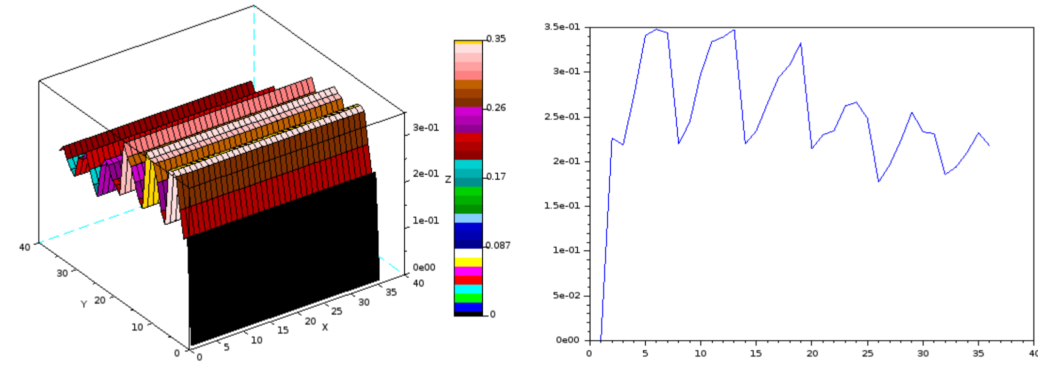

FIGURE 7.APPROXIMATION OF THE REFERENCE SOLUTION z^ϵ, FOR $\epsilon = 0.01$ AT $t = \frac{1}{3}$, 3D IN THE LEFT AND 2D IN THE RIGHT, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.. Approximation of the reference solution z^ϵ, for $\epsilon = 0.01$ at $t = \frac{1}{3}$, 3D in the left and 2D in the right, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.

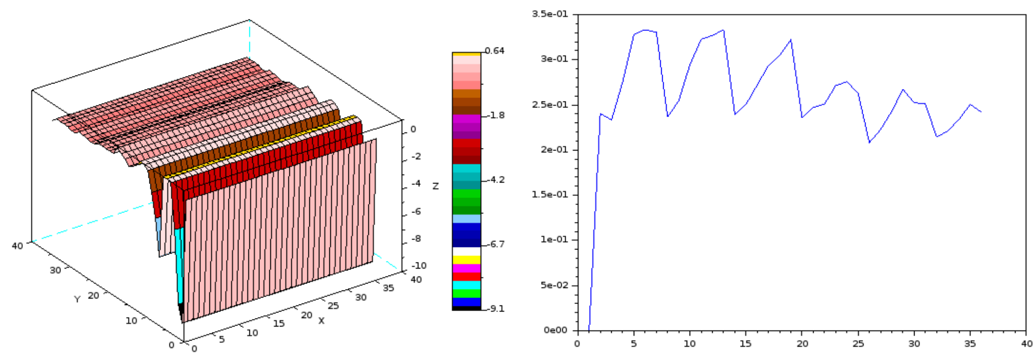

FIGURE 8.APPROXIMATION OF THE REFERENCE SOLUTION z^ϵ, FOR $\epsilon = 0.01$ AT $t = \frac{1}{4}$, 3D IN THE LEFT AND 2D IN THE RIGHT, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.. Approximation of the reference solution z^ϵ, for $\epsilon = 0.01$ at $t = \frac{1}{4}$, 3D in the left and 2D in the right, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.

The second sinusoidal expression of velocity and height variation fields are given by

$$\mathcal{U}(t, \theta, x_1, x_2) = 2\frac{agk}{\sigma} \sin(kx_1) \sin(\sigma\theta) \tag{4.2}$$

and

$$\mathcal{M}(t, \theta, x_1, x_2) = 2a \cos(kx_1) \cos(\sigma\theta) \tag{4.3}$$

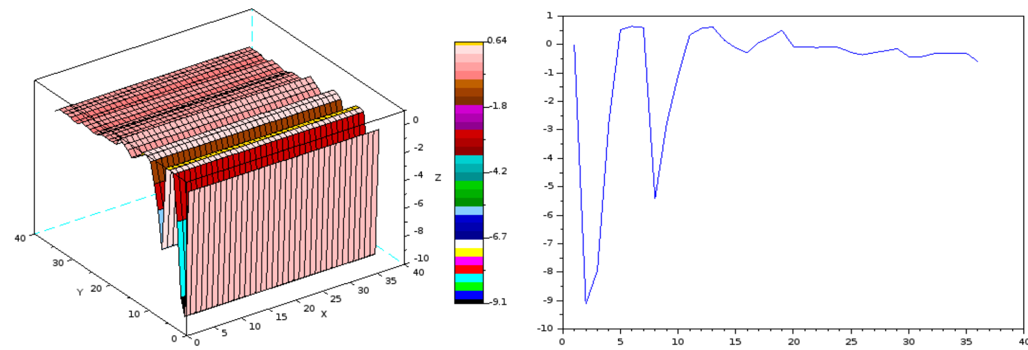

FIGURE 9. APPROXIMATION OF THE REFERENCE SOLUTION z^{ϵ}, FOR $\epsilon = 0.01$ AT $t = \frac{1}{8}$, 3D IN THE LEFT AND 2D IN THE RIGHT, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.. Approximation of the reference solution z^{ϵ}, for $\epsilon = 0.01$ at $t = \frac{1}{8}$, 3D in the left and 2D in the right, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.

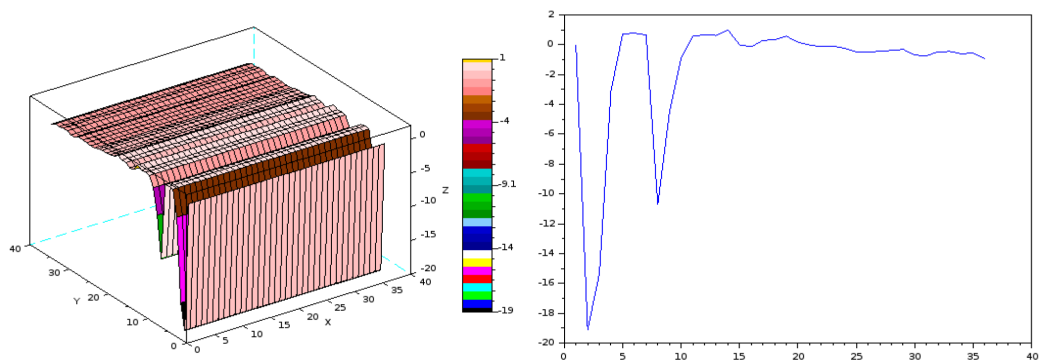

FIGURE 10. APPROXIMATION OF THE REFERENCE SOLUTION z^{ϵ}, FOR $\epsilon = 0.01$ AT $t = \frac{1}{10}$, 3D IN THE LEFT AND 2D IN THE RIGHT, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.. Approximation of the reference solution z^{ϵ}, for $\epsilon = 0.01$ at $t = \frac{1}{10}$, 3D in the left and 2D in the right, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.

where a is the amplitude wave , g the gravity fields, wave number $k = \frac{2\pi}{L}$, wave lenght $L = T \times C$, tide period $T = 13$hours, wave speed $C = \sqrt{g \times H}$ where water height H and tide frequency $\sigma = \frac{2\pi}{T}$.

The θ-evolution and space distribution of $\mathcal{M}(t, \theta, x_1, x_2)$, are given in figure 11 and figure 12.

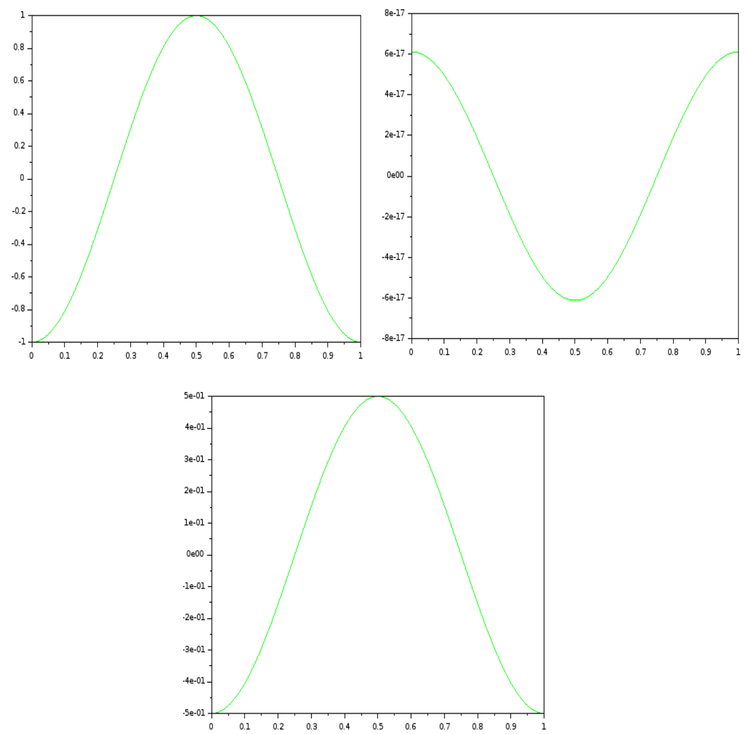

FIGURE 11. $\theta-$EVOLUTION OF $\mathcal{M}$ FOR $x_1 = \frac{1}{2},\ x_2 = \frac{1}{2}$ IN THE LEFT, $x_1 = \frac{1}{4},\ x_2 = \frac{1}{2}$ IN THE MIDDLE AND $x_1 = \frac{1}{3},\ x_2 = \frac{1}{4}$ IN THE RIGHT.. $\theta-$evolution of $\mathcal{M}$ for $x_1 = \frac{1}{2},\ x_2 = \frac{1}{2}$ in the left, $x_1 = \frac{1}{4},\ x_2 = \frac{1}{2}$ in the middle and $x_1 = \frac{1}{3},\ x_2 = \frac{1}{4}$ in the right.

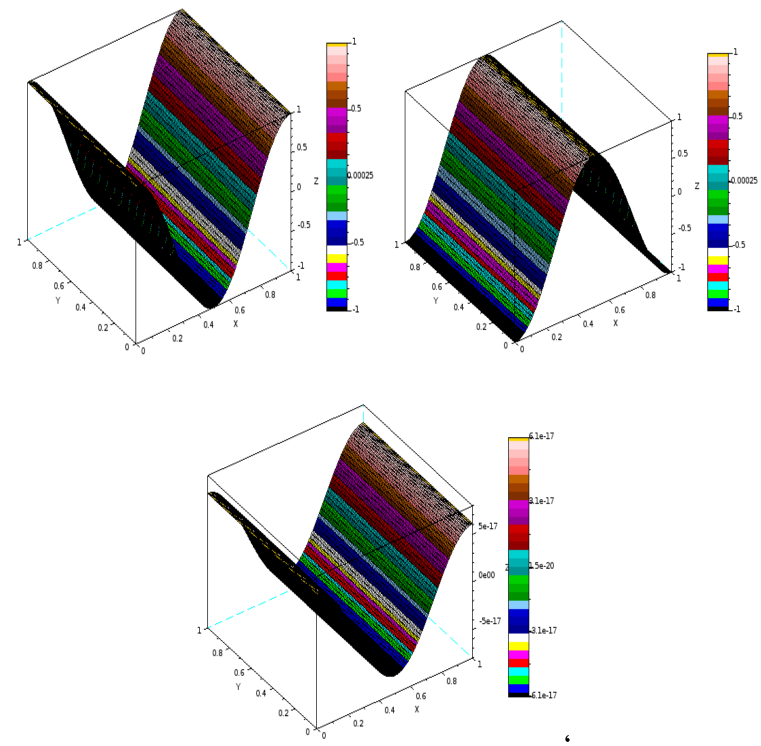

FIGURE 12.SPACE DISTRIBUTION OF $\mathcal{M}$ FOR $t = 1$ IN THE LEFT, $t = \frac{1}{2}$ IN THE MIDDLE AND $t = \frac{1}{4}$ IN THE RIGHT AND $\epsilon = 0.01$.. Space distribution of $\mathcal{M}$ for $t = 1$ in the left, $t = \frac{1}{2}$ in the middle and $t = \frac{1}{4}$ in the right and $\epsilon = 0.01$.

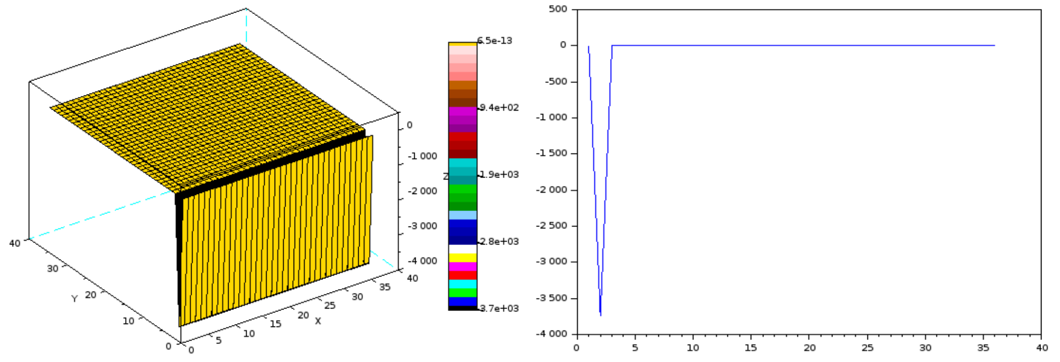

FIGURE 13.DISTRIBUTION OF THE FIRST TERM CORRECTOR $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D IN THE LEFT, 2D IN THE RIGHT AT $t = 1$ AND $\epsilon = 0.01$.. Distribution of the first term corrector $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D in the left, 2D in the right at $t = 1$ and $\epsilon = 0.01$.

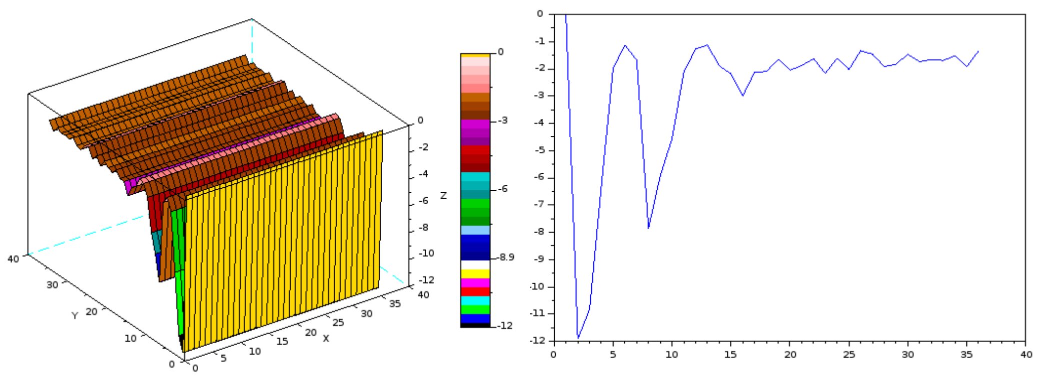

FIGURE 14.DISTRIBUTION OF THE FIRST TERM CORRECTOR $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D IN THE LEFT, 2D IN THE RIGHT AT $t = \frac{1}{3}$ AND $\epsilon = 0.01$.. Distribution of the first term corrector $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D in the left, 2D in the right at $t = \frac{1}{3}$ and $\epsilon = 0.01$.

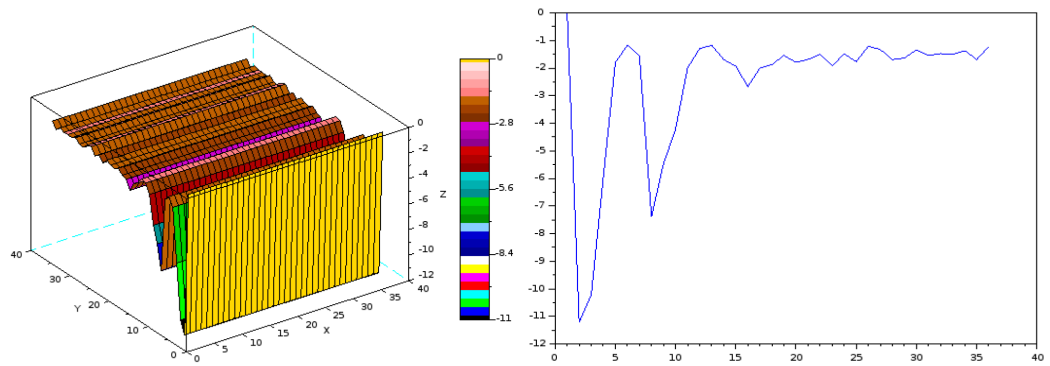

FIGURE 15.DISTRIBUTION OF THE FIRST TERM CORRECTOR $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D IN THE LEFT, 2D IN THE RIGHT AT $t = \frac{1}{4}$ AND $\epsilon = 0.01$.. Distribution of the first term corrector $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D in the left, 2D in the right at $t = \frac{1}{4}$ and $\epsilon = 0.01$.

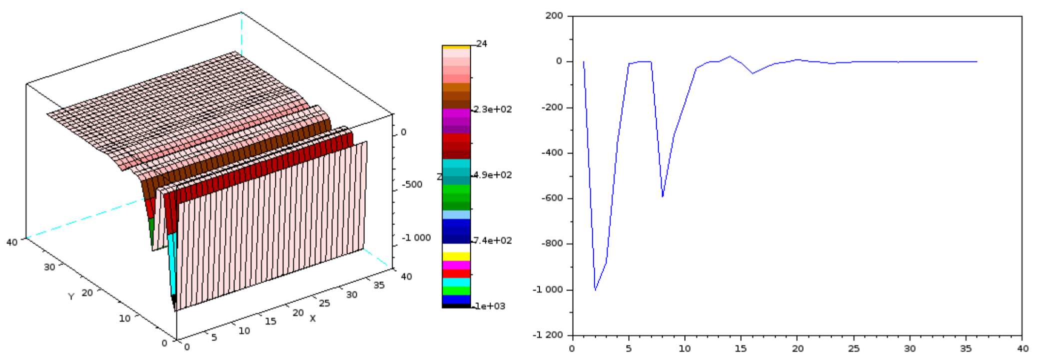

FIGURE 16.DISTRIBUTION OF THE FIRST TERM CORRECTOR $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D IN THE LEFT, 2D IN THE RIGHT AT $t = \frac{1}{8}$ AND $\epsilon = 0.01$.. Distribution of the first term corrector $Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$ 3D in the left, 2D in the right at $t = \frac{1}{8}$ and $\epsilon = 0.01$.

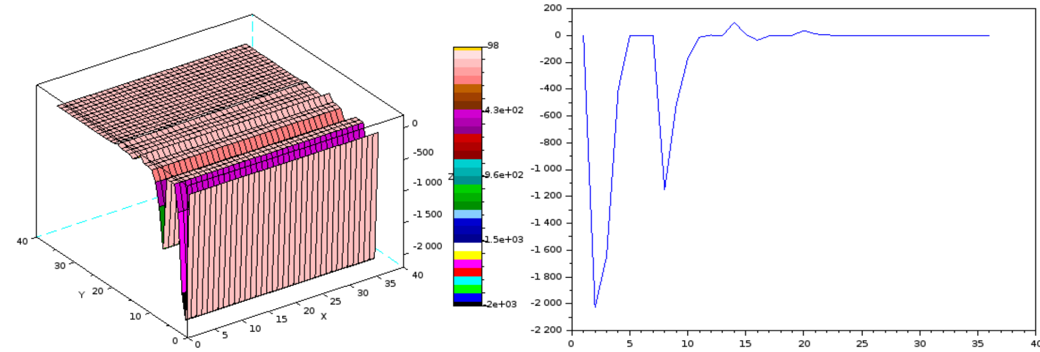

FIGURE 17.DISTRIBUTION OF THE FIRST TERM CORRECTOR $Z^1(t,\frac{t}{\epsilon},\cdot,\cdot)$ 3D IN THE LEFT, 2D IN THE RIGHT AT $t=\frac{1}{10}$ AND $\epsilon=0.01$.. Distribution of the first term corrector $Z^1(t,\frac{t}{\epsilon},\cdot,\cdot)$ 3D in the left, 2D in the right at $t=\frac{1}{10}$ and $\epsilon=0.01$.

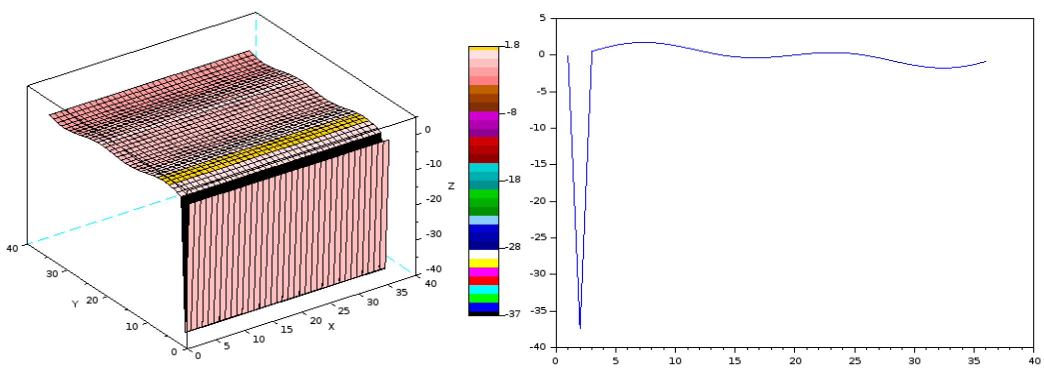

FIGURE 18.APPROXIMATION OF THE REFERENCE SOLUTION z^ϵ, FOR $\epsilon=0.01,\ t=1$ AND 3D IN THE LEFT AND 2D IN THE RIGHT, $Z^0(t,\frac{t}{\epsilon},\cdot,\cdot)+\epsilon Z^1(t,\frac{t}{\epsilon},\cdot,\cdot)$.. Approximation of the reference solution z^ϵ, for $\epsilon=0.01,\ t=1$ and 3D in the left and 2D in the right, $Z^0(t,\frac{t}{\epsilon},\cdot,\cdot)+\epsilon Z^1(t,\frac{t}{\epsilon},\cdot,\cdot)$.

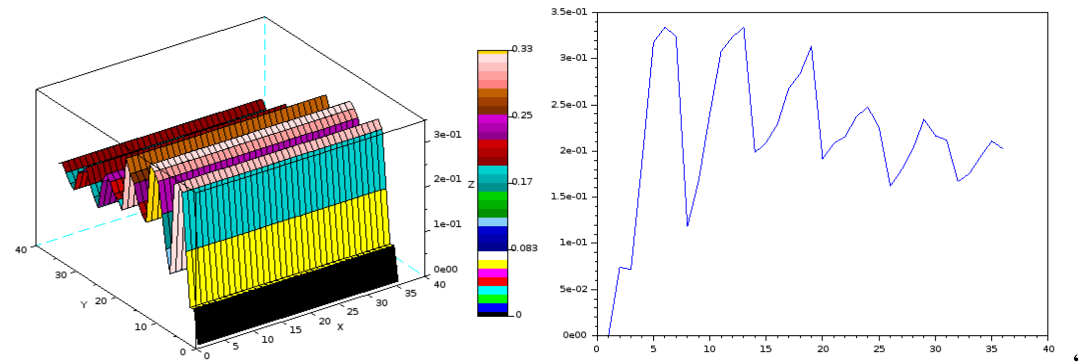

FIGURE 19.APPROXIMATION OF THE REFERENCE SOLUTION z^ϵ, FOR $\epsilon = 0.01,\ \ t = \frac{1}{3}$ AND 3D IN THE LEFT AND 2D IN THE RIGHT, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.. Approximation of the reference solution z^ϵ, for $\epsilon = 0.01,\ \ t = \frac{1}{3}$ and 3D in the left and 2D in the right, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.

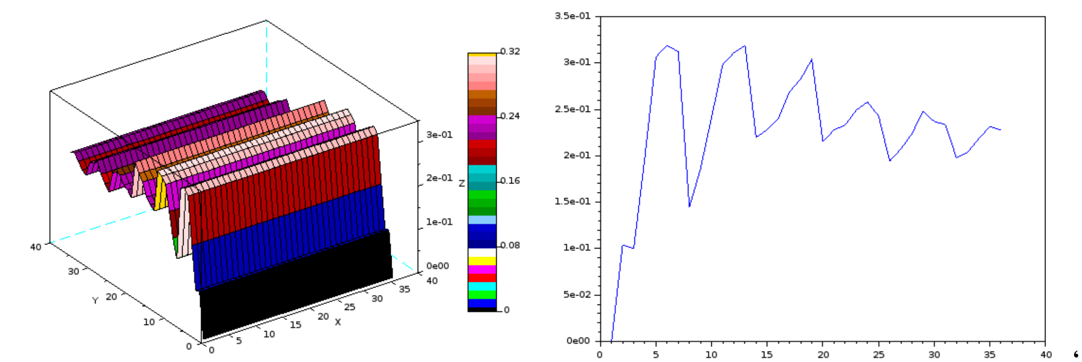

FIGURE 20.APPROXIMATION OF THE REFERENCE SOLUTION z^ϵ, FOR $\epsilon = 0.01,\ \ t = \frac{1}{4}$ AND 3D IN THE LEFT AND 2D IN THE RIGHT, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.. Approximation of the reference solution z^ϵ, for $\epsilon = 0.01,\ \ t = \frac{1}{4}$ and 3D in the left and 2D in the right, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.

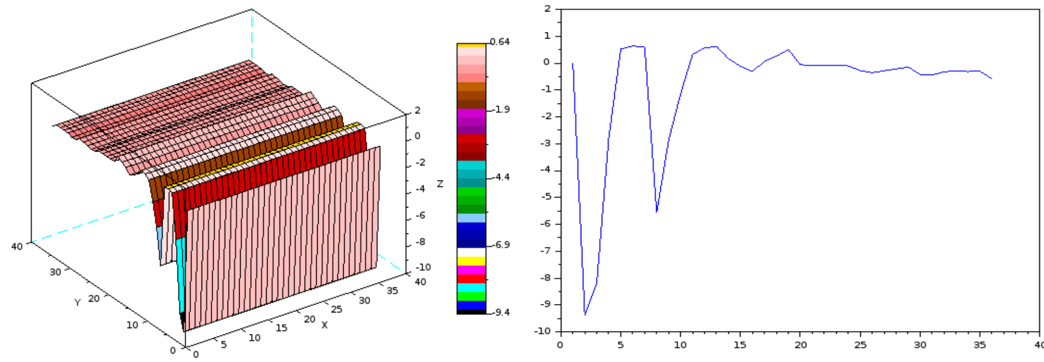

FIGURE 21. APPROXIMATION OF THE REFERENCE SOLUTION z^ϵ, FOR $\epsilon = 0.01,\ \ t = \frac{1}{8}$ AND 3D IN THE LEFT AND 2D IN THE RIGHT, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.. Approximation of the reference solution z^ϵ, for $\epsilon = 0.01,\ \ t = \frac{1}{8}$ and 3D in the left and 2D in the right, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.

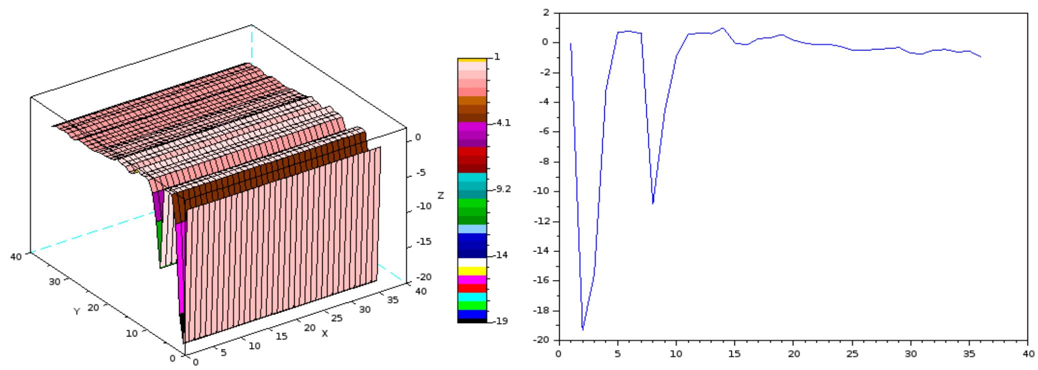

FIGURE 22. APPROXIMATION OF THE REFERENCE SOLUTION z^ϵ, FOR $\epsilon = 0.01,\ \ t = \frac{1}{10}$ AND 3D IN THE LEFT AND 2D IN THE RIGHT, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.. Approximation of the reference solution z^ϵ, for $\epsilon = 0.01,\ \ t = \frac{1}{10}$ and 3D in the left and 2D in the right, $Z^0(t, \frac{t}{\epsilon}, \cdot, \cdot) + \epsilon Z^1(t, \frac{t}{\epsilon}, \cdot, \cdot)$.

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