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CHAPTER 23

A Simple Round-up on the Validity of a.s Convergence after Partial Modification of the Probability Law and Application, *by G.S. Lo*

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Abstract. We make a simple Round-up on the validity of different type of convergences when the sequence of random variables is partialla altered and provide an application in extreme value theory.

Keywords. Types of convergence of random variables; convergence in probability; convergence in distribution; convergence in almost sure; partial change of margins; preservation of type of limits; order statistics of uniform random variables; partial sums of standard exponential;

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Full Abstract. Let U_1, U_2, ... be a sequence of independent and uniformly distributed random variables on $(0,1)$ defined on the same probability space. Let $U_{1,n} \leq ... \leq U_{n,n}$ be the order statistics of the sample U_1, U_2,...U_n of size $n \geq 1$. Let $(k(n))_{n\geq 1}$ be a sequence of integers such that $1 \leq k(n) \leq n$ and $k(n) \longrightarrow +\infty$. We prove that $nU_{k(n),n}/k(n) \longrightarrow 1$ a.s as $n \longrightarrow +\infty$. We take the opportunity to make a simple Round-up on the validity of different type of convergences when the sequence of random variables is replaced by another sequence preserving parts of the probability law of the original sequence.

1. Introduction and Motivation

The following question concerning the a.s. behavior of the $k(n)^{th}$ order statistic of a sample of $\boldsymbol{U}(0,1)$ random variables of size n, when $k(n)$ is unbounded was brought, to me by a colleague working in Extreme Value Theory. I tried to bring an answer to the question, but I seize the opportunity to make a round-up on similar questions.

Let C be an arbitrary bi-dimensional copula and let (U,E) be a bi-dimensional random vector of copula C where U and E respectively follow a uniform distribution on $(0,1)$ denoted by $U \sim \boldsymbol{U}(0,1)$ and a standard exponential distribution, denoted by $E \sim \boldsymbol{E}(1)$. By a Kolmogorov Theorem there exists a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ holding an infinite sequence of independent copies of (U,E) denoted (U_n, E_n), $n \geq 1$ (See for example Lo (2018a), Chapter 9, problem 2, p.313). Let us denote for any $n \geq 1$, the ordered statistics by $U_{1,n} \leq ... \leq U_{1,n}$ based on the sample of U_1 ... U_n of size n. As well, let us denote : $S_n = E_1 + ... + E_n$, $n \geq 1$ and $S_0 = 0$.

Let $k = k(n)$ be a sequence of positive integers such that

$$\forall n \geq 1, 1 \leq k = k(n) \leq n \ and \ k = k(n) \longrightarrow +\infty \ a.s. \ n \longrightarrow +\infty. \ (K)$$

The question which is asked is: do we have

$$(n/k(n))\, U_{k(n),n} \longrightarrow 1, \ a.s. \ as \ n \longrightarrow +\infty \ ?$$

One would think that the following argument should lead to a positive answer. For one side, if we have, for each $n \geq 1$, an array $E_1^{(n)}$, $E_2^{(n)}$,..., $E_{n+1}^{(n)}$ of $n+1$ independent and standard exponentially distributed random variables defined on the same probability space and we denote $S_j^{(n)} = E_1^{(n)} + E_2^{(n)} + ... + E_j^{(n)}$, $1 \leq j \leq n+1$, then we have the following representation

$$\text{(1.1)} \qquad \forall n \geq 1, \left(U_{1,n}, \cdots, U_{k(n),n}, \cdots, U_{n,n}\right) =_d \left(\frac{S_1^{(n)}}{S_{n+1}^{(n)}}, \cdots, \frac{S_{k(n)}^{(n)}}{S_{n+1}^{(n)}}, \cdots, \frac{S_n^{(n)}}{S_{n+1}^{(n)}}\right),$$

where $=_d$ stands for the equality in distribution. On the other side, if we have an infinite sequence $(E_n)_{n\geq 1}$ of independent and standard exponentially distributed random variables defined the same probability space and if we denote $S_0 = 0$, $S_n = E_1 + \cdots + E_n$, $n \geq 0$, we have

$$\text{(1.2)} \quad \forall n \geq 1, \left(\frac{S_1}{S_{n+1}}, \cdots, \frac{S_{k(n)}}{S_{n+1}}, \cdots, \frac{S_n}{S_{n+1}}\right) =_d \left(\frac{S_1^{(n)}}{S_{n+1}^{(n)}}, \cdots, \frac{S_{k(n)}^{(n)}}{S_{n+1}^{(n)}}, \cdots, \frac{S_n^{(n)}}{S_{n+1}^{(n)}}\right),$$

and by the Kolmogorov's strong law of large numbers for a sequence of independent and identically distributed and centered random variables, $S_n/n \to 0$ *a.s.* and hence

$$\frac{S_n}{S_{k(n)}} \times \frac{k(n)}{n} \to 1, \ a.s.$$

(See Loève (1997), Statement B, page 251 or Lo (2018a), Chapter 7, Theorem 17, page 234). Yet the argument based on the two equalities (1.1) and (1.2) is not enough to imply that

$$\frac{S_n^{(n)}}{S_{k(n)}^{(n)}} \times \frac{k(n)}{n} \to 1, \ a.s.$$

So that argument given above is misleading. We have to go in the methods of the available proofs to answer the question. Indeed we will see the answer is positive.

But more generally, we wish to take the opportunity to make a round-up on topic of the preservation of types of limits when we change a sequence of random variables with marginal and partial equalities in distribution. And since we want to have a self-reading document, we provide the proof of the representation (1.1) in Part C of the Appendix, as in Lo *et al.* (2018b), which in turn is an adaptation of proofs in Shorack and Wellner (1986).

The rest of the paper is organized as follows. In Section 2, we deal with general conditions for the preservation of types of limits under distributional equalities between sequences of random variables. In Section 2, we deal with other conditions for preservation of the *a.s.* limits. In Section 5, we deal with the question we described in the first page of the paper and give a positive answer. We conclude the paper by a section of concluding remarks. Finally, the proof of Formula (1.1) and other important results are provided in order to make the paper self-contained.

2. General conditions for validity of types of limits after a marginal change of the sequence

Let us proceed by steps.

2.1. Weak Convergence. Let us begin by defining $C_b(\mathbb{R})$, the class of real-valued continuous and bounded functions defined on $\mathbb{R}$.

By definition, a sequence of random variables $\{X_1, X_2, ...\}$ defined on arbitrary probability spaces $(\Omega_n, \mathcal{A}_n, \mathbb{P}_n)$, $n \geq 1$, converges weakly or vaguely to a random variable X_∞ defined on some probability space $(\Omega_\infty, \mathcal{A}_\infty, \mathbb{P}_\infty)$ if and only if

$$\forall f \in C_b(\mathbb{R}), \int f d\mathbb{P}_n X_n^{-1} \longrightarrow \int f d\mathbb{P}_\infty X_\infty^{-1} \tag{2.1}$$

The name of vague or weak convergences comes from that the definition (2.1) uses only the marginal probability laws $\mathbb{P}_{X_n} = \mathbb{P}_n X_n^{-1}$, $n \geq 1$ of the elements of the sequence and the probability law of the limit $\mathbb{P}_{X_\infty} = \mathbb{P}_\infty X_\infty^{-1}$, regardless of the spaces on which the random variables are defined.

Rule 01. Here, it is clear and straightforward that two sequences of real-valued random variables $\{X_1, X_2, ...\}$ and $\{Y_1, Y_2, ...\}$ with equal margins in distribution, that is

$$\forall n \geq 1, \ \mathbb{P}_{X_n} = \mathbb{P}_{Y_n} \tag{2.2}$$

or equivalently

$$\forall n \geq 1, X_n =_d Y_n \tag{2.3}$$

do have the same weak limit (uniquely determined in distribution) or fail to convergence weakly at the same time.

2.2. Convergence in Probability. By definition, a sequence of random variables $\{X_1, X_2, ...\}$ defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathbb{P}$-a.s. finite, converges in probability to a $\mathbb{P}$-a.s. finite random variable X_∞ defined on the same probability space, if and only if

$$\forall \varepsilon > 0,\ \mathbb{P}\left(|X_n - X_\infty| > \varepsilon\right) \longrightarrow 0, n \longrightarrow +\infty. \tag{2.4}$$

Let us denote $B_2(\varepsilon) = \{(x,y) \in \mathbb{R}^2 : |x-y| > \varepsilon\} \subset \mathbb{R}^2$ and the closed ball $B\left(a, \varepsilon\right)$ centered at $a \in \mathbb{R}^2$ and radius $\varepsilon > 0$. Condition (2.4) becomes

$$\forall \varepsilon > 0,\ \mathbb{P}_{X_n, X_\infty}\left(B_2(\varepsilon)\right) \longrightarrow 0, n \longrightarrow +\infty. \tag{2.5}$$

and is, when $X_\infty = a$ is real number,

$$\forall \varepsilon > 0,\ \mathbb{P}_{X_n, X_\infty}\left(B(a, \varepsilon)^c\right) \longrightarrow 0, n \longrightarrow +\infty \tag{2.6}$$

Rule 02. From formulas (2.5) and (2.6), we infer the following conclusion. If two sequences of random variables $\{X_1, X_2, ...\}$ and $\{Y_1, Y_2, ...\}$ which are such that

$$\forall n \geq 1,\ (X_n, X_\infty) =_d \left(Y_n, Y_\infty\right), \tag{2.7}$$

do have the same limit in probability X_∞ (uniquely determined in distribution) or fail to convergence in probability at the same time.

If $X_\infty = a$, based on Formula (2.6), two sequences of random variables $\{X_1, X_2, ...\}$ and $\{Y_1, Y_2, ...\}$ with equal margins in distribution, do have the same limit in probability to a real number or fail to convergence in probability to a real number.

2.3. Convergence almost sure (a.s.). Let us keep it simple by focusing on almost sure convergence to real numbers. By definition, a sequence of random variables $\{X_1, X_2, ...\}$ defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathbb{P}$-a.s. finite, converges in probability to a real number $a \in \mathbb{R}$ if and only if

$$\forall \varepsilon > 0,\ \mathbb{P}\left(\bigcap_{n\geq 1} \bigcup_{p \geq n} (|X_p - a| > \varepsilon)\right) = 0 \tag{2.8}$$

Let us denote the following subset of $\mathbb{R}^{\mathbb{N}^*}$, where $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$ and $\varepsilon > 0$,

$$B_\infty(a,\varepsilon) = \{x = (x_p)_{p\geq 1} \in \mathbb{R}^{\mathbb{N}^*} : \forall n \geq 1, \exists p \geq n, |X_p - a| > \varepsilon\}.$$

Let us consider the infinite product space $\left(\mathbb{R}^{\mathbb{N}^*}, B(\mathbb{R})^{\otimes \mathbb{N}^*}\right)$ which is metrizable with the metric defined for $x = (x_p) \in \mathbb{R}^{\mathbb{N}^*}$ and $y = (y_p) \in \mathbb{R}^{\mathbb{N}^*}$:

$$d(x,y) = \sum_{p\geq 1} 2^{-p} \frac{|x_p - y_p|}{1 + |x_p - y_p|}.$$

The measurable space $\left(\mathbb{R}^{\mathbb{N}^*}, B(\mathbb{R})^{\otimes \mathbb{N}^*}\right)$ is the Borel space associated with $\left(\mathbb{R}^{\mathbb{N}^*}, d\right)$. This metric space is Polish space, that is a separable and complete space. By Kolmogorov's Theorem, a probability measure $\mathbb{Q}$ on $\left(\mathbb{R}^{\mathbb{N}^*}, B(\mathbb{R})^{\otimes \mathbb{N}^*}\right)$ is characterized by its finite marginal distributions

$$\mathbb{Q}_{(1,\dots,n)} = \mathbb{Q}\pi^{-1}_{(1,\dots,n)}, \ n \geq 1$$

where for each $n \geq 1$, $\pi^{-1}_{(1,\dots,n)}$ is the pojection defined by

$$x = (x_p) \in \mathbb{R}^{\mathbb{N}^*} \mapsto \pi_{(1,\dots,n)}(x) = (x_1, \dots, x_n)^t$$

and where A^t stands for the transpose of a matrix A. This implies that the probability law $\mathbb{P}_X$ of $X = (X_p)_{p\geq 1}$, when considered as a stochastic process $X \in \mathbb{R}^{\mathbb{N}^*}$, is characterized by its finite marginal probability laws

$$\mathbb{P}_{(X_1,\dots,X_n)}, n \geq 1.$$

Rephrasing Formula (2.8), the a.s. convergence of $\{X_1, X_2 \dots\}$ to $a \in \mathbb{R}$ holds if and only if

$$\forall \varepsilon > 0, \ \mathbb{P}_X(B_\infty(a,\varepsilon)) = 0 \tag{2.9}$$

Based on the elements given above, we get the following

Rule 03. If two sequences of real random variables $\{X_1, X_2 \dots\}$ and $\{Y_1, Y_2 \dots\}$ have the same law as stochastic processes, that is they have the same finite marginal laws, they converge a.s. at the same time to a real number or fail both to converge a.s. to a real number.

2.4. Conclusion. We conclude as follows:

(a) The equality in distribution of individual marginal laws between two sequences of real random variables is not enough to preserve the a.s. limit to a real number of one them when this one is replced by the other.

(b) If two infinity sequences $\{X_1, X_2...\}$ and $\{Y_1, Y_2...\}$, the elements of each of them being defined on the same probability space, have the same finite marginal laws converge or not to a same real simultaneously.

(c) The letter point does not cover equality in distribution between arrays, as for example, in the case

$$\forall n \geq 1, \ \left(S_1^{(n)}, ..., S_{n+1}^{(n)}\right) =_d (S_1, ..., S_{n+1}) \tag{2.10}$$

However, Formula implies

$$\forall n \geq 1, \ (U_{1,n}, ..., U_{n,n}) \ =_d \ \left(\frac{S_1^{(n)}}{S_{n+1}^{(n)}}, ..., \frac{S_n^{(n)}}{S_{n+1}^{(n)}}\right) \tag{2.11}$$

$$=_d \ \left(\frac{S_1}{S_{n+1}}, ..., \frac{S_n}{S_{n+1}}\right) \tag{2.12}$$

Which ensures that

$$\forall n \geq 1, (n/k(n))\, U_{k(n),n} =_d \left(\frac{n+1}{n}\right)\left(\frac{S_{n+1}}{n+1}\right)\left(\frac{S_{k(n)}}{k(n)}\right).$$

By *Rule 02* in Subsection and by the classical operations on convergences in probability to real numlbers, we have

$$\forall n \geq 1, (n/k(n))\, U_{k(n),n} \longrightarrow_{\mathbb{P}} 1 \ asn \longrightarrow +\infty.$$

Fortunately, the a.s. limit preservation does not necessarily need the equality of finite marginal laws as we are going to see it soon.

3. Other sufficient conditions for conservation of the a.s. limits

In that section, we review two general results and a particular one of a.s. convergences which are stable under partial equality in distribution.
(1) **a.s. Convergence based on Part I of Borel- Cantelli's Lemma.**

Suppose that the a.s. convergence of the sequence $\{X_1, X_2, ...\}$ to a real number a is the result of the first part of the classical Borel-Cantelli in the form: there exists a sequence of non-negative real numbers $(r_n)_{n\geq 1}$ such that $r_n \longrightarrow 0$ as $n \longrightarrow +\infty$ and

$$\sum_{n\geq 1} \mathbb{P}\left(|X_n - a| \geq r_n\right) < +\infty.$$

It follows that any other sequence $\{Y_1, Y_2, ...\}$ whose individual margins are equal to those of $\{X_1, X_2, ...\}$ converges to a a.s..

We have to remind that the result

$$\mathbb{P}\left(\lim_{n\rightarrow+\infty} inf\, |X_n - a| < r_n\right) = 1$$

is derived from

$$\mathbb{P}\left(\lim_{n\rightarrow+\infty} inf\, |X_n - a| < r_n\right) \geq 1 - \lim_{p\rightarrow+\infty} \sum_{p\geq 1} \mathbb{P}\left(|X_p - a| < r_p\right) = 0,$$

Whose right-hand member is unchanged by a change of the sequence with equal distribution of individual margins.

(2) **Special case of sums of independent random variables with constant means and variances.**

Let X_1,X_2, ... be a sequence of independent random variables, defined on the same probability space with mean zero and variance one and let us define $S_n^* = X_1 + ... + X_n,\ n \geq 1$. The method of perfect squares for proving that

$$\frac{S_n^*}{n} \longrightarrow 0,\ asn \longrightarrow +\infty,$$

is based on the two following principles. First, we have, for $0 < \beta < \frac{1}{2}$,

$$\sum_{n\geq 1} \mathbb{P}\left(\left|\frac{S_{n^2}^*}{n^2}\right| \geq n^{-\beta}\right) \leq \sum n^{2(\beta)-1} < \infty.$$

which ensures that $S_{\ell(n)}/\ell(n)$ converges to 0 a.s. when $(\ell(n))_{n\geq 1}$ is a subsequence of the sequence of square integers $(n^2)_{n\geq 1}$ Secondly, in the general case, we have

$$\forall n \geq 1,\ \exists m \geq 0,\ m(n)^2 \leq n \leq (m(n)+1)^2.$$

By denoting this unique integer $m(n)^2$ by $\ell(n)$, we get

$$\forall n \geq 1,\ \exists m \geq 0,\ \ell(n) \leq n \leq \left(\sqrt{\ell(n)}+1\right)^2.$$

and we have the second principle in

$$\sum_{n\geq 1} \mathbb{P}\left(\left|\frac{1}{n}\left(S_n^* - S_{\ell(n)}^*\right)\right| > n^{-\beta}\right) \leq 3\sum_{n\geq 1} n^{-(\frac{3}{2}-2\beta)} < \infty,$$

Whenever $\beta < 1/4$. When combined, the two principles give for $\beta = 1/8$,

$$\begin{aligned}
\sum_{n\geq 1} \mathbb{P}\left(\left|\frac{S_n^*}{n}\right| > 2n^{-1/8}\right) &\leq \sum_{n\geq 1} \mathbb{P}\left(\frac{\ell(n)}{n}\left|\frac{S_{\ell(n)}}{\ell(n)}\right| > 2n^{-1/8}\right)\\
&+ \sum_{n\geq 1} \mathbb{P}\left(\left|\frac{S_n - S_{\ell(n)}}{n}\right| > n^{-1/8}\right)\\
&\leq \sum_{n\geq 1} \ell(n)^{-7/4}\left(1+\ell(n)^{-1}+2\ell(n)^{-1/2}\right)^{1/4}\\
&+ 3\sum_{n\geq 1} n^{-5/4} < +\infty, \qquad (3.1)
\end{aligned}$$

that is

$$\sum_{n\geq 1} \mathbb{P}\left(\left|\frac{S_n^*}{n}\right| > 2n^{-1/8}\right) < +\infty. \qquad (3.2)$$

If we apply this result to $X_n = (E_n - 1)$ and $S_n^* = S_n - n$, $n \geq 1$, we have

$$\frac{S_n^{(n)}}{n} \longrightarrow 1 a.s.\ as\ n \longrightarrow +\infty.$$

(3) **Hájèk-Rényi strong law of large numbers.**

Let us remind that this law does not require independence of the variables whose partial sums are considered and generalize the Kolmogorov strong law of large numbers which in turn include law (3.1) as a special case.

The Hájèk-Rényi strong law of large numbers is stated as follows.

THEOREM 65. *Let X_1,X_2,..., be a sequence of square integrable random variables with mean zero and $S_n^* = X_1 + ... + X_n$, $n \geq 1$.*
Let a_n and b_n two sequences of real numbers and let $r > 0$. Let us suppose that $(b_n)_{n\geq 1}$ is a increasing and non-bounded sequence of positive real numbers. Suppose that

$$\sum_n \frac{a_n}{b_n} < +\infty$$

and there exists $C > 0$ such that for all $n \in \mathbb{N}$ and for all $\varepsilon > 0$

$$\mathbb{P}\left(\max_{m\leq n} |S_m^*| \geq \varepsilon\right) \leq C\varepsilon^{-r} \sum_{m\geq n} a_m.$$

Then we have

$$\lim_{n\rightarrow+\infty} \frac{S_n}{b_n} = 0 \ a.s.$$

The proof of this important general strong law of large numbers is given in Appendix (B). We can see there that the a.s. convergence is based of the following fact

$$\mathbb{P}\left(A_{m,n}\right) < C\varepsilon^{-r} \sum_{k=1}^{n} a_k \beta_k^{-r}$$

where, for all $n \geq 1$, we have

$$A_{m,n} = \{\sup_{k\leq n} \left|\frac{S_k^*}{\beta_k}\right| > \varepsilon_m\}$$

As a result, when applied to $X_n = (E_n - 1)$ and $S_n^* = S_n - n$, $a_n = Var\left(X_n\right)$, $b_n = n$, $n \geq 1$, we see again that anything remains unchanged when, for each $n \geq 1$, $\{S_1^*, S_n^*\}$ is replaced by $\{S_1^{(n)}, S_n^{(n)}\}$.

Here again, it follows that

$$\frac{S_n^{(n)}}{n} \longrightarrow 1, a.s. \ as \ n \ \longrightarrow +\infty.$$

Partial conclusion. We keep reminding that the strong law of large numbers for independent random variable with common finite mean and variance and the Hájèk-Rényi strong law of large numbers remain valid for sequences $(S_{nn})_{n\geq 1}$, when each S_{nn} is the total sum of arrays $\{X_{1n}, ..., X_{nn}\}$ under the specific conditions of each array $(X_{kn})_{1\leq k\leq n}$.

4. Main Problem

Let us consider for each $n \geq 1$

$$\{U^*_{j,n}, 1 \leq j \leq n\} = \{\frac{n+1}{j}\left(\frac{S_j}{S_{n+1}} - \frac{j}{n+1}\right), 1 \leq j \leq n\}$$

We have for $n \geq 1$

$$U^*_{k(n),n} = \frac{(n-k+1)\, S_k - k\,(X_{k+1} + ... + X_{n+1})}{k(n) S_{n+1}},$$

$$\mathbb{E}U^*_{k(n),n} = 0 \text{ and } \mathbb{V}\left(U^*_{k(n),n}\right) = \tfrac{n+1-k(n)}{k(n)(n+1)} \leq \tfrac{1}{k(n)}.$$

Denote for $n \geq 1$,

$$U^{1*}_{k(n),n} = \frac{(n-k+1)\, S_k - k\,(X_1 + ... + X_{n+1})}{k(n)(n+1)},$$

It follows from Formulas (3.2) that

$$\sum_{n\geq 1} \mathbb{P}\left(\left|\frac{S_{n+1}}{n+1} - 1\right| > (n+1)^{-1/8}\right) < +\infty.$$

Let $n_1 \geq 1$ such for all $n \geq n_1$, $(n+1)^{-1/8} \leq 1/2$. We have for $n \geq n_1$,

$$\begin{aligned}
\sum_{n\geq n_1} \mathbb{P}\left(\left|U^*_{k(n),n}\right| > k(n)^{-1/8}\right) &\leq \sum_{n\geq n_1} \mathbb{P}\left(\left|\frac{S_{n+1}}{n+1} - 1\right| > (n+1)^{-1/8}\right)\\
&+ \sum_{n\geq n_1} \mathbb{P}\left(\left|U^{1*}_{k(n),n}\right| > k(n)^{-1/8}/2\right)\\
&\leq \sum_{n\geq n_1} \mathbb{P}\left(\left|\frac{S_{n+1}}{n+1} - 1\right| > (n+1)^{-1/8}\right)\\
&+ 4\sum_{n\geq n_1} k(n)^{-3/4} < +\infty,
\end{aligned}$$

whenever $(k(n))_{n\geq 1}$ is a sub-sequence of square integer numbers $(n^2)_{n\geq 1}$ such that

$$\forall n \geq 1, \ell(n) \leq k(n)\ell(n) + 2\sqrt{\ell(n)} + 1.$$

We have for all $n \geq 1$,

$$U^{1*}_{k(n),n} = \frac{\ell(n)}{k(n)}\frac{n+1}{\ell(n)}\left(\frac{S_{\ell(n)}}{S_{n+1}} - \frac{\ell(n)}{n+1}\right) + \frac{1}{S_{n+1}/(n+1)}\frac{S_{k(n)} - S_{\ell(n)}}{k(n)} - \left(1 - \frac{\ell(n)}{k(n)}\right)$$

and next for

$$U^{2*}_{k(n),n} = \left|\left(\frac{S_{\ell(n)}}{S_{n+1}} - \frac{\ell(n)}{n+1}\right)\right| + \frac{1}{|S_{n+1}/(n+1)|}\frac{\left|S_{k(n)} - S_{\ell(n)}\right|}{k(n)} + 3k(n)^{-1}\ell(n)^{1/2},$$

we get

$$\left|U^{1*}_{k(n),n}\right| \leq U^{2*}_{k(n),n}$$

Since for all $n \geq 1$,

$$\mathbb{E}\left(\frac{S_{k(n)} - S_{\ell(n)}}{k(n)}\right) = 0$$

and

$$\mathbb{E}\left(\frac{S_{k(n)} - S_{\ell(n)}}{k(n)}\right)^2 = \frac{k(n) - \ell(n)}{k(n)^2} \leq 4\ell(n)^{-3/2},$$

we have

$$\mathbb{P}\left(\frac{\left|S_{k(n)} - S_{\ell(n)}\right|}{k(n)} > \ell(n)^{-1/8}\right) \leq 4\ell(n)^{-5/4},$$

and next for n_2 such that $\ell(n)^{-1/8} < 3k(n)^{-1}\ell(n)^{1/2} \sim 3\ell(n)^{-1/2}$ for all $n \geq n_2$, we have for $n_3 = \max(n_1, n_2)$,

$$\begin{aligned}
\sum_{n\geq n_3} \mathbb{P}\left(U^{2*}_{k(n),n} > 4\ell(n)^{-1/8}\right) &\leq \sum_{n\geq n_3} \mathbb{P}\left(\left|U^{*}_{\ell(n),n}\right| > \ell(n)^{-1/8}\right) \\
&+ \sum_{n\geq n_3} \mathbb{P}\left(\left|\frac{S_{n+1}}{n+1} - 1\right| > (n+1)^{-1/8}\right) \\
&+ \sum_{n\geq n_3} \mathbb{P}\left(\frac{\left|S_{k(n)} - S_{\ell(n)}\right|}{k(n)} > \ell(n)^{-1/8}\right) \\
&< +\infty
\end{aligned}$$

We have

$$\sum_{n\geq 1}\mathbb{P}\left(\left|U^{*}_{k(n),n}\right| > 4\ell(n)^{-1/8}\right) < +\infty. \tag{4.1}$$

We conclude from the equality in distribution (2.4) which implies

$$\forall n\geq 1, \{U^{*}_{j,n}, 1\leq j\leq n\} =_d \{\frac{n+1}{j}U_{j,n}-1, 1\leq j\leq n\};$$

and Formula (4.1) together establish that

$$\frac{(n+1)U_{k,n}}{k(n)} \longrightarrow 1, a.s.\ as\ n \longrightarrow +\infty$$

whenever (K) holds.

5. Appendix

(A) Full Proof of Theorem. To keep the notations easy, let us drop $*$ in the partial sums. We set

$$Y_n = S_{n^2}/n^2,\ n \geq 1,$$

that is, we only consider the elements of the sequence $(S_k/k)_{k\geq 1}$ corresponding to a square index $k = n^2$. Remark that $\mathbb{V}ar\,(Y_n) = n^{-2}$, $n \geq 2$. Fix $0 < \beta < 1/2$. By Chebychev's inequality, we have

$$\mathbb{P}\left(|Y_n| > n^{-\beta}\right) \leq n^{2(1-\beta)}$$

and thus,

$$\sum_n \mathbb{P}\left(|Y_n| > n^{-\beta}\right) \leq \sum_n n^{2(1-\beta)} < \infty.$$

By Borel-Cantelli'sLemma, we conclude that

$$\mathbb{P}\left(\lim_{n\to+\infty} inf\, |Y_n| \leq n^{-\beta}\right) = 1.$$

Let us remind that

$$\Omega_0 = \lim_{n\to+\infty} inf\left(|Y_n| \leq n^{-\beta}\right) = \bigcup_{n\geq 0}\bigcap_{r\geq n}\left(|Y_r| \leq r^{-\beta}\right).$$

Hence, for all $\omega \in \Omega_0$, there exists $n(\omega) \geq 0$ such that for any $r \geq n$,

$$|Y_r| \leq r^{-\beta}.$$

By the sandwich's rule, we conclude that, for any $\omega \in \Omega_0$, we have

$$Y_m(\omega) \longrightarrow 0.$$

This means that

$$\Omega_0 \subset (Y_n \longrightarrow 0).$$

We conclude that $\mathbb{P}\,(Y_n \longrightarrow 0) = 1$ and hence $Y_n \longrightarrow 0, a.s.$

To extend this result to the whole sequence, we use the decomposition of $\mathbb{N}$ by segments with perfect squares bounds. We have

$$\forall\,(n \geq 0)\,, \exists m \geq 0, m^2 \leq n \leq (m+1)^2\,.$$

By denoting this unique integer m^2 by $k(n)$, we get

$$\forall(n \geq 1), k(n) \leq n \leq \left(\sqrt{k(n)}+1\right)^2.$$

We have

$$\mathbb{E}\left(\frac{1}{n}\left(S_n - S_{k(n)}\right)\right) = 0$$

and

$$\mathbb{V}ar\left(\frac{1}{n}\left(S_n - S_{k(n)}\right)\right) = \frac{1}{n^2}\mathbb{E}\sum_{i=k(n)+1}^{n} X_i^2 \leq \frac{1}{n^2}\left(2\sqrt{k(n)}+1\right) \leq 3n^{-3/2}.$$

Hence,

$$\sum_n \mathbb{P}\left(\left|\frac{1}{n}\left(S_n - S_{k(n)}\right)\right| > n^{-\beta}\right) \leq 3\sum_n n^{-(\frac{3}{2}-2\beta)} < +\infty$$

whenever $\beta < 3/4$. We conclude as previously that

$$\frac{1}{n}\left(S_n - S_{k(n)}\right) \longrightarrow 0 a.s.$$

Finally we have

$$\frac{S_n}{n} = \frac{S_n - S_{k(n)}}{n} + \frac{S_{k(n)}}{k(n)} \times \frac{K(n)}{n} \longrightarrow 0 \ a.s.$$

since

$$1 \leq \frac{n}{k(n)} < 1 + \frac{2}{\sqrt{k(n)}} + \frac{1}{k(n)}$$

and

$$\frac{k(n)}{n} \longrightarrow 1.$$

We just finished to prove that

$$\frac{S_n}{n} \longrightarrow 0 \ a.s.\blacksquare$$

(B) Arbitrary Kolmogorov-Hájèk - Rényi Strong Law of large Numbers. We are going to state the Hájèk - Rényi theorem whose proof is based on the Fazekas and Klesov (1998)'s Lemma which follows.

THEOREM 66. *Let $r>0$ be a positive real number. Let a_n and b_n be two sequence of positive real numbers such that the sequence $(b_n)_{n\geq 1}$ is increasing and unbounded and*

$$\sum_n \frac{a_n}{b_n} < +\infty$$

and there exists $C>0$ such that for $n \in \mathbb{N}$ and for all $\varepsilon > 0$

$$\mathbb{P}\left(\max_{m\leq n} |S_m| \geq \varepsilon\right) \leq C\varepsilon^{-r} \sum_{m\geq n} a_m.$$

Then we have

$$\lim_{n\to+\infty} \frac{S_n}{b_n} = 0 \ a.s.$$

Now let us state the Fazekas and Klesov (1998)'s lemma.

LEMMA 46. *Let $r>0$ be a positive real number. Let a_n and b_n be two sequence of positive real numbers such that the sequence $(b_n)_{n\geq 1}$ is increasing and unbounded. Suppose that*

$$\sum_n \frac{a_n}{b_n} < +\infty.$$

Then there exists an increasing and unbounded sequence β_n of positive real number such that

$$\lim_{n\to+\infty} \frac{\beta_n}{b_n} = 0 \ et \ \sum_n \frac{a_n}{\beta_n^r} < +\infty.$$

Proof of Theorem. By Lemma 4, there exists β_n as in the cited lemma for which

$$\lim_{n\to+\infty} \frac{\beta_n}{b_n} = 0 \ and \ \sum_n \frac{a_n}{\beta_n^r} < +\infty. \tag{5.1}$$

Hence, we get

$$\mathbb{P}\left(\max_{n\leq m} |S_n|\, \beta_n^{-1} \geq \varepsilon\right) \leq C\varepsilon^{-r} \sum_{n\leq m} a_n \beta_n^{-r}.$$

Let $(\varepsilon_m)_{m\geq 1}$ be an increasing and unbounded sequence of positive real numbers. Let us set

$$A_m = \{\sup_k \left|\frac{S_k}{\beta_k}\right| > \varepsilon_m\}$$

and

$$A_{m,n} = \{\sup_{k \leq n} \left|\frac{S_k}{\beta_k}\right| > \varepsilon_m\}$$

We remark (1) that $A_{m,n} \subset A_m$, (2) next $A_{m,n} \subset A_{m,n+1}$, for all $n \geq 1$ and (3) finally, $\bigcup_{n \geq 1} A_{m,n} \subset A_m$. We also have

$$\sup_{k \leq n} \left|\frac{S_k}{\beta_k}\right| \nearrow \sup_k \left|\frac{S_k}{\beta_k}\right|,$$

which implies

$$\omega \in A_m \leftrightarrow \sup_k \left|\frac{S_k}{\beta_k}\right|(\omega) > \varepsilon_m$$

Hence there exists n_0 such that

$$\sup_{k \leq n_0} \left|\frac{S_k}{\beta_k}\right|(\omega) > \varepsilon_m$$

and thus, $\omega \in A_{m,n_0}$ and next $A_m \subset \bigcap_{n \geq n_0} A_{m,n}$. By combining the remarks above, we get

$$A_{m,n} \uparrow A_m \ as \uparrow +\infty.$$

Now for $m \geq 1$ fixed, we have

$$\mathbb{P}\left(A_{m,n}\right) < C\varepsilon_m^{-r} \sum_{k=1}^{n} a_k \beta_k^{-r}$$

and by using the continuity of the probability measure, we get

$$\lim_{n \to +\infty} \mathbb{P}\left(A_{m,n}\right) < C\varepsilon_m^{-r} \sum_{k=1}^{+\infty} a_k \beta_k^{-r}.$$

Since we have $\mathbb{P}\left(A_m\right) = \lim_{n \to +\infty} \mathbb{P}\left(A_{m,n}\right)$ for all $m \geq 1$, we conclude for all $m \geq 1$ that

$$\mathbb{P}\left(\sup_{k \leq n} \left|\frac{S_k}{\beta_k}\right| > \varepsilon_m\right) \leq C\varepsilon_m^{-r} \sum_{k=1}^{+\infty} a_k \beta_k^{-r}.$$

From this and from the first part of formula (5.1), we derive

$$\lim_{n\rightarrow+\infty} \mathbb{P}\left(\sup_{k\leq n}\left|\frac{S_k}{\beta_k}\right| > \varepsilon_m\right) = 0. \tag{5.2}$$

Let us prove that we have

$$\sup_{k\leq n}\left|\frac{S_k}{\beta_k}\right| < +\infty \ a.s.$$

Indeed, let us set

$$\Omega_0 = \{\sup_{k\leq n}\left|\frac{S_k}{\beta_k}\right| = +\infty\}.$$

We remark that for all $\omega \in \Omega_0$ and for $m \geq 1$, we have

$$\omega \in \sup_{k\leq n}\left|\frac{S_k}{\beta_k}\right| > \varepsilon_m.$$

Hence $\omega \in A_m$ for all $m \geq 1$ and next

$$\mathbb{P}\left(\Omega_0\right) < \mathbb{P}\left(A_m\right),$$

for all $m \geq 1$. By applying Formula (5.2) to the right-hand member, we obtain, as $m \longrightarrow +\infty$

$$\mathbb{P}\left(\Omega_0\right) = 0.$$

Next for all $\omega \in \Omega_0^c$

$$\begin{aligned}\left|\frac{S_k}{b_k}\right|(\omega) &= \left|\frac{S_k}{\beta_k}\right|(\omega) \times \frac{\beta_k}{b_k}\\ &\leq \frac{\beta_k}{b_k} \times \sup_k \left|\frac{S_k}{\beta_k}\right|\end{aligned}$$

This, by the second Formula (5.1), we get that for all $\omega \in \Omega_0^c$,

$$\left|\frac{S_k}{b_k}\right|(\omega) \longrightarrow 0.$$

Hence,

$$\mathbb{P}\left(\left|\frac{S_k}{b_k} \nrightarrow 0\right|\right) \leq \mathbb{P}\left(\Omega_0\right).$$

We conclude that

$$\frac{S_k}{b_k} \quad \longrightarrow^{a.s} \quad 0.$$

(C) Around Malmquist's Representation.

This section is intended to provide representations of order statistics $X_{1,n} \leq X_{2,n} \leq ... \leq X_{n,n}$, $n \geq 1$, of any n independent random variables $X_1, ..., X_n$ with common distribution function F in that of standard uniform or exponential independent random variables.

We remind again that in this section, all the random variables are defined on the same probability space $(\Omega, A, \mathbb{P})$.

We begin by recalling the density probability function of the order statistics from a density probability function h.

5.1. Density of the order statistics.

Let us begin with this lemma.

LEMMA 47. *Let $Z_1, Z_2, ..., Z_n$ be n independent copies of an absolutely continuous random variable Z of probability density function h and probability distribution function H, defined on the same probability space $(\Omega, A, \mathbb{P})$. Let $1 \leq r \leq n$, $1 \leq n_1 < n_2 < ... < n_r$. Then the r order statistics $Z_{n_1,n} < Z_{n_2,n} < ... < Z_{n_r,n}$ have the joint probability density function in $(z_1, ..., z_r)$,*

$$n! \prod_{j=1}^{r+1} \frac{h(z_j)\,(F(z_j) - F(z_j))^{n_j - n_{j-1} - 1}}{(n_j - n_{j-1} - 1)!} 1_{(z_1 < ... < z_r)}, \tag{5.3}$$

with by convention $n_0 = 0$ and $n_r = n + 1$, $z_0 = -\infty$ and $z_{r+1} = +\infty$.

Proof. Suppose that the assumptions of the proposition holds. Let us find the joint density probability functions of r order statistics $Z_{n_1,n} < Z_{n_2,n} < ... < Z_{n_r,n}$, with $1 \leq r \leq n$, $1 \leq n_1 < n_2 < ... < n_r$.

Since Z is an absolutely continuous random variable, the observations are distinct almost surely and we have $Z_{n_1,n} < Z_{n_2,n} < ... < Z_{n_r,n}$. Then for dz_i small enough and for $z_1 < z_2 < ... < z_r$, the event

$$(Z_{n_i,n} \in \,]z_i - dz_i/2, z_i + dz_i/2[\,, 1 \leq i \leq r)$$

A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia. **Lo G.S. (2018). A Simple Round-up on the Validity of a.s Convergence after Partial Modification of the Probability Law and Application**. Pages 437 — 465.

occurs with $n_1 - 1$ observations of the sample $Z_1, ..., Z_n$ falling at left of z_1, one point in $]z_1 - dz_1/2, z_1 + dz_1/2[$, n_2-n_1-1 between $z_1+dz_1/2$ and $z_2-dz_2/2$, one point in $]z_2 - dz_2/2, z_2 + dz_2/2[$, etc and $n - k_k$ points at right of z_r.

This is illustrated in Figure 1 for $r = 3$.

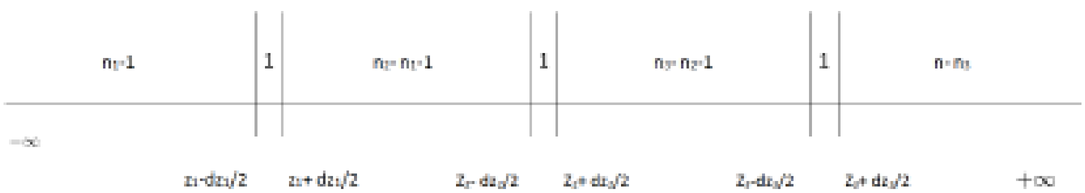

FIGURE 1.How are placed the observations with respect to $z_1 < ... < z_r$. How are placed the observations with respect to $z_1 < ... < z_r$

.

By definition, the probability density function $f_{(Z_{n_1,n}, Z_{n_2,n}, ..., Z_{n_r,n})}$, whenever it exists, satisfies

$$\begin{aligned} & \frac{\mathbb{P}\left(Z_{n_i,n} \in \left]z_i - dz_i/2, z_i + dz_i/2\right[, 1 \leq i \leq r\right)}{dz_1 \times ... \times dz_r} \\ (5.4) \qquad = \ & f_{(Z_{n_1,n}, Z_{n_2,n}, ..., Z_{n_r,n})}\left(z_1, ..., z_r\right)\left(1 + \varepsilon(dz_1, ..., dz_r)\right), \end{aligned}$$

where $\varepsilon(dz_1, ..., dz_r) \longrightarrow 0$ as each $dz_i \longrightarrow 0$ $(1 \leq i \leq r)$. Now, by using the independence $Z_1, ..., Z_n$, $\mathbb{P}\left(Z_{n_i,n} \in \left]z_i - dz_i/2, z_i + dz_i/2\right[, 1 \leq i \leq r\right)$ is obtained as a multinomial probability. Using in addition the fact that h is the common probability density function of Z, we get

$$\begin{aligned} & \frac{\mathbb{P}\left(Z_{n_i,n} \in \left]z_i - dz_i/2, z_i + dz_i/2\right[, 1 \leq i \leq r\right)}{dz_1 \times ... \times dz_r} \\ = \quad & n! \times \frac{h(z_1)^{n_1-1}}{(n_1-1)!} \times \frac{(F(z_2)-F(z_1))^{n_2-n_1-1}}{(n_2-n_1-1)!} \\ \times ... \times \quad & \frac{(F(z_j)-F(z_{j-1}))^{n_j-n_{j-1}-1}}{(n_j-n_{j-1}-1)!} \\ \times ... \times \quad & \frac{(F(z_r)-F(z_{r-1}))^{n_r-n_{r-1}-1}}{(n_r-n_{r-1}-1)!} \\ \times \quad & \frac{(1-F(z_r))^{n-n_r}}{(n-n_r)!} \\ \times \quad & \prod_{i=1}^{r} \frac{\mathbb{P}\left(Z_{n_i,n} \in \left]z_i - dz_i/2, z_i + dz_i/2\right[\right)}{1!dz_i} \end{aligned}$$

The last factor in latter product is

$$\prod_{i=1}^{r} h(z_i)(1+dz_i).$$

By setting $n_0 = 0$ and $n_r = n+1$ and for $-\infty = z_0 < z_1 < ... < z_r < z_{r+1} = +\infty$,

$$f_{(Z_{n_1,n}, Z_{n_2,n}, ..., Z_{n_r,n})}(z_1, ..., z_r) = n! \prod_{j=1}^{r} \frac{h(z_j)\left(F(z_j)-F(z_{j-1})\right)^{n_j-n_{j-1}-1}}{(n_j-n_{j-1}-1)!},$$

we see that $f_{(Z_{n_1,n}, Z_{n_2,n}, ..., Z_{n_r,n})}$ satisfies (5.4). ■

Now, let us apply this lemma to the whole order statistics. We get this proposition.

PROPOSITION 23. *Let $Z_1, Z_2, ..., Z_n$ be n independent copies of an absolutely continuous random variable Z with common probability density function h, and defined on the same probability space $(\Omega, A, \mathbb{P})$. The associated order statistic*

$$(Z_{1,n}, Z_{2,n}, ..., Z_{n,n})$$

has the joint probability density function

$$h_{(Z_{1,n},Z_{2,n},\ldots,Z_{n,n})}(z_1,\ldots,z_n) = n!\prod_{i=1}^{n} h(z_i)1_{(z_1\leq\ldots\leq z_n)}.$$

Proof. Let us apply lemma 46 with $r = n$ and $n_1 = 1$, $n_2 = 2$, ..., $n_n = n$. Since the numbers $n_j - n_{j-1} - 1$ vanish in (5.3). It comes that $Z_{1,n} < Z_{2,n} < \ldots < Z_{n,n}$ have the joint probability density

$$n!\prod_{j=1}^{n} h(z_j)1_{(z_1<\ldots<z_r)}\blacksquare$$

Now, we are focusing on the relation between standard uniform and exponential order statistics.

PROPOSITION 24. *Let $n \geq 1$ be a fixed integer and $U_{1,n} \leq U_{2,n} \leq \ldots \leq U_{n,n}$ be the order statistics associated with $U_1, U_2, \ldots, U_n$, which are n independent random variables uniformly distributed on $(0,1)$. Let E_1, E_2, ..., E_n, E_{n+1}, $(n+1)$ independent random variables following the standard exponential law, that is*

$$\forall x \in \mathbb{R},\ \mathbb{P}(E_i \leq n) = \left(1-e^{-x}\right)1_{(x\geq 0)},\ i = 1,\ldots,n+1.$$

Let $S_j = E_1 + \ldots + E_j$, $1 \leq j \leq n+1$. Then we have the following equality in distribution

$$(U_{1,n}, U_{2,n}, \ldots, U_{n,n}) =_d \left(\frac{S_1}{S_{n+1}}, \ldots, \frac{S_n}{S_{n+1}}\right).$$

Proof. On one hand, by (1), the probability density function (pdf) of $U = (U_{1,n}, U_{2,n}, \ldots, U_{n,n})$ is given by

$$\forall (u_1,\ldots,u_n) \in \mathbb{R}^n,\ f_U(u_1,\ldots,u_n) = n!1_{(0\leq u_1\leq\ldots\leq u_n\leq 1)}.$$

We are going to find the distribution of $Z^*_{n+1} = (S_1, S_2, \ldots., S_n, S_{n+1})$ given $S_{n+1} = t$, $t > 0$. We have for $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$,

$$\begin{aligned} f^{S_{n+1}=t}_{Z^*_n}(y) &= \frac{f_{(Z^*_n,S_{n+1})}(y,t)}{f_{S_{n+1}}(t)} \\ &= \frac{f_{Z^*_n}(y,t)}{f_{S_{n+1}}(t)}1_{(0\leq y_1\leq\ldots\leq y_n\leq t)}. \end{aligned} \tag{5.5}$$

But S_{n+1} follows a gamma law of parameters $n+1$ and 1, that is $S_{n+1} \sim \gamma(n+1, 1)$, and its probability density function is

$$f_{S_{n+1}}(t) = \frac{t^n e^{-t}}{\Gamma(n+1)} 1_{(t \geq 0)} = \frac{t^n}{n!} e^{-t} 1_{(t \geq 0)}. \tag{5.6}$$

The distribution function of $(S_1, ..., S_{n+1})$ comes from the transformation

$$\begin{pmatrix} E_1 \\ . \\ . \\ . \\ . \\ . \\ E_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ -1 & 1 & & & & \\ 0 & -1 & 1 & & & \\ . & . & . & . & & \\ . & . & . & . & & \\ . & . & . & . & . & \\ . & . & . & . & -1 & 1 \end{pmatrix} \begin{pmatrix} S_1 \\ . \\ . \\ . \\ . \\ . \\ S_{n+1} \end{pmatrix}$$

Let B be the matrix on the formula above. The Jacobian determinant in absolute value is $|B| = 1$ and

$$B \begin{pmatrix} y_1 \\ . \\ . \\ . \\ . \\ . \\ y_{n+1} \end{pmatrix} = (y_1, y_2 - y_1, ..., y_{n+1} - y_n).$$

Thus, the density of $(S_1, ..., S_{n+1})$ is then given by

$$\begin{aligned} f_{Z_{n+1}^n}(y_1, ..., y_{n+1}) &= f_{(E_1,...,E_{n+1})}(B(y_1, ..., y_{n+1})) 1_{(0 \leq y_1 \leq ... \leq y_n \leq y_{n+1})} \\ &= \prod_{i=1}^{n+1} e^{-(y_i - y_i - 1)} 1_{(0 \leq y_1 \leq ... \leq y_n \leq y_{n+1})} \\ &= e^{-y_{n+1}} 1_{(0 \leq y_1 \leq ... \leq y_n \leq y_{n+1})}. \end{aligned}$$

$y_0 = 0$ by convention. Going back to (5.5) and (5.6), we get, with $y = (y_1, y_2, ..., y_n)$,

(5.7) $$f_{Z_n^*}^{S_{n+1}=t}(y) = \frac{t^n}{n!} 1_{(0 \leq y_1 \leq ... \leq y_n \leq y_{n+1})}.$$

Now, for $u = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$,

$$f^{S_{n+1}=t}_{\left(\frac{S_1}{S_{n+1}},...,\frac{S_n}{S_{n+1}}\right)}(u) = f^{S_{n+1}=t}_{\left(\frac{S_1}{t},...,\frac{S_n}{t}\right)}(u_1, u_2, ..., u_n).$$

This probability density function is obtained from (5.7) by the transform

$$(y_1, y_2, ..., y_n) = t(u_1, u_2, ..., u_n) \leftrightarrow (u_1, u_2, ..., u_n) = \frac{1}{t}(y_1, y_2, ..., y_n)$$

with Jacobian determinant t^n. Then

$$\begin{aligned} f^{S_{n+1}=t}_{\left(\frac{S_1}{S_{n+1}},...,\frac{S_n}{S_{n+1}}\right)}(u_1, u_2, ..., u_n) &= f^{S_{n+1}=t}_{Z_n^*}(t(u_1, u_2, ..., u_n)) t^n 1_{(0 \leq tu_1 \leq ... \leq tu_n \leq t)} \\ &= n! 1_{(0 \leq u_1 \leq ... \leq u_n \leq 1)}. \end{aligned}$$

This is exactly (5.5). Then the conditional distribution of $Z = \left(\frac{S_1}{S_{n+1}}, ..., \frac{S_n}{S_{n+1}}\right)$ given $S_{n+1} = t$ does not depend on t. So, its conditional distribution is also its unconditional distribution function and then

$$\left(\frac{S_1}{S_{n+1}}, ..., \frac{S_n}{S_{n+1}}\right)$$

has the same law as $U = (U_{1,n}, U_{2,n}, ..., U_{n,n})$ and it is independent of S_{n+1}. This puts an end to the proof.

We formalize the last conclusion in the following lemma.

LEMMA 48. *Let* $E_1, E_2, ..., E_n$, E_{n+1}, $n \geq 1$, *be independent standard exponential random variables defined on the same probility space. Let* $S_i = E_1 + E_2 + ... + E_i$, $1 \leq i \leq n+1$ *then*

$$\left(\frac{S_1}{S_{n+1}}, ..., \frac{S_n}{S_{n+1}}\right)$$

is independent of S_{n+1}.

The latter proposition exposed representations of order statistics of standard uniform random variables into that of standard exponential random variables. The following proposition reverses the situation.

PROPOSITION 25. *Assume the notations of Proposition 24 hold. Then for any* $n \geq 1$,

$$(-logU_{n,n}, ..., -logU_{1,n}) =_d (E_{1,n}, ..., E_{n,n}),$$

where $E_{1,n} \leq ... \leq E_{n,n}$ *are the order statistics of* $E_1, E_2, ..., E_n$, *which are* n *independent and exponent distributed with intensity one.*

Proof. By Proposition 23, the pdf of $E_{1,n} \leq ... \leq E_{n,n}$ is

$$f_Z(z) = n! e^{-\sum_{i=1}^n z_i} 1_{(0 \leq z_1 \leq ... \leq z_n)}, \ z = (z_1, z_2, ..., z_n) \in \mathbb{R}^n, \tag{5.8}$$

where $Z = (E_{1,n}, ..., E_{n,n})$. The distribution of $Z^* = (-log(1-U_{1,n}), ..., -log(1-U_{n,n}))$ comes from that of $U = (U_{1,n}, ..., U_{n,n})$ by the diffeomorphism $(z_1, ..., z_n) = (-log(1-u_1), ..., -log(1-u_n))$ which preserves the order of the arguments and has a Jacobian determinant in absolute value equal to

$$\begin{aligned}\left|\frac{\partial U_i}{\partial z_j}\right| &= \left|\frac{\partial e^{-z_j}}{\partial z_j}\right| = \left|diag\left(-e^{-z_1}, ..., -e^{-z_n}\right)\right| \\ &= e-\sum_{i=1}^n z_i.\end{aligned}$$

Then, the pdf of Z^* is

$$\begin{aligned}f_{Z^*}(z_1, ..., z_n) &= f_U\left(-e^{-z_1}, ..., -e^{-z_n}\right) e^{-\sum_{i=1}^n z_i} 1_{(0 \leq z_1 \leq ... \leq z_n)} \\ &= n! e^{-\sum_{i=1}^n z_i} 1_{(0 \leq z_1 \leq ... \leq z_n)}.\end{aligned}$$

This pdf is that of $(E_{1,n}, ..., E_{n,n})$ by (5.8). We get

$$(-log\,(1-U_{1,n})\,, ..., -log\,(1-U_{n,n})) =_d (E_{1,n}, ..., E_{n,n})\,.$$

To conclude, we use the equality $U =_d (1-U)$ to have for any $n \geq 1$, we have

$$(U_{n,n}, ..., U_{1,n}) =_d (1-U_{1,n}, ..., 1-U_{n-i+1,n}, ..., 1-U_{n,n})\,.$$

The proof is over. □

We may build on this the following result. Denote

$$\alpha_{i,n} = -log\,(1-U_{i,n})\,,\ 1 \leq i \leq n.$$

Consider the transformation for $n \geq 1$,

$$\begin{pmatrix} n\alpha_{1,n} \\ (n-1)(\alpha_{2,n}-\alpha_{1,n}) \\ . \\ . \\ (n-i+1)(\alpha_{i,n}-\alpha_{i-1,n}) \\ . \\ . \\ 1(\alpha_{n,n}-\alpha_{n-1,n}) \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \\ . \\ . \\ V_i \\ . \\ . \\ V_n \end{pmatrix}$$

We have

$$\begin{pmatrix} \alpha_{1,n} \\ \alpha_{2,n} \\ . \\ . \\ . \\ . \\ \alpha_{n,n} \end{pmatrix} = \begin{pmatrix} V_1/n & & & & & & & & \\ V_1/n & + & V_2/(n-1) & & & & & & \\ . & & & & & & & & \\ . & & & & & & & & \\ . & & & & & & & & \\ . & & & & & & & & \\ V_1/n & + & V_2/(n-1) & + & ... & + & V_{n-1}/2 & + & V_n/1 \end{pmatrix}.$$

The probability density function of $(V_1, ..., V_n)$ is given by

$$\begin{aligned} & f_V(v_1, ..., v_n) \\ = \ & f_{(\alpha_{1,n},...,\alpha_{n,n})}(v_1/n, v_1/n + v_2/(n-1), ..., v_1/n + v_2/(n-1) + ... + v_{n-1}/2 + v_n) \\ \times \ & |J(v)| \times 1_{D_V}(v). \end{aligned}$$

The Jacobian determinant in absolute value of this transform is

$$|J(v)| = \frac{1}{n!}$$

and the domain of V is

$$D_V = \mathbb{R}^n_+.$$

We conclude by using (5.8) which gives the joint *pdf* of $(\alpha_{1,n}, ..., \alpha_{n,n})$, and by denoting $s_i = v_1/n + v_2/(n-1) + ... + v_{n-i+1}/(n-i+1)$, $i = 1, ..., n$. We get

$$\begin{aligned} f_V(v_1, ..., v_n) &= \frac{1}{n!} \times n! e^{-\sum_{i=1}^n s_i} 1_{(v_1 \geq 0, ..., v_n \geq 0)} \\ &= e^{-\sum_{i=1}^n s_i} 1_{(v_1 \geq 0, ..., v_n \geq 0)}. \end{aligned}$$

(5.9)

We may check that $s_1 + ... + s_n = v_1 + ... + v_n$. We arrive at

$$f_V(v_1, ..., v_n) = \prod_{i=1}^{n} e^{-v_i} 1_{(v_i \geq 0)}.$$

This says that $(V_1, ..., V_n)$ has independent standard exponential coordinates. We summarize our finding in:

PROPOSITION 26. *Let* $\alpha_{i,n} = -log(1 - U_{i,n})$, $i = 1, ..., n$. *Then the random variables*

$$n\alpha_{1,n}, (n-1)(\alpha_{2,n} - \alpha_{1,n}), ..., (n-i+1)(\alpha_{i,n} - \alpha_{i-1,n}), ..., (\alpha_{n,n} - \alpha_{n-1,n})$$

are independent standard exponential random variables.

Let us do more and put for any $1 \leq i \leq n$,

$$(n-i+1)(\alpha_{i,n} - \alpha_{i-1,n}) = (n-i+1)log\left(\frac{1 - U_{i-1,n}}{1 - U_{i,n}}\right).$$

By our previous results we have that the random variables

$$E^*_{n-i+1} = (n-i+1)(\alpha_{i,n} - \alpha_{i-1,n}) = log\left(\frac{1 - U_{i-1,n}}{1 - U_{n-i,n}}\right)^{(n-i+1)}$$

are independent and standard exponential random variables. We may and do change $U_{n-i,n}$ to $U_{i+1,n}$ to arrive at this celebrated representation.

PROPOSITION 27. *(Malmquist representation). Let* U_1,U_2,...,U_n *be standard uniform random variables for* $n \geq 1$. *Let* $0 \leq U_{1,n} < U_{2,n} < ... < U_{n,n} \leq 1$ *be their associated order statistics. Then the random variables*

$$log\left(\frac{U_{i+1,n}}{U_{i,n}}\right)^{i}, \ i = 1, ..., n$$

are independent standard exponential random variables.

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