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CHAPTER 24

Floquet theory and averaging methods : Useful tools for equilibrium stability of dynamical model of Typha proliferation, *by M. L. Diagne and T. Sari*

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Abstract. Motivated by the fact that Floquet theory and averaging methods were been rarely used to study the stability of linear periodic systems in continuous time. The theory of averaging is based on the construction of approximate solutions essentially first-order differential equation with rapidly oscillating ordinary. A condition of stability of the trivial equilibrium of the switching system is given.

keywords. Floquet theory; averaging methods; Typha profilation; switching system .
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A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia. **Diagne M.L. and Sari T.(2018). Floquet theory and averaging methods : Useful tools for equilibrium stability of dynamical model of *Typha* proliferation**. Pages 469 — 486.

Full Abstract. Motivated by the fact that Floquet theory and averaging methods used to study the stability of linear periodic systems in continuous time, we formulate and analyze the dynamics of a nonlinear and non-autonomous system of ordinary differential equations describing the dynamics of an invasive reproductive plant: the *Typha*. Its two modes of reproduction namely; sexual (via seeds) and asexual (via rizhomes) are included into the hybrid system which combines the features of classical continuous time and discrete time systems. Stability of the null equilibrium is investigated via the basic reproduction rate Ro of the model in the absence of *Typha* is computed. For $R_{0,\alpha} < 1$ the a useful too which can be applied to analyze the stability of models with seasonality. The theory of averaging is based on the construction of approximate solutions essentially first-order differential equation with rapidly oscillating ordinary. A condition of stability of the trivial equilibrium of the switching system is given. Numerical simulations to support the analytical results are provided.

1. Introduction

The proliferation of *Typha* is recognized as a potential threat to wetland ecosystems around the world Mallik and Wein (1986); Ball (1998). Replacement of existing vegetation by dense, mono-specific stands of *Typha* can result in significant loss of habitat structure and function Smith and Kaldlec (1985). Accordingly, the study of conditions that facilitate such invasions are of considerable importance to wetland preservation and management. The growth of *Typha* is regulated by a number of factors, including hydrology temperature Adriano *et al.* (1980), plan competition Emery and Parry (1996); Mal and Doust (1997) , and nutriments Bonnewell and Pratt (1978); Davis and Valk (1983), Reddy textit et al. (1998); Miao and Sklar (1998). With modest eco-hydrological requirements, *Typha* spp. colonizes areas with rhizomes multiplication and seasonal seeds dissemination by reproductive adults. We consider a nonlinear and non-autonomous tri-dimensional system of ordinary differential equations describing the dynamics of reproductive plants and the two young plants categories from *Typha*'s sexual (seeds) and asexual (rhizomes) reproduction. Thus we obtain a switching system of three dimensions Diagne *et al.* (2012). A switched system consists of a set of subsystems and a rule that describes switching among them. The subsystems may be continuous-time or discrete-time systems and the switching rule may depend on time or states of individual subsystems. Switched systems arise when dynamics of systems undergo abrupt changes due to component failures, parameter changes, switching elements, or switching controllers. Such systems have been studied extensively in the literature. A recent survey of switched systems can be found in Liberzon and Morse (1999) and various applications of switched systems are discussed in Morse (1997).

In this paper, a switched systems of dynamical model of *Typha* proliferation developed in Diagne *et al.* (2012) is considered. More specifically, we consider a periodically switched system. The goal of this paper is to investigate stability of trivial equilibrium of switched system. The main tools used for this purpose are the Floquet theory Richards (1983); Rugh (1996) and averaging method. Floquet theory provides a numerical result while the method of averaging gives an analytic result. It should be mentioned that although Floquet theory has been used in the literature to investigate stability of periodically time varying linear systems and the averaging method is a useful computational technique, the method is the classical methods in analyzing nonlinear oscillations Nicolas (2007). Although Floquet theory and averaging method have a wide range of potential uses in ecological, biological and evolutionary modeling and is relatively easy to implement, its use in ecology and biology has been extremely limited Kooi and Troost (2006).

Averaging is the procedure of replacing a vector field by its average with the goal to obtain asymptotic approximations of the original system and to obtain periodic solutions. We aim at constructing asymptotic approximations of the solution of model and proving their asymptotic validity. We use the method of averaging to obtain approximations valid for all time. Averaging was developed for ordinary differential equations, see Sanders *et al.* (2007).

Therefore, the other goal of this paper is to promote the wider use of Floquet theory and averaging methods which are useful tools for studying the effects of temporal variability on ecological systems.

This paper is organized as follows. The model is presented in Section 2. The Floquet theory and its application to the model is reviewed in Section 3. In Section 4, the averaging method and its application to the model are presented and illustrated. Conclusions are given in Section 5.

2. Model

The model of the spread of *Typha* studied in this paper has been introduced in Diagne *et al.* (2012). It is described by the system (2.5) used to model the proliferation of *Typha*. The system variables E_s, E_r and A stand, respectively, for young *Typha* from sexual reproduction, young plant from asexual reproduction and adults in which the plant reproduces by the two reproduction modes. Denoting by K the limited host medium and $Y(t) = E_s(t) + E_r(t) + A(t)$ the total number of plants at time t capacity, the model is obtained in the form of an autonomous nonlinear and system of ordinary differential equations (for more details, see Diagne *et al.* (2012)):

$$\begin{cases} \dot{E}_s = \tilde{c}_s(t)A\,(1 - Y/K) - (\gamma_s + \mu_s)\,E_s, \\ \dot{E}_r = c_r A\,(1 - Y/K) - (\gamma_r + \mu_r)\,E_r, \\ \dot{A} = \gamma_s E_s + \gamma_r E_r - \mu_a A, \end{cases} \tag{2.1}$$

Let us denote by γ_s and γ_r, respectively, the transition rates from compartment E_s to A and from E_r to A. The death in the compartments E_s, E_r and A are characterized respectively by the per-capita mortality rates μ_s, μ_r and μ_a.

The term $(1-X/K)$ is the probability to find an available space of emergence in the domain of *Typha*'s growth. The term $\tilde{c}_s(t)$ defines the instantaneous rate at which the population of adults reproduces young plants from seeds without space of emergence constrains. We assume that c_r is constant and positive ($c_r > 0$) for all time $t \geq 0$.
For simplicity, we consider that the sexual emergence rate $\tilde{c}_s(t)$ is function defined by

$$\tilde{c}_s(t) = \begin{cases} c_s & \text{if } t \in [0, \alpha T[\text{ mod } T \\ 0 & \text{if } t \in [\alpha T, T[\text{ mod } T \end{cases} \tag{2.2}$$

where $c_s > 0$ is a constant and α; $0 \leq \alpha \leq 1$ is the fraction of each year in which there is emergency from sexual reproduction. In this case, the non-autonomous and non linear model (Eq.2.1) can be written as two 3-dimensional and nonlinear subsystem which are according to the presence or not of emergency from sexual reproduction. Indeed, for $t \in \mathbb{R}_+$, we have

- presence of seasonal emergency from seed subsystem equations, if $0 \leqslant t < \alpha T$:

$$\begin{cases} \dot{E}_s = c_s A(1 - Y/K) - (\gamma_s + \mu_s)E_s, \\ \dot{E}_r = c_r A(1 - 1/K) - (\gamma_r + \mu_r)E_r, \\ \dot{A} = \gamma_s E_s + \gamma_r E_r - \mu_a A, \end{cases} \tag{2.3}$$

- absence of seasonal emergency from seed subsystem equations, if $\alpha T \leqslant t < T$:

$$\begin{cases} \dot{E}_s = -(\gamma_s + \mu_s)E_s, \\ \dot{E}_r = c_r A(1 - Y/K) - (\gamma_r + \mu_r)E_r, \\ \dot{A} = \gamma_s E_s + \gamma_r E_r - \mu_a A, \end{cases} \tag{2.4}$$

We introduce proportions related to occupied space with capacity K by setting $e_s = \dfrac{E_s}{K}, e_r = \dfrac{E_r}{K}$, $a = \dfrac{A}{K}$ and $y(t) = e_s(t) + e_r(t) + a(t)$. So, we derive from Eq.2.1 the dimensionless switching model of *Typha* proliferation:

$$\begin{cases} \dot{e}_s = c_s(t)a(1 - y) - (\gamma_s + \mu_s)e_s, \\ \dot{e}_r = c_r a(1 - y) - (\gamma_r + \mu_r)e_r, \\ \dot{a} = \gamma_s e_s + \gamma_r e_r - \mu_a a, \end{cases} \tag{2.5}$$

For simplicity and without loss generality, we chose a simple occurrence of seasonal emergency rate function $\tilde{c}_s(t)$ which is T periodic (with $T = 12$ months) and defined on $[0, T]$ by

$$\tilde{c}_s(t) = \begin{cases} c_s & \text{if } t \in [0, \alpha T[\mod T \\ 0 & \text{if } t \in [\alpha T, T[\mod T \end{cases} \tag{2.6}$$

where α $(0 \leq \alpha \leq 1)$ is constant and denotes the fraction of the year in which emergencies from seed take place.

It is known that the system of the switching model of *Typha* proliferation is a combination of two 3-dimensional non linear autonomous systems. Here, we determine model characteristics with biological interest such as equilibrium and reproduction number for each sub-dynamics according to the presence or not of seasonal emergency from seed. From the computation of the equilibrium of (Eq.2.3), it appears that the reproduction number of the sub-dynamics with presence of both emergency types defined by

$$R_0 = \frac{c_s\gamma_s}{\mu(\gamma_s+\mu_s)} + \frac{c_r\gamma_r}{\mu(\gamma_r+\mu_r)}$$

The parameter R_0 is, in an obvious manner, the sum of emergency from seed $R_{0,s} = \dfrac{c_s\gamma_s}{\mu_a(\gamma_s+\mu_s)}$ and emergency from rhizom $R_{0,r} = \dfrac{c_r\gamma_r}{\mu_a(\gamma_r+\mu_r)}$. Also, we find $R_{0,r}$ computing equilibrium of (Eq.2.4). Those reproduction numbers R_0 and $R_{0,r}$ guide the existence of equilibrium of related subsystem in the manner established in the following results.

3. Floquet theory

Floquet theory transforms a linear periodically time varying system into a linear time invariant system through a Lyapounov transformation. Hence, the stability of the former system can be inferred from that of the latter. Below is a brief review of the Floquet theory .
Consider the linear time varying system

$$\begin{cases} \dot{X}(t) = A(t)X(t) \quad t \geq t_0 \\ X(t_0) = X_0 \end{cases} \tag{3.1}$$

where $X(t) \in \mathbb{R}^n$, the matrix $A(t) \in \mathbb{R}^{n\times n}$ is piecewise continuous, bounded, and periodic with period T. Although its parameters $A(t)$ vary periodically, the solutions of Eq.3.1 are typically not periodic, and despite its linearity, closed form solutions of Eq.3.1 typically cannot be found.
The matrix system associated with the system Eq.3.1 is

$$\dot{Z}(t) = A(t)Z(t), \tag{3.2}$$

where Z is an $n \times n$ matrix. A non-singular matrix solution of Eq.3.2 is called a fundamental matrix. Given a fundamental matrix $\phi(t)$, every solution of Eq.3.1 can be written as $\phi(t)c$ for a constant vector c. It is usual to take Eq.3.2 with the initial condition $\phi(0) = I$, where I is the $n \times n$ identity matrix. The Floquet theory Richards (1983) Rugh (1996) consists in showing that every fundamental matrix $\phi(t)$ can be rewritten under the following form

$$\phi(t) = P(t)exp^{Bt}$$

where B is constant matrix, $P(t)$ has the same period as $A(t)$ and $P(t_0) = I$.

Application. Our switched system provide a natural context for applying classical Floquet theory. By linearization of fields of vector of the switching system in the neighborhood of the origin we obtain the following linear switching system :

$$\dot{x} = A(t)x. \tag{3.3}$$

Here,

$$A(t) = \begin{cases} A_1 & \text{if } t \in [0, \alpha T), \quad \text{mod } T \\ A_2 & \text{if } t \in [\alpha T, T], \quad \text{mod } T \end{cases}$$

where

$$A_1 = \begin{pmatrix} -(\gamma_s + \mu_s) & 0 & c_s \\ 0 & -(\gamma_r + \mu_r) & c_r \\ \gamma_s & \gamma_r & \mu_a \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} -(\gamma_s + \mu_s) & 0 & 0 \\ 0 & -(\gamma_r + \mu_r) & c_r \\ \gamma_s & \gamma_r & \mu_a \end{pmatrix}.$$

We define the average of basic number of reproduction $R_{0,\alpha}$, by

$$R_{0,\alpha} = \alpha R_0 + (1-\alpha)R_{0,r} = \alpha(R_0 - R_{0,r}) + R_{0,r} = \alpha R_{0,s} + R_{0,r}.$$

Local stability of the origin E_0 of the switching system Eq.2.5 can also be characterized using the following theorem.

THEOREM 67. *The switching system Eq.* (2.5) *is locally stable in the origin* E_0 *if* $R_{0,\alpha} < 1$.

Proof. The assumptions of the Floquet theorem are satisfied. Then we obtain : for $0 \leqslant t < \alpha T$, mod T, $\dot{x}(t) = A_1 x(t)$. Thus $x(t) = exp^{[A_1 t]} x(t_0) = exp^{[A_1 t]} x_0$.

Similarly, for $\alpha T \leqslant t < T$, mod T,. Hence $\dot{x}(t) = A_2 x(t) t$

$$\begin{aligned} x(t) &= exp^{[A_2(t-\alpha T]} x(\alpha T) \\ &= exp^{[A_2(t-\alpha T]} exp^{[A_1 \alpha T]} x_0 \end{aligned}$$

Thus, it follows that

$$M = \phi(T) = exp^{[A_2(T-\alpha T)]} exp^{[A_1 \alpha T]} x_0.$$

To show the stability of the origin we apply the result of Floquet theory. Therefore we show that the spectral radius of the monodromy matrix $\rho(M)$ is less than 1.

In the following numerical simulations, we chooses : $\gamma_s = \dfrac{1}{8}$, $\gamma_r = \dfrac{1}{6}$, $\mu_s = \mu_r = \dfrac{1}{24}$, $\mu_a = \dfrac{1}{72}$, $c_s = 0.02$ and $c_r = 0.01$.

By the dichotomy method we determined the value of α such that $\rho(M) = 1$. Thus, for $\alpha = 0.4275$ we get $\rho(M) = 1$.

For any value of α such that $\alpha < 0.4275$ the switching system converges to the trivial solution.

In the sequel, we will represent the curves $R_{0,\alpha}$ and $\rho(M)$ in terms of α and we compare them.

In Figures 3, the right figure is a zoom of the left one, made in the vicinity of the intersections. We note that the curves do not intersect at the

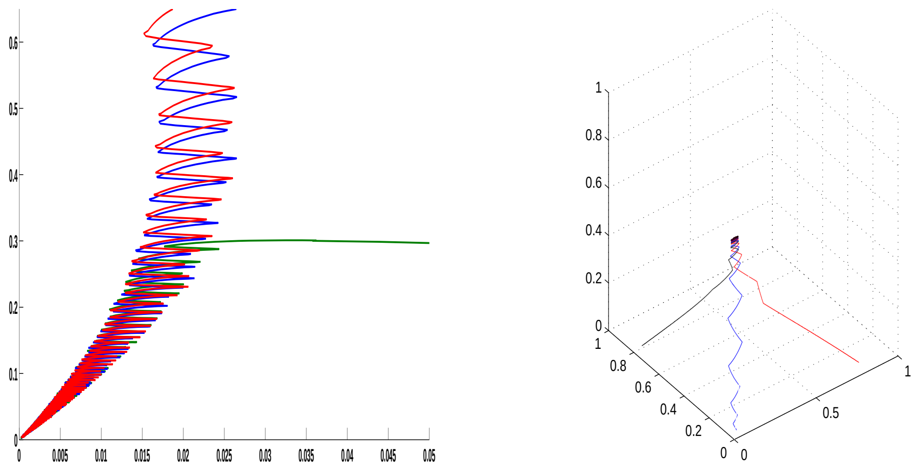


FIGURE 1. PHASE PORTRAIT OF THE SWITCHING SYSTEM (2.5) WHEN $\rho(M) < 1$. WE ILLUSTRATE THE CONVERGENCE OF SOLUTIONS OF THE SYSTEM TO SWITCH TO THE ZERO BALANCE WHEN $\alpha T = 3$. WITH THIS VALUE $\rho(M) = 0.9979 < 1, R_0 = 1.6200$ AND $R_{0,r} = 0.54$. Phase portrait of the switching system (2.5) when $\rho(M) < 1$. We illustrate the convergence of solutions of the system to switch to the zero balance when $\alpha T = 3$. With this value $\rho(M) = 0.9979 < 1, R_0 = 1.6200$ and $R_{0,r} = 0.54$

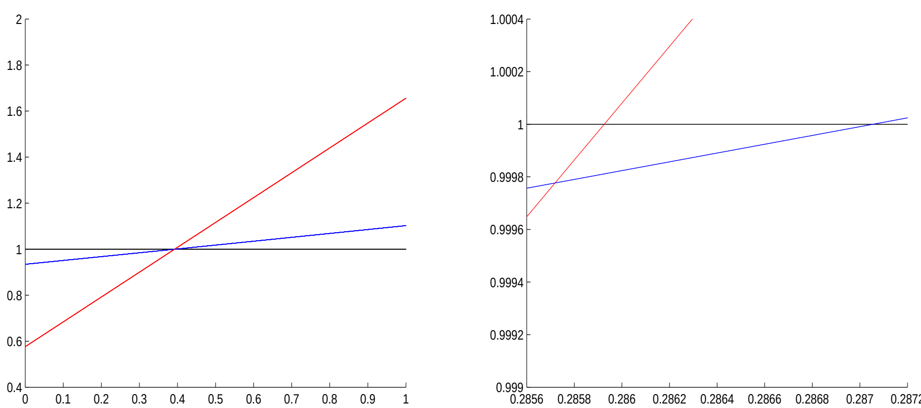


FIGURE 2. $\rho(M)$ AND $R_{0,\alpha}$ IN TERMS OF α. $\rho(M)$ and $R_{0,\alpha}$ in terms of α

same place. The curve $R_{0,\alpha}$ intersects the line $y = 1$ before the curve $\rho(M)$. Therefore, we conclude, for the chosen parameters, that if $R_{0,\alpha} < 1$ then, $\rho(M) < 1$. Thus $R_{0,\alpha} < 1$ is a sufficient condition for stability of the trivial equilibrium of the switching system. We will show in the following section that this result is a consequence of the averaging theory.

4. Averaging method

Consider an ordinary differential equation of the type

$$\dot{x} = \varepsilon f(x,t,\varepsilon) \quad x(0) = x_0, \quad x,\ x_0 \in D \subset \mathbb{R}^n, \tag{4.1}$$

where D is an open set with compact (that is, closed and bounded) closure, on which f is defined. The parameter ε is assumed to be small. The equation often arises by expansion in the neighborhood of an equilibrium. The vector field f is assumed to be differentiable with respect to all variables, but this can be relaxed.

Since f depends explicitly on time t , equation (4.1) is a non-autonomous differential equation. This type of equation is usually very difficult to analyze, so one is interested in finding an autonomous system, the solutions of which approximate the original system, where the accuracy of the approximation is a function of ε .

Setting $\varepsilon = 0$ is not of much interest : it will give us an approximation that is valid on the interval $0 \leq t \leq L$ for some constant L, that is on time scale 1. On a longer time scale, for instance $\frac{1}{\varepsilon}$, this is a singular perturbation problem, that is to say, the solution of the unperturbed problem (with $\varepsilon = 0$) is not an approximation of the solution of the full problem 4.1.

On this longer time $\frac{1}{\varepsilon}$ another natural idea works better: average the right hand side over the time t. Assume, for simplicity, that f is periodic in t with period T. Then define the average

$$\bar{f}(x) = \frac{1}{T}\int_0^T f(x,s,0)\,ds. \tag{4.2}$$

More rigorously, we state in the following, the result of the asymptotic method called "Averaging Method" given by Maurice Roseau in Maurice (1975).

Lets be a K is a compact, $J =]0,a] \subset \mathbb{R}_+,\ a > 0$ a real interval and f a function defined on $\mathbb{R}_+ \times K \times J$

THEOREM 68. , *(Maurice (1975), Chapter 4). Suppose that f is measurable in t for all fixed (x,ε) , continuous in x for all fixed t , almost everywhere* *on* $\mathbb{R}_+$.

Let Y *be the solution of the overage problem and* $I = [0, w[, 0 < w \leq \infty$*. Then, for all* L *in* I *and all* $\delta > 0$ *if* ε *is small enough, there exists* $\varepsilon_0 = \varepsilon_0(L, \delta)$*such that for any* ε *in* $]0, \varepsilon 0]$*,* x *any solution of the equation (4.1) Initial value* y_0 *at* $t = 0$ *is defined at least on the interval* $[0, L]$ *and satisfies the inequality* $|x(t) - y(t)| < \ delta$ *for all* t *in* $[0, L]$*.*

THEOREM 69. *(Sanders et al. (2007), Chapter 5 and 6). When the averaged system* $\dot{y} = \overline{f}(y)$ *has an hyperbolic equilibrium* $\overline{y}$*, then the initial system (4.1) has a periodic solution in a vicinity of the balance and stability is the same as the equilibrium* $\bar{y}$*.*

Application. In this part, we are interested to the study of the global stability of the trivial equilibrium. Indeed, finding a condition that provides the asymptotic stability of the equilibrium is ecologically important. Since reducing the speed of *Typha* is important. The switching system is transformed to rapidly oscillating model. To apply the principle of averaging developed previously, we rewrite the model in the standard form (4.1). Recall that the system in study is (see system 2.5) :

$$(4.3) \qquad \begin{cases} \dot{e}_s = \tilde{c}_s(t)a(1-y) - (\gamma_s + \mu_s)e_s \\ \dot{e}_r = c_r a(1-y) - (\gamma_r + \mu_r)e_r \\ \dot{a} = \gamma_s e_s + \gamma_r e_r - \mu_a a \end{cases}$$

where $\tilde{c}_s$ is a periodic function of period T.
Since the parameters of the system are small (because the maximum is $\gamma_r = \dfrac{1}{6}$), we assume the coefficients of system (4.3) are written in the following form

$$\tilde{c}_s = \varepsilon \tilde{c}_s^*, \quad c_r = \varepsilon c_r^*, \quad \gamma_s = \varepsilon \gamma_s^*, \quad \mu_s = \varepsilon \mu_s^*, \quad , \gamma_r = \varepsilon \gamma_r^*, \quad \mu_r = \varepsilon \mu_r^*, \quad \mu_a = \varepsilon \mu_a^*.$$

with $\varepsilon > 0$ small. We obtain the following system

$$\begin{cases} \dot{e}_s = \varepsilon\Big[\tilde{c}_s^*(t)a(1-y) - (\gamma_s^* + \mu_s^*)e_s\Big] \\ \\ \dot{e}_r = \varepsilon\Big[c_r^* a(1-y) - (\gamma_r^* + \mu_r^*)e_r\Big] \\ \\ \dot{a} = \varepsilon\Big[\gamma_s^* e_s + \gamma_r^* e_r - \mu_a^* a\Big] \end{cases}$$

To avoid burdening the notation, we can remove the stars and write again this system as follows

$$\begin{cases} \dot{e}_s = \varepsilon\Big[\tilde{c}_s(t)a(1-y) - (\gamma_s + \mu_s)e_s\Big] \\ \\ \dot{e}_r = \varepsilon\Big[c_r a(1-y) - (\gamma_r + \mu_r)e_r\Big] \\ \\ \dot{a} = \varepsilon\Big[\gamma_s e_s + \gamma_r e_r - \mu_a a\Big] \end{cases} \tag{4.4}$$

The main result of the averaging (see theorem 68), is applied in the case where ε is small $(\varepsilon < 1)$.. The system (4.4) has as averaged system :

$$\begin{cases} \dot{e}_s = \varepsilon\Big[\bar{c}_s a(1-y) - (\gamma_s + \mu_s)e_s\Big] \\ \\ \dot{e}_r = \varepsilon\Big[c_r a(1-y) - (\gamma_r + \mu_r)e_r\Big] \\ \\ \dot{a} = \varepsilon\Big[\gamma_s e_s + \gamma_r e_r - \mu a\Big] \end{cases} \tag{4.5}$$

with

$$\bar{c}_s = \frac{1}{T}\int_0^T \tilde{c}_s(t)dt = \alpha c_s.$$

In terms of time scaling, we introduce the new time $\tau = \varepsilon t$. Note that τ is much slower than t. we have

$$x_i' = \frac{dx_i}{d\tau} = \frac{dx_i}{dt}\frac{dt}{d\tau} = \frac{1}{\varepsilon}\dot{x}_i.$$

By using $\tau = \varepsilon t$, the system (4.5) can be written as follows

$$(4.6)\qquad \begin{cases} e_s' = \bar{c}_s a(1-y) - (\gamma_s + \mu_s)e_s \\ \\ e_r' = c_r a(1-y) - (\gamma_r + \mu_r)e_r \\ \\ a' = \gamma_s e_s + \gamma_r e_r - \mu a \end{cases}$$

We have that the asymptotic stability of the system is governed by

$$R_{0,\alpha} = \frac{\bar{c}_s\gamma_s}{\mu_a(\gamma_s+\mu_s)} + \frac{c_r\gamma_r}{\mu_a(\gamma_r+\mu_r)} = \alpha\frac{c_s\gamma_s}{\mu_a(\gamma_s+\mu_s)} + \frac{c_r\gamma_r}{\mu_a(\gamma_r+\mu_r)}.$$

Thus, for an infinite time we have the global stability of System (4.6).

THEOREM 70. *If $R_{0,\alpha} \leq 1$ then trivial equilibrium of system (4.4) is global and asymptotically stable. Else trivial equilibrium of system is unstable.*

Numerical simulations. Lets consider the following parameter

$$c_s = 0.002; \quad c_r = 0.012, \quad \gamma_r = \frac{1}{6}, \quad \gamma_s = \frac{1}{8}, \quad \mu_s = \mu_r = \frac{1}{24}, \quad \mu_a = \frac{1}{72}, \quad T = 12, \quad \alpha = \frac{1}{3}.$$

We have

$$\tilde{c}_s(t) = \begin{cases} c_s, & \text{si } t \in [0,4,) \quad \text{mod } 12 \\ 0, & \text{si } t \in [4,12] \quad \text{mod } 12, \end{cases}$$

For these coefficients, $R_{0,moyen} = 0.7272 < 1$.

We can write the system again as $\dot{x} = \varepsilon f(t, x)$

$$\begin{cases} \dot{e}_s = \dfrac{1}{6}\left[\tilde{c}_s(t)a(1-y) - \left(\dfrac{3}{4} + \dfrac{1}{4}\right)e_s\right] \\ \\ \dot{e}_r = \dfrac{1}{6}\left[0.002a(1-y) - \left(1 + \dfrac{1}{4}\right)e_r\right] \\ \\ \dot{a} = \dfrac{1}{6}\left[\dfrac{3}{4}e_s + e_r - \dfrac{1}{12}a\right] \end{cases}$$

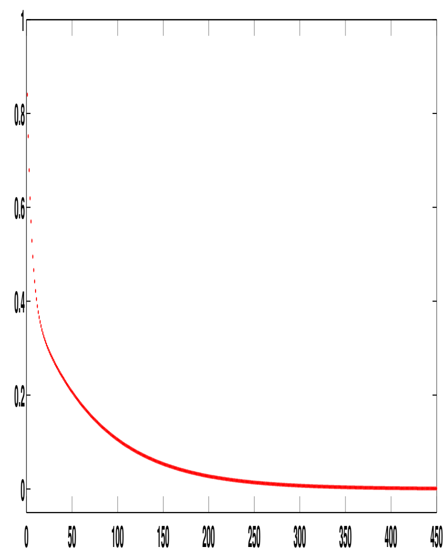
Thus the averaging method applies ($\varepsilon = \frac{1}{6} < 1$ and predicted that, the system is approximated by the averaged system. The following simulations illustrate this result.

Let us comment the table of figures 4. Figure (a) shows the evolution over time of the total population $y = e_s + e_r + a$ average the system with the initial condition $Y_0 = (0.12, 0.45, 0.27)$. Figure (b) shows the evolution over time of the total population y of switching system with the initial condition$Y_0 = (0.12, 0.45, 0.27)$. Figure (c) is thet superposition of the two curves (a) and (b). Figure (d) illustrates the evolution over time of the approximation error of the two solutions from the same initial condition. With the current parameter values, we obtain $R_0 = 0.108, R_{0,r} = 0.6912$ and $R_{0,\alpha} = 0.7272$. Note that in Figure (d) the ordinate scale is multiplied by 10^{-4}. Also Figure (d) shows that the upper bounds of the approximation error defined in Theorem 68 are power 10^{-4} and reach over time smaller powers. Thus, the asymptotic approximation becomes more accurate.

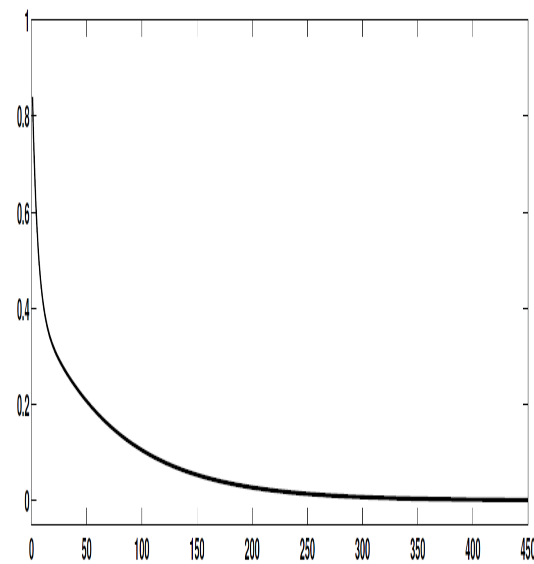
In the graphical comparison, one can clearly see that the approximation by averaging provides a satisfactory result if $R_{0,\alpha} < 1$.

5. Conclusion

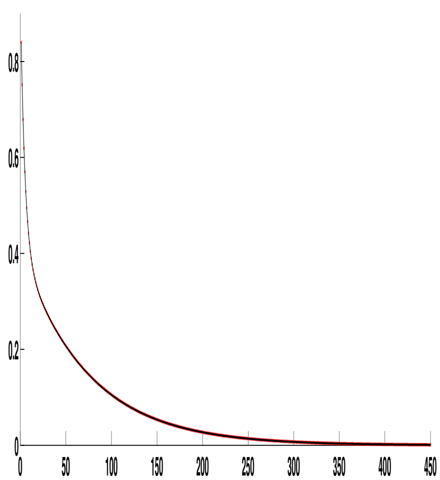
This example of this document shows that the Floquet theory and the method of averaging are versatile tool for the study of the ecology and evolution of periodic systems . The Floquet theory defines fitness in periodic h



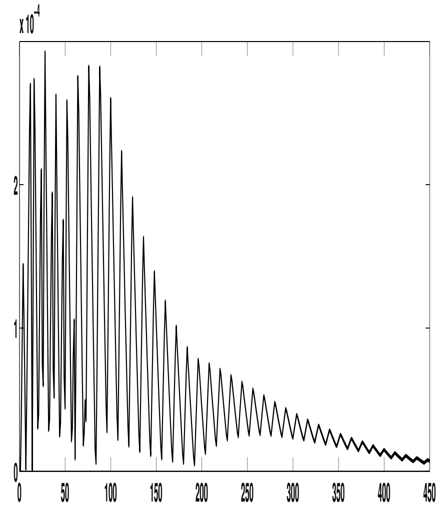
(a)



(b)



(c)



(d)

environments can numerically calculate the convergence criteria of a common two sub system equilibrium Nicolas and Dads (2012), and can be used to test the stability of solutions of this cycle. The method of averaging in some cases gives us theoretical results. Given these various uses and the omnipresence of two structured populations and periodic systems in nature, the Floquet theory and the method of averaging will be a useful addition to theorists toolboxes. Although the theory is a linear Floquet theory, nonlinear models can be linearized near solutions limit cycle to enable the use of the Floquet theory. We have shown that the method of averaging is a method to obtain approximate solutions of periodic systems. Elsewhere,we proved that the solutions converge to a trivial equilibrium, and ε-close to the trivial equilibrium solution of the system means.

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