A COLLECTION OF PAPERS IN MATHEMATICS AND RELATED SCIENCES,
A FESTSCHRIFT IN HONOUR OF THE LATE GALAYE DIA
EDITORS : PROFS HAMET SEYDI, GANE SAMB LO, ABOUBAKARY DIAKHABY
SPAS-EDS (SAINT-LOUIS, CALGARY, 2018), WWW.STATPAS.ORG/SPASEDS/
IN EUCLID (WWW.PROJECTEUCLID.ORG)
DOI : 10.16929/SBS/2018.100

CHAPTER 22

Gaussian Approximations and Related Questions for the Spacings process, *by G. S. Lo*

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Abstract. The strong approximation of the empirical process based on k-spacings of *iid* $(0,1)$-random variables is studied. A better rate of convergence, if not the better, the the existence ones is provided. Other related strong laws are exposed also.

Keywords. Spacings; empirical process; oscillation modulus; strong and weak approximation; order statistics; gamma distribution and function; law of the iterated logarithm.

AMS 2010 Mathematics Subject Classification. 60G30; 60F15; 60B10

Cite the chapter as :
Lo, G.S. (2018). Gaussian Approximations and Related Questions for the Spacings process.
In *A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia* (Editors : Seydi H., Lo G.S. and Diakhaby A.). Spas Editions, Euclid Series Book, pp. 413 – 435.
Doi : 10.16929/sbs/2018.100-04-05

A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia. **Lo G.S. (2018). Gaussian Approximations and Related Questions for the Spacings process**. Pages 413 — 435.

Foreword. This paper was part of the Science Doctorate of the candidate (Cheikh Anta Diop University, 1991) that was not yet published in a peer-reviewed journal. A slightly different version was published in *Rapports Techniques, LSTA, Université Paris VI*, 48, 1986, under the same title.

Full Abstract. All the available results on the approximation of the k-spacings process to Gaussian processes have only used one approach, that is the Shorack and Pyke's one. Here, it is shown that this approach cannot yield a rate better than $(N/\log\log N)^{-\frac{1}{4}}(\log N)^{\frac{1}{2}}$. Strong and weak bounds for that rate are specified both where k is fixed and where $k \to +\infty$. A Glivenko-Cantelli Theorem is given while Stute's result for the increments of the empirical process based on independent and identically distributed random variables is extended to the spacings process. One of the Mason-Wellner-Shorack cases is also obtained.

1. Introduction

The non-overlapping uniform k-spacings are defined by

$$D_{i,n}^{k} = U_{ik,n} - U_{(i-1)k,n}, \quad 1 \leq i \leq \left[\frac{n+1}{k}\right] = N,$$

where $0 \equiv U_{0,n} \leq U_{1,n} \leq ... \leq U_{n,n} \leq U_{n+1,n} \equiv 1$ are the order statistics of a sequence $U_1, ..., U_n$ of independent random variables (r.v.'s) uniformly distributed on $(0,1)$ and $[x]$ denotes the integer part of x. The study of these r.v.'s have received a great amount of attention in recent years (see Aly *el al.* (1984), Deheuvels (1984), Pyke (1965) and Stute (1982)). Particularly the related empirical process

$$\beta_N(x) = N^{\frac{1}{2}}\{F_N(x) - H_k(x)\}, 0 \leq x \leq +\infty,$$

where

$$F_N(x) = \#\left\{i, 1 \leq i \leq N,\ NkD_{i,n}^{k} \leq x\right\}/N$$

and

$$H_k(x) = \int_0^x \frac{t^k e^{-t}}{(k-1)!}dt, \quad x \geq 0.$$

plays a fundamental role in many areas instatistics (see Deheuvels (1984)). All its aspects are have described by various authors.

(i) For the convergence of statistics based on spacings, it is helpful to have a Glivenko-Cantelli Theorem for $F_N(.)$. Such results for the overlapping case are available in Beirlant and Van Zuijlen (1985).

(ii) The limiting law of the spacings statistics may follow from suitable approximations of β_N to Gaussian processes. It is clear that the better the rates of those approximations are the less restrictive the conditions on the underlying random variables ($r.v.$). Such approximations also yield Kolmogorov-Smirov's tests.

(iii) Finally, the oscillation modulus of β_N has been studied in Lo (1987), where is established the weak behaviour of the oscillation moduli of β_N is equivalent to that of the empirical process based on a sequence of independent and identically distributed ($i.i.d$) random variables.

Our aim is to give strong versions of weak characterizations of the oscillation moduli that we have already given in Lo (1987). As to the approximation of β_n to Gaussian processes, we will show that the rate given in Lo (1987) is, in fact, a strong one. Our best achievement is that this rate is the best attainable for the approach used until now and we provide the corresponding bounds. With respect to Aly (1985) and Aly *el al.* (1984), we do not let k fixed. We allow it to go to infinity. Finally we give the Glivenko-Cantelli Theorem for F_N with almost the same condition as in Beirlant and Van Zuijlen (1985) for the overlapping case.

2. The Gaussian approximation.

Approximations of β_N to Gaussian processes are available since Shorack (1972). The best rates among those already given are due to Aly (1985) and to Aly *el al.* (1984). Among other results, Aly *el al.* (1984) proved the following theorem and corollary.

THEOREM 59. . *There exists a probability space carrying a sequence* $U_1, U_2, ...$ *of independent r.v.'s uniformly distributed on* $(0,1)$ *and a sequences of Gaussian processes* $\{W_N(x), 0 \leq x \leq +\infty\}$, $N = 1, 2, ...$ *satisfying*

$$\forall N > 1,\ \mathbb{E}\left(W_N(x)\, W_N(y)\right)$$

$$(2.1) \qquad = \min\left(H_k(x),\ H_k(y)\right) - H_k(x)\,H_k(y) - k^{-1}xyH'_k(x)\,H'_k(y)$$

such that

$$\lim_{N\to+\infty}\sup\,(\log N)^{-\frac{3}{4}}\,N^{\frac{1}{4}}\sup_{0\leq x\leq+\infty}\left|\beta_N(x)-W_N(x)\right|<+\infty, a.s.$$

whenever k is fixed. Here $H'_k(x) = dH_k(x)/dx.$

Remark 24. *From now on, we will say according to the wording of Theorem 59 at the place of There exist a probability space ... such that.*

Definition 9. *A Gaussian process whose covariance function is given by (2.1) will be called a Shorack process of parameter* k *or a* k*-Shorack process.*

Corollary 19. *According to wording of Theorem 59, we have*

$$N^{\frac{1}{4}}(\log N)^{-\frac{1}{2}}(\log\log N)^{-\frac{1}{4}}\sup_{0\leq x<+\infty}\left|\beta_N(x)-W_N(x)\right| = 0_p(1)\,,\ \text{as } N\to+\infty.$$

This means that $a_N^o = (\log N)^{\frac{3}{4}}N^{-\frac{1}{4}}$ *is a strong rate of convergence while* $a_N = (\log N)^{\frac{1}{2}}(2\log\log N)^{\frac{1}{4}}N^{-\frac{1}{4}}$ *is a weak one. In fact Aly (1985) has showed*

Theorem 60. . *There exist another sequence of processes* β_N^1, $N=1,2,...$ *and a sequence of k-Shorack processes* W_N^1, $N=1,2,...$ *such that, for k fixed, the two following assertions hold :*

$$\text{(i)} \qquad \beta_N^1 =^d \beta_N,\ \forall N\geq 1$$

$$\text{(ii)} \qquad \sup_{0\leq x<+\infty}\left|\beta_N^1(x)-W_N^1(x)\right| a.s. = 0\left(a_N\right)N\to+\infty,\ \text{a.s.}\ .$$

All these results are based on representations of spacings by exponential r.v.'s. Namely, when $n+1=kN$,

$$\left\{D_{i,n}^{k}, 1 \leq i \leq N\right\} =^{d} \left\{ \frac{\left(\sum_{j=(i-1)k}^{j=ik} E_j\right)}{S_{n+1}}, 1 \leq i \leq N \right\}$$

$$=: \left\{Y_i/S_{n+1}, 1 \leq i \leq N\right\}, \tag{2.2}$$

where $E_1,\ E_2, ...$ *is a sequence of independent exponential rv's with mean one and whose partial sums are* $S_n, n \geq 1$. *If* $\mu_N = \delta_n = S_{n+1}/Nk$*, it follows that*

$$\left\{\beta_N\left(x\right), 0 \leq x < +\infty\right\} =^{d} \left\{N^{\frac{1}{2}}\left(\xi_N \mu_N\left(x\right) - H_k\left(x\right)\right) + 0\left(N^{\frac{1}{2}}\right)\right\}$$

$$= \left\{\Lambda_N\left(x\right) + R_N\left(x\right), 0 \leq x < +\infty\right\} =: \left\{\beta_N^{*}\left(x\right), 0 \leq x < +\infty\right\}, \tag{2.3}$$

where $\xi_N\left(.\right)$ *(resp.* $\Lambda_N\left(.\right)$*) is the empirical distribution function (resp. empirical process) based on* $Y_1, ..., Y_N$*. The cited results are derived from simultaneous approximations of* Λ_N *and* R_N*.*

First, we establish that the best rate attainable through this approach is that of Aly (1985) even when $k \to +\infty$*.*

THEOREM 61. *According to the wording of Theorem 60, for any k satisfying*

$$\exists \delta_0 < 0,\ \forall 0 < \delta < \delta_0, kN^{-\delta} \to 0 \text{ as } N \to +\infty, \tag{L}$$

we have

$$\lim_{N \to +\infty} \sup a_N^{-1} \sup_{0 \leq x < +\infty} \left|\beta_N^{*}\left(x\right) - W_N^{*}\left(x\right)\right| a.s. = \begin{cases} K\left(k\right) = \left(k^{k+\frac{1}{2}} e^{-k}/k!\right)^{\frac{1}{2}}, & \text{(k fixed)} \\ K_0 = \left(2\pi\right)^{-\frac{1}{4}}, & \left(k \to +\infty\right). \end{cases}$$

Our second result is an improvement of Theorem 1 of Aly el al. (1984).

THEOREM 62. . *According to the wording of Theorem 59, we have for any k such that for some $\delta_0, 0<\delta_0<\frac{1}{4}, kN^{-\frac{1}{4}+\delta_0} \rightarrow 0$ as $N \rightarrow +\infty$,*

$$\lim_{N\rightarrow+\infty} \sup a_N^{-1} \sup_{0\leq x<+\infty} |\beta_N(x) - W_N(x)| \leq \begin{cases} K(k), \text{ (k fixed)} \\ K_0 \text{ (k} \rightarrow +\infty\text{)} \end{cases} a.s.,$$

Proof of Theorem 62. From (2.3), we have $\beta_N =^d \beta_N^*$ for all $N \geq 1$. Furthermore,

$$\begin{aligned} \beta_N^*(x) &= \Lambda_N(x) + N^{\frac{1}{2}}(H_k(\mu_N x) - H_k(x)) - \{\Lambda_N(\mu_N x) - \Lambda_N(x)\} + 0\left(N^{-\frac{1}{2}}\right) \\ &= : \Lambda_N(x) + R_{N1}(x) + R_{N2}(x) + R_{N3}(x). \end{aligned}$$

We shall proceed by steps, approximating each of the R_{Ni}'s.

LEMMA 42. *Let $N_p = [(1+\rho)^p]$, $p>0$, $p=1,2,...,\varepsilon>0$ and*

$$C_{N_p} = \bigcup_{N=N_p}^{N=N_{p+1-1}} \left\{ \sup_{0\leq x<+\infty} \left| R_{N1}(x) - N^{\frac{1}{2}}\ xH_n'(x)(\mu_N - 1) \right| > \varepsilon a_N K(k)/4 \right\}.$$

Then if $k/N \rightarrow 0$ as $N \rightarrow +\infty$, $\sum_p \mathbb{P}\left(C_{N_p}\right) < +\infty$.

Proof of Lemma 42 Apply the mean value theorem twice and get

$$A_{N1} = R_{N1}(x) - N^{\frac{1}{2}}(\mu_N - 1)xH_k'(x) = N^{\frac{1}{2}}(\mu_N - 1)^2 x^2 H_k''(x_N), \tag{2.4}$$

Where $0 < |x_N/x| < \max(1, \mu_N)$. First, it may be easily seen that

$$\sup_{0\leq x<+\infty} \frac{xH_k'(x)}{k^{\frac{1}{2}}} = \frac{k^{\frac{1}{2}+k}e^{-k}}{k!} = K(k)^2, \tag{2.5}$$

$$\lim_{k\rightarrow+\infty} \sup_{0\leq x<+\infty} \left| x\ H_k'(x)/k^{\frac{1}{2}} \right| = K_0^2, \tag{2.6}$$

and

$$0 < M = \sup_{k \geq 1} \sup_{0 \leq x < +\infty} \left| x^2 H_k''(x)/k \right| < +\infty. \tag{2.7}$$

Recall that for all $\varepsilon > 0$,

$$\sum_p \mathbb{P}\left(\max(1, \mu_N) > 1 + \varepsilon\right) \leq \sum_N \mathbb{P}\left(|\mu_N| > 1 + \varepsilon\right) < +\infty, \tag{2.8}$$

by the strong law of large numbers (SLLN) and

$$\sum_p \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \left(\frac{Nk}{2 \log \log nk} \right) |\mu_N - 1| > 1 + \varepsilon \right) < +\infty \tag{2.9}$$

by the law of the iterated logarithm (loglog-law). We show in the Appendix how to adapt the classical SLLN and loglog-law to these cases.

Now by (2.4), (2.5) and (2.6)

$$\mathbb{P}\left(C_{N_p}\right) \leq \sum_{N=N_p}^{N=N_{p+1-1}} \mathbb{P}\left(\max(1, \mu_N)^2 > 1 + \varepsilon\right)$$

$$+ \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} |\mu_N - 1|^2 \left(\frac{Nk}{2 \log \log Nk} \right) > c e_N \right), \tag{2.10}$$

with $c = \varepsilon K(k)^2 / 4M(1+\varepsilon), e_N = (\log \log N)^{\frac{1}{4}} N^{\frac{1}{4}} (\log N)^{\frac{1}{2}} (2 \log \log Nk)^{-\frac{1}{2}}$. But $\log \log Nk = (\log N)(1 + o(1))$, $K(k)$ is bounded and thus $ce_N > (1+\varepsilon)^2$ for large N. Thus we can apply (2.8) and (2.9) to (2.10) and, by this, complete the proof.

Lemma 43. *Let* $\varepsilon > 0$ *and*

$$D_{N_p} = \left\{ \bigcup_{N=N_p}^{N=N_{p+1-1}} \left(\sup_{0 \leq x < +\infty} |R_{N2}(x)| > (1 + \varepsilon/4)\, a_N\ K(k) \right) \right\},\ p = 1, 2, \ldots$$

Then for any $k = k(N)$ *such that* $k/N \to 0$ *as* $N \to +\infty$, $\sum_p p\left(D_{N_p}\right) < +\infty$.

Proof of Lemma 43 The mean value theorem implies

$$|H_k(\mu_N x) - H_k(x)| \leq |\mu_N - 1| \ K(k)^2 \max(1, \ \mu_N)\, k^{\frac{1}{2}}. \tag{2.11}$$

By proceeding similarly to (2.10), we get

$$\mathbb{P}\left(D_{N_p}\right) \leq \sum_{N=N_p}^{N=N_{p+1-1}} \mathbb{P}\left(\max(1, \mu_N) > (1+\varepsilon/4)^{1/3}\right)$$

$$+\mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \left\{\sup_{|H_k(x)-H_k(y)|<c_N} |\Lambda_N(x) - \Lambda_N(y)| > (1+\varepsilon/4)\, a_N K(k)\right\}\right)$$

$$= R_{N21} + R_{N22}, \tag{2.12}$$

with $c_N = K(k)^2 k^{\frac{1}{2}} |\mu_N - 1| (1+\varepsilon/4)^{1/3}$. Now,

$$R_{N22} \leq \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \left\{|\mu_N - 1| > (1+\varepsilon/4)^{1/3} \left(\frac{2\log\log N}{Nk}\right)^{\frac{1}{2}}\right\}\right) \tag{2.13}$$

$$+\mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \left\{\sup_{|H_k(x)-H_k(y)|\leq b_N} |\Lambda_N(x) - \Lambda_N(y)| > (1+\varepsilon/4)\, a_N K(k)\right\}\right),$$

where $b_N = \left(\frac{2\log\log\ N}{N}\right)^{\frac{1}{2}} K(k)^2 (1+\varepsilon/4)^{2/3}$. Let $\gamma_N(.)$ be the empirical process based on $U_1, ..., U_N$ and P_{N_p} be the second term of the right member of the inequality (2.13). Thus (2.2) implies

$$P_{N_p} \leq \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \left\{\sup_{0\leq u\leq 1-b_N} \frac{\gamma_N(u) - \gamma_N(u+b_N)}{\left(2b_N \log b_N^{-1}\right)^{\frac{1}{2}}} > 1+\varepsilon_1\right\}\right), \tag{2.14}$$

$1+\varepsilon_1 < (1+\varepsilon)^{2/3}$, where we have used the fact that $\left(2b_N \log b_N^{-1}\right)^{\frac{1}{2}} / a_N k(k) \to (1+\varepsilon)^{1/3}$ as $k/N \to 0$, as $N \to +\infty$. Finally, from line 14, p.95 and line 23, p.98 in Stute (1982), we get $\sum_p P_{N_p} < +\infty$. This and (2.11), (2.12), (2.13) and (2.14) together imply Lemma 43.

LEMMA 44. *(Komlós, Májor,Tusnády, 1975). There exist a probability space carrying a sequence $Y_1, Y_2, \ldots$ as defined in (2.2) and a sequence of Brownian bridges*

$$B_N^1(s), 0 \leq s \leq 1, \ N = 1, 2, \ldots$$

such that

$$\forall N \geq N_1, \mathbb{P}\left(\sup_{0 \leq x < +\infty} \left|\Lambda_N(x) - B_N^1(H_k(x))\right| > \frac{A \log\ N + x}{N^{\frac{1}{2}}}\right) \leq Be^{-\lambda x},$$

for all sequence $(k = k(N))_{N \geq 1}$ and for all x, where N_1, A, B and λ are absolute positive constants.

Proof of Lemma 44 This doesn't need to be proved. It is directly derived from Komlós *et al.* (1975) and Corollary 4.4.4 of Csőrgö and Révèsz (1981).

Proof of Theorem 61 continued. On the probability space of Lemma 44, Lemmas 42 and 43 combined with the fact $R_{N3} \leq N^{-\frac{1}{2}}$ imply that

$$\sum_p \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \sup_{0 \leq x \leq +\infty} |\beta_N^*(x) - \beta_N^{**}(x)| > (1 + 3\varepsilon/4)\, a_N K(k)\right) < +\infty, \tag{2.15}$$

where $\beta_N^{**}(x) = \Lambda_N(x) - N^{\frac{1}{2}}$ x $H_k'(x)(\mu_N - 1), 0 \leq x < +\infty$. Hence, the proof will be complete if we approximate β_N^{**} in the right way. But by Lemma 44, for any $\varepsilon > 0$, for large N,

$$\mathbb{P}\left(\sup_{0 \leq x < +\infty} \left|\Lambda_N(x) - B_N^1(H_k(x))\right| > A_1 (\log N)^2 N^{-\frac{1}{2}}\right) \leq N^{-1-\varepsilon}, \tag{2.16}$$

where A_1 is some absolute constant. From Lemma 3.1 of Aly *el al.* (1984)

$$N^{\frac{1}{2}}(\mu_N - 1) = N^{\frac{1}{2}} k \frac{S_{n+1} - Nk}{Nk} + k^{-1} \int_0^{+\infty} \left\{\Lambda_N(x) - B_N^1(H_k(x))\right\} dx$$

$$+ k^{-1} \int_0^{+\infty} B_N^1(H_k(x))\, dx. \tag{2.17}$$

Let $t_N = N^{\frac{1}{4}-\delta}, 0 \leq \delta \leq \delta_0$. On the one hand, one has for large N.

$$\mathbb{P}\left(\left|\int_0^{t_N}\left\{\Lambda_N(x)-B_N^1(x)\right\}dx\right|>\varepsilon a_N/12\right)$$

$$\leq \mathbb{P}\left(\sup_{0\leq x<+\infty}\left|\Lambda_N(x)-B_N^1(H_k(x))\right|>\frac{\varepsilon(2\log\log N)^{\frac{1}{4}}(\log N)^{\frac{1}{2}}}{12N^{\frac{1}{4}-\delta}}\right)$$

$$\leq \mathbb{P}\left(\sup_{0\leq x<+\infty}\left|\Lambda_N(x)-B_N^1(H_k(x))\right|>A_1\log N/N^{\frac{1}{2}}\right).$$

This and (2.6) together imply

$$\mathbb{P}\left(\sup_{0\leq x<+\infty}\left|xH_k'(x)k^{-1}\int_0^{t_N}\left\{\Lambda_N(t)-B_N^1(H_k(t))\right\}dt\right|>\varepsilon a_N K(k)/12\right) \leq N^{-1-\varepsilon}, \tag{2.18}$$

for N large enough. On the other hand, as $N\to+\infty$,

$$\mathbb{P}\left(\sup_{0\leq x<+\infty}\left|\int_{t_N}^{+\infty}\left\{\Lambda_N(t)-B_N^1(H_k(t))\right\}dt\right|>N^{-\frac{1}{2}}\right) \leq N^{\frac{1}{2}}\exp\left(-N^{\frac{1}{4}-\delta}/4\right). \tag{2.19}$$

To see that, we apply Markov's inequality with

$$\mathbb{E}\int_{t_N}^{+\infty}\left|\Lambda_N(x)-B_N^1(H_k(x))\right|dx \leq \int_{t_N}^{+\infty}4k^{-1}e^{-x/2}\frac{x^{(k-1)/2}}{(k-1)!}dx \leq 4k^{-1}t_N^k\exp(-t_N/2).$$

Since $k = o\left(N^{\frac{1}{4}-\delta}o\right)$, as $N\to+\infty$, (2.19) follows. Finally for large N,

$$\mathbb{P}\left(\sup_{0\leq x<+\infty}\left|xH_k'(x)/k^{\frac{1}{2}}\frac{S_{Nk}-S_{n+1}}{(Nk)^{\frac{1}{2}}}\right|>\varepsilon a_N K(k)/16\right)$$

$$\leq \mathbb{P}\left(S_k > N^{\frac{1}{2}}k^{\frac{1}{2}}\right) = 1-H_k\left(N^{\frac{1}{2}}k^{\frac{1}{2}}\right).$$

Integrating by parts we have : $k/x \leq \frac{1}{2} \Rightarrow 1-H_k(x) \leq 2x^{k-1}e^{-x}/(k-1)!$. Then if $k/N \leq \frac{1}{2}$ for large N, we get by Sterling's formula,

$$1-H_k\left(k^{\frac{1}{2}}N^{\frac{1}{2}}\right)\leq Const.\ \exp\left(-k^{\frac{1}{2}}N^{\frac{1}{2}}\left(1+(k/N)^{\frac{1}{2}}\log\left(k/N\right)\right)\right). \tag{2.20}$$

Thus,

$$\mathbb{P}\left(\sup_{0\leq x<+\infty} xH_k'\left(x/k\right)\left(\left(S_{Nk}-S_{n+1}\right)/N^{\frac{1}{2}}\right)>\varepsilon a_N K\left(k\right)/12\right) \tag{2.21}$$

$$\leq const.\ \exp\left(-\frac{1}{4}k^{\frac{1}{2}}N^{\frac{1}{2}}\right), \tag{2.22}$$

ultimately as $N\to+\infty$ whenever $k/N\to 0$ as $N\to+\infty$. Put together (2.16), (2.17), (2.18), (2.19) and (2.22) to get

$$\sum_N \mathbb{P}\left(\sup_{0\leq x<+\infty}\left|\beta_N^{**}\left(x\right)-W_N^{**}\left(x\right)\right|>\varepsilon a_N K\left(k\right)/4\right)<+\infty, \tag{2.23}$$

where $W_N^{**}\left(x\right)=B_N^1\left(H_k\left(x\right)\right)-xk^{-1}H_k'\left(x\right)\int_0^{+\infty}t\ dB_N^1\left(H_k\left(t\right)\right),\ x\geq 0$. And combine (2.15) with (2.23) to have

$$\sum_p \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}}\left\{\sup_{0\leq x<+\infty}\left|\beta_N^{*}\left(x\right)-W_N^{**}\left(x\right)\right|>\left(1+\varepsilon\right)a_N K\left(k\right)\right\}<+\infty\right). \tag{2.24}$$

This together with Lemma 4.4.4. of Csőrgö and Révèsz (1981) completes the proof.

Proof of Theorem 61. As in the proof of Theorem 62, the spacings are always defined on the probability space of Lemma 44. We shall study each of the R_{Ni}'s once again. First we put together (2.4), (2.5), (2.6) and (2.7) to get, as $N\to+\infty$

$$\sup_{0\leq x<+\infty}\left|R_{N1}\left(x\right)-N^{\frac{1}{2}}\left(\delta_n-1\right)xH_k'\left(x\right)\right|=0\left(N^{-\frac{1}{2}}\log\log N\right),\ a.s.. \tag{2.25}$$

Now Lemma 43 says nothing else but

$$\lim_{N\to+\infty}\sup\ \sup_{0\leq x<+\infty}\left|R_{N2}\left(x\right)/a_N\right|\leq K\left(k\right)\ \text{ or } K_0, a.s., \tag{2.26}$$

whenever k is fixed or $k \to +\infty$ while $k/N \to 0$ as $N \to +\infty$. And the proof will be completed through our fundamental Lemma which is the following. □

Lemma 45. *Under the assumptions of Theorem 61, we have*

$$\lim_{N\to+\infty} \sup \sup_{0\leq x<+\infty} \left|a_N^{-1} R_{N2}(x)\right| \geq K(k) \text{ or } K_0, a.s.,$$

according whether k is fixed or $k \to +\infty$ and satisfies (L).

Proof of Lemma 45. Let $\psi(x) = ((k-1)!)^{-1} x^k e^{-x}$, $x \geq 0$. By the mean value theorem,

$$|\psi(x) - \psi(k)| \leq he^h k^{k-1} (1-1/k)^{k-1} ((k-1)!)^{-1}, \text{ if } |x-k| \leq h \leq 1.$$

By Sterling's formula we can find a constant $\tau > 0$ such that

$$\sup_{|x-k|\leq h\leq 1} k^{\frac{1}{2}} |\psi(x) - \psi(k)| \leq \tau h k^{-1}, \text{ for all } k \geq 1. \tag{2.27}$$

Now, we have

$$\begin{aligned} &A_N(x) = H_k(\delta_n x) - H_k(x) = (\delta_n - 1)\psi(x_n)(x_n/x), \\ &0 \leq x_n/x \leq \max(1, \delta_n). \end{aligned} \tag{2.28}$$

If $|x-k| \leq h \leq 1, |x_n - k| \leq k + (k+h)|1-\delta_n|$, and thus by (2.27),

$$|x-k| \leq h \leq 1 \Rightarrow A_N(x) = (1+o(1)) k^{\frac{1}{2}} (\delta_n - 1)$$

$$\times \{K(k) + 0(\{h + (h+k)|1-\delta_n|\}/k)\}, a.s.$$

Let $h = h(N) \to 0$ as $N \to +\infty$. Then by the loglog-law, there exists $\Omega^1 \subset \Omega$ and a sequence $\left(N_{j(\omega)}\right)$ extracted from (N) (let n_j and k_j be the corresponding sub-sequences) satisfying

$$\mathbb{P}\left(\Omega^1\right) = 1, \ \forall\omega \in \Omega^1, \ A_{N_j}(x) = ((2\log\log n_j)/N_j)^{\frac{1}{2}} K(k_j)^{\frac{1}{2}} (1+o(1))$$

$$=: (1+o(1)) d_{N_j}, \tag{2.29}$$

uniformly in x, $k_j - h_j \leq x \leq k_j - h_j$, where $h_j = h(N_j)$ as $N \to +\infty$. Thus we have uniformly in $x \in [k_j - h_j,\ k_j + h_j] = I_k$,

$$\left|R_{N_{j^2}}(x)\right| \ d_{=} \ \left|\gamma_{N_j}\left(H_{k_j}(x) + d_{N_j}(1+o(1)) - \gamma_{N_j}\left(H_{k_j}(x)\right)\right)\right|$$

$$= \ : \left|R^*_{N_{j^2}}(x)\right|. \tag{2.30}$$

Now, we have to prove that

$$\exists\Omega \subset \Omega^1,\ \mathbb{P}(\Omega_0) = 1,\ \forall\omega \in \Omega_0,\ \lim_{j\to+\infty} \inf \sup_{x \in I_{k_j}} \left\{\left|R^*_{N_{j^2}}(x)/b\left(d_{N_j}\right)\right|\right\} \geq 1, \tag{2.31}$$

where $b(s) = (2s \log\log s^{-1})^{\frac{1}{2}},\ 0 < s < 1$.

Proof of (2.31). Let

$$C_{N_1}(p) = \sup_{0 \leq v \leq d_N/p} \sup_{0 \leq s \leq 1-v} |\gamma_N(s) - \gamma_N(s+v)| / b(d_N),\ p \geq 1.$$

By Theorem 0.2 of Stute (1982),

$$\forall p \geq 1,\ \exists\Omega_p \subset \Omega,\ P(\Omega_p) = 1,\ \forall\omega \in \Omega_p,\ \lim_{N\to+\infty} \sup C_{N_1}(p)(\omega) < p^{-\frac{1}{2}}. \tag{2.32}$$

Let

$$\Omega = \Omega^1 \bigcap \bigcup_{p=1}^{p=+\infty} \Omega_p.$$

Obviously $\mathbb{P}(\Omega^2) = 1$. And for any $\omega \in \Omega^2$, $C_{N_{j^2}}(\omega) =$

$$\sup_{0 \leq x < +\infty} \gamma_{N_j}\left(H_{k_j}(x) + d_{N_j}(1+o(1)) - \gamma_{N_j}\left(H_{k_j}(x) + d_{N_j}\right)\right) = o\left(b\left(d_{N_j}\right)\right),$$

This, together with the following, as $j \to +\infty$,

$$\forall x \in I_{k_j},\ R^*_{N_{j^2}}(x) = \gamma_{N_j}\left(H_{k_j}(x) + d_{N_j}\right) - \gamma_{N_j}\left(H_k(x)\right)$$

$$+\gamma_{N_j}\left(H_{k_j}(x) + d_{N_j}(1+o(1))\right) - \gamma_{N_j}\left(H_{k_j}(x) + d_{N_j}\right), \tag{2.33}$$

implies that

$$\sup_{x\in I_{k_j}} R^*_{N_{j^2}}(x) \geq \sup_{x\in I_{k_j}} \gamma_{N_j}\left(H_{k_j}(x)+d_{N_j}\right)-\gamma_{H_j}\left(H_{k_j}(x)\right)+o\left(b\left(d_{N_j}\right)\right)$$

$$\geq: C_{N_{j^3}}\left(h\left(N_j\right)\right)+o\left(b\left(d_{N_j}\right)\right). \tag{2.34}$$

Now put $J_k = H_k(I_k)$ and remark that the length of J_k is $\rho(J_k) = 2K(k)^2 nk^{-\frac{1}{2}}(1+o(1))$.

For any $p \geq 1$, choose $h = h(N,\ p) = h_p$ (with $h_{j,p} = h(N_j, p)$ such that $2K(k)^2 h_p k^{-\frac{1}{2}} d_N^{-1/4p} = 1+o(1)$, as $N \to +\infty$. Thus, $h \to 0$ as $N \to +\infty$ when (L) holds. Also $m_N = \max\{i,\ i \geq 0,\ H_k(k-h_p)+id_N \in J_k\} \to +\infty$ as $N \to +\infty$. Therefore we may use the lines of the proof of Lemma 2.9 in Stute (1982) to conclude that for any $p \geq 1$,

$$\mathbb{P}(D_N) = \mathbb{P}\left(\max_{1\leq i\leq m_N}\left\{\gamma_N\left(C^N_{i+1}\right)-\gamma_N\left(C^N_i\right)\right\}/b(d_N) \leq (1-1/p)^{\frac{1}{2}}\right)$$

$$= 0\left(N^{\frac{1}{2}}\exp\left(-m_N d_N^{1-1/2p}\right)\right),$$

as $N \to +\infty$, where $C^N_i = H_k(k-h_p)+id_N,\ i = 1,\dots,m_N$. But $m_N d_N = \left(2K(k)^2 h_p k^{-\frac{1}{2}} x d_N^{-1/4p}\right) d_N^{1/4p} = d_N^{1/4p}(1+o(1))$. Hence $\mathbb{P}(D_N) = 0\left(d_N^{-1/8p}\right)$ for large N. Thus $\sum_N \mathbb{P}(D_N) < +\infty$, that is

$$\forall p \geq 1,\ \exists\Omega'_p,\ \mathbb{P}\left(\Omega'_p\right) = 1,$$

$$\forall \omega \in \Omega'_p,\quad \lim_{N\to+\infty}\inf C_{N3}(h_p)/b(d_N) \geq (1-1/p)^{\frac{1}{2}}.$$

Letting

$$\Omega'_0 = \Omega^2 \bigcup \bigcup_{p=1}^{p=+\infty},$$

we get $\mathbb{P}(\Omega'_0) = 1$ and for all $\omega \in \Omega'_0$,

$$\lim_{j\to+\infty}\inf\sup_{x\in I_{j_k}}\left|R^*_{N_{j^2}}(x)\right|/b(d_N) \geq 1. \tag{2.35}$$

We have used in (2.30) that representation for commodity reasons as it has appeared in the proof. The same may be done, step by step, following

Stute's results (see Stute (1982)) to get the version of (2.35) for $R_{N_{j^2}}$ itself. This remark completes the proof of (2.31). □

Continuation of the proof of of Lemma 45. Remark that

$$\begin{aligned}
& \limsup_{N\to+\infty} \sup_{0\le x<+\infty} \left|R_{N2}(x)\right|/b\left(d_N\right) \\
\geq\ & \limsup_{j\to+\infty} \sup_{0\le x<+\infty} \left\{\left|R_{N_{j^2}}(x)\right|/b\left(d_{N_j}\right)\right\} \\
\geq\ & \liminf_{j\to+\infty} \sup_{0\le x<+\infty} R_{N_{j^2}}(x)/b\left(d_{N_j}\right) \\
\geq\ & \liminf_{j\to+\infty} \sup_{x\in I_k} R_{N_{j^2}}(x)/b\left(d_{N_j}\right).
\end{aligned}$$

This combined with (2.31) and with the fact that $b\left(d_N\right)=K\left(k\right)a_N\left(1+o\left(1\right)\right)$ as $N\to+\infty$ proves the Lemma 45. The proof is complete. ■

Conclusion. It is clear by Theorem 61 that the approach used until now cannot yield a rate better than a_N. The problem is now : what new approach would be used to reach, if possible, the very best rate, that of Komlós *et al.* (1975) which is $N^{-\frac{1}{2}}\log N$.

3. The Glivenko-Cantelli Theorem

For the overlapping case, Beirlant and Van Zuijlen (1985) obtained a Glivenko-Cantelli theorem when the step satisfies $kN^{-1+a}\to 0$ as $N\to+\infty$ for some $0<a<1$. As to the overlapping case only fixed steps have been handled in Aly *el al.* (1984). We give the general result in

THEOREM 63. . *Let $k\geq 1$ be fixed or $k\to+\infty$ while $k/N\to 0$ as $N\to+\infty$. Then*

$$\lim_{N\to+\infty}\sup_{0\le x<+\infty}\left|F_N\left(x\right)-H_k\left(x\right)\right|=0, a.s.$$

on the probability space where the spacings are defined.

Proof of Theorem 63. We have

$$\forall N \geq 1, \{F_N(x) - H_k(x), 0 \leq x < +\infty\}$$

$$=^d \left\{\xi_N(x) - H_k(x) + R_{N4}(x) + N^{-\frac{1}{2}} R_{N2}(x) + 0\left(N^{-\frac{1}{2}}\right), 0 \leq x < +\infty\right\}. \tag{3.1}$$

First, it follows from Lemma 43 that for all $\varepsilon > 0$,

$$\sum_p \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} N^{-\frac{1}{2}} \sup_{0 \leq x < +\infty} |R_{N2}(x)| > \varepsilon/4\right) < +\infty.$$

Next,

$$\mathbb{P}\left(\sup_{0 \leq x < +\infty} |R_{N4}(x)| > \varepsilon/4\right) \leq \mathbb{P}\left(|1 - \mu_N| k^{\frac{1}{2}} K(k)^2 > \varepsilon/4\right).$$

And direct calculations imply that for all $\lambda > 1$, we have

$$\mathbb{P}\left(|1 - \mu_N| k^{\frac{1}{2}} K(k)^2 > \varepsilon/4\right) \leq \mathbb{P}\left(|1 - \mu_N| \left(\frac{Nk}{2 \log \log Nk}\right)^{\frac{1}{2}} > \lambda\right)$$

for large N. Thus by (2.9)

$$\sum_p \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \sup_{0 \leq x < +\infty} |R_{N4}(x)| > \varepsilon/4\right) < +\infty$$

whenever $k/N \to 0$ as $N \to +\infty$. Finally,

$$\mathbb{P}\left(\sup_{0 \leq x < +\infty} |\xi_N(x) - H_k(x)| > N^{-\frac{1}{4}}\right) = \mathbb{P}\left(\sup_{0 \leq s < 1} N^{-\frac{1}{2}} |\gamma_N(s)| > N^{-\frac{1}{4}}\right)$$

$$\leq 2N \max_{0 \leq i \leq N} \mathbb{P}\left(U_{i,N} - \frac{i}{N} > N^{-\frac{1}{4}} - N^{-1}\right) = J_N,$$

by the fact that $\gamma_N(.)$ has stationary increments. Using now a representation of γ_N by a Poisson process and an approximation of a Poisson distribution by a Gaussian one (see Lemmas 2.7 and 2.9 in Stute (1982)) to get for large N that

$$J_N \leq const.\ N^{3/2} \mathbb{P}\left(N(0,1) > N^{-\frac{1}{4}} const.\right) \leq const.\ N^{5/4} \exp\left(-N^{1/8}\right).$$

Thus $\sum_N J_N < +\infty$. And the proof of Theorem 63 is now complete. ■

4. The oscillation moduli

The oscillation modulus of a function $R(s)$, $0 \leq s < 1$, is defined by

$$\kappa(d, R) = \sup_{0 \leq h \leq d} \sup_{0 \leq s < 1-h} |R(s+h) - R(s)|, 0 < d < 1.$$

That of the empirical process pertaining to iid rv's has been studied for several choices of d in Mason *et al.* (1983) and Stute (1982). It is remarkable that the weak versions of all those results are inherited by the reduced spacings process $\alpha_N(s) = \beta_N\left(H_k^{-1}(s)\right), 0 \leq s < 1$, (see Lo (1987)). For the strong case, we obtain these two results.

THEOREM 64. *I. The Stute's case.*

If $(d_N)_{N \geq 1}$ *is a sequence of non-increasing positive reals such that*

(S1) $$N d_N \to +\infty,$$

(S2) $$\left(\log d_N^{-1}\right) / (N d_N) \to 0,$$

(S3) $$\left(\log d_n^{-1}\right) / \log\log N \to +\infty,$$

(S4) $$\left(2 d_N \log d_N^{-1}\right)^{\frac{1}{2}} / a_N =: q_N / a_N \to +\infty, \text{ as } N \to +\infty,$$

then for $k \geq 1$ *fixed or* $k = k(N) \to +\infty$ *as* $N \to +\infty$ *and satisfying*

(4.1) $$\exists N_o, \delta > 2, \ \forall N \geq N_o, 0 < d_N < k^{k(\delta-2)} \exp\left(-\frac{1}{2} k^\delta\right).$$

We have $\lim_{N \to +\infty} \sup \kappa(d_N, \alpha_N) / q_N = 1 \ \ a.s.$

II. A Mason-Wellner-Shorack case.

Let $a_N = \alpha(\log N)^{-c}, \alpha > 0, c > 0$. *Then under the same assumptions on k used in Part I, we have* $\lim_{N \to +\infty} \sup \kappa(d_N, \alpha_N) / q_N \leq (1+c)^{\frac{1}{2}}, a.s.$

Proof of Part I of Theorem 64. We have by Lemmas 42 and 43,

$$\forall N \geq 1, \{\alpha_N(s), 0 \leq s < 1\} \, d_= \left\{\Lambda_N\left(H_k^{-1}(s)\right) + R_{N5}(s) + R_{N6}(s), 0 \leq s < 1\right\}$$

$$=: \{\bar{\alpha}_N(s), 0 \leq s < 1\}, \tag{4.2}$$

with

$$R_{N5}(s) = N^{\frac{1}{2}}(\mu_N - 1) H_k^{-1}(s) H_k'\left(H_k^{-1}(s)\right) =: N^{\frac{1}{2}}(\mu_N - 1)\phi(s), 0 \leq s < 1,$$

and

$$\sum_p \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \sup_{0 \leq s < 1} |R_{N6}(s)| > (1+\varepsilon) a_N K(k)\right) < +\infty, \tag{4.3}$$

by (4.3) and $(S4)$, we have

$$\sum_p \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \kappa(d_N, R_{N6}) > \varepsilon q_N/3\right) < +\infty. \tag{4.4}$$

By Lemma A4 in Lo (1987), $\kappa(d_N, \phi) = (1 + o(1)) q_N^2$ as $N \to +\infty$ for all k satisfying $(S5)$. Thus, by the loglog-law,

$$\sum_p \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \kappa(d_N, R_{N5}) > \varepsilon q_N/3\right) < +\infty,$$

whenever

$$\lim_{N \to +\infty} k^{-1} d_N \log\log(1/d_N) \log\log \; Nk = 0 \tag{4.5}$$

is satisfied. This obviously follows from $(S1)$, $(S2)$, $(S3)$, $(S4)$ and $(S5)$. By the results of Stute (1982) as recalled in (2.14), for $\varepsilon > 0$,

$$\sum_p \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \kappa\left(d_N, \Lambda_N\left(H_k^{-1}\right)\right) > (1+\varepsilon/3)\, q_N\right) < +\infty, \tag{4.6}$$

when $(S1)$, $(S2)$ and $(S3)$ hold. Since ε is arbitrary and since $(S1)$ and $(S3)$ imply (4.5), we get

$$\lim_{N\to+\infty} \sup q_N^{-1} \kappa\left(d_N, \alpha_N\right) \leq 1, a.s. \tag{4.7}$$

To get the other inequality, define for $0 < c_1 < c_2 < +\infty, 0 < d < 1$, for any function $R(s), 0 \leq s < 1$,

$$\kappa'(d,\ R) = \sup_{c_1 d < u-t < c_2 d} |R(u) - R(t)| / \sqrt{u-t}, 0 \leq u,\ t \leq 1. \tag{4.8}$$

Let $R_N(.) = R_{N5}(.) + R_{N6}(.)$ and $r_N(.) = \Lambda_N\left(H_k^{-1}(.)\right)$. Now remark that for all $\varepsilon_1 > 0$, there exists $\varepsilon_2 > 0$ such that for

$$a = \left((1-\varepsilon_1)\log d_N^{-1}\right)^{\frac{1}{2}}$$

and

$$b = \left(\varepsilon_2 \log d_N^{-1}\right)^{\frac{1}{2}},$$

$$a+b = \left(\left(1-\varepsilon_1+\varepsilon_2+2\left(\varepsilon_2\left(1-\varepsilon_1\right)\right)^{\frac{1}{2}}\right)\log d_N^{-1}\right)^{\frac{1}{2}} = \left((1-\varepsilon_3)\log d_N^{-1}\right)^{\frac{1}{2}}$$

with $\varepsilon_3 > 0$, $\varepsilon_3, \varepsilon_2 \to 0$ as $\varepsilon_1 \to 0$. Thus,

$$\mathbb{P}\left(\kappa'\left(d_N, \alpha_N\right) \leq a\right) \leq \mathbb{P}\left(\left\{\kappa'\left(d_N, \bar{\alpha}_N\right) \leq a\right\} \bigcup \left\{\kappa'\left(d_N, R_N\right) > b\right\}\right)$$

$$+\mathbb{P}\left(\left\{\kappa'\left(d_N, \bar{\alpha}_n\right) \leq a\right\} \prod \left\{\kappa'\left(d_N, R_N\right) \leq b\right\}\right)$$

$$\leq \mathbb{P}\left(\kappa'\left(d_N, R_N\right) > b\right) + \mathbb{P}\left(\kappa'\left(d_N, r_N\right) \leq a+b\right), \tag{4.9}$$

By (4.3) and (4.4)

$$\sum_{p} \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}} \kappa'\left(d_N, R_N\right) > b\right) < +\infty, \tag{4.10}$$

for all $\varepsilon_2 > 0$. Thus by (4.2), (4.8), (4.9) and (4.10) and Lemma 2.9 of Stute (1982) and some straightforward considerations, we get $\lim_{N\rightarrow+\infty} \inf \kappa'\left(d_N, \alpha_N\right) \geq 1, a.s.$, under $(S1)$, $(S2)$, $(S3)$ and $(S4)$. Letting $c_1 = c_2 = 1$,

$$\liminf_{N\rightarrow+\infty} \kappa\left(d_N, \alpha_N\right) \geq \lim_{N\rightarrow+\infty} \inf \kappa'\left(d_N, \alpha_N\right) \geq 1, a.s. \tag{4.11}$$

(4.7) and (4.11) together complete the proof of Part I of Theorem 64.

Proof of Part II of Theorem 64.

Here $(S3)$ and $(S4)$ are satisfied. It suffices thus to write again the proof of the part one where one should use the probability inequality (2.4) of Mason *et al.* (1983). It must be noticed that Part III of Theorem 1 in Mason *et al.* (1983) holds for the general case where $a_N = \alpha\left(\log_N\right)^{-c}, 0 < \alpha, 0 < c$.

APPENDIX. PROOFS OF STATEMENTS (2.8) AND (2.9).

a) **Proof of Statement (2.8).**

Tchebychev's inequality yields $\alpha > 1$ and $\beta > 1$ such that $\mathbb{P}\left(S_n 2/n^2 > 1+\varepsilon\right) \leq A_2 n^{-\alpha}$ and $\mathbb{P}\left(\left|S_n - S_{m(n)}\right| > n\varepsilon/2\right) \leq A_3 n^{-\beta}$ as $n \to +\infty$, where

$$m\left(n\right) = \max\left\{j^2, j^2 \leq n,\ j = 1, 2, ...\right\}.$$

Thus

$$\mathbb{P}\left(\left|\mu_N - S_{n+1}/\left(n+1\right)\right| > \varepsilon/2\right) + \mathbb{P}\left(S_{n+1} \geq 1 + \varepsilon/2\right)$$

(4.12)

$$\leq \mathbb{P}\left(\left|\mu_N - S_{n+1}/\left(n+1\right)\right| > \varepsilon/2\right) + \left(A_2 + o\left(1\right)\right) k^{-\alpha} N^{-\alpha} + \left(A_3 + o\left(1\right)\right) k^{-\beta} N^{-\beta},$$

since $(n+1) \sim Nk$ as $N \to +\infty$. Furthermore, by Tchebychev's inequality,

$$\mathbb{P}\left(S_{Nk}/Nk - \left(Nk\right)^2 > \varepsilon/8\right) \leq 64 N^{-3} k^{-3}/\varepsilon^2$$

$$\mathbb{P}\left(S_k/\left(Nk\right)^2 - \left(Nk\right)^2 > \varepsilon/8\right) \leq 64 N^{-4} k^{-3}/\varepsilon^2$$

and

$$\mathbb{P}\left(\left|\mu_N - S_{n+1}/\left(n+1\right)\right| \geq \varepsilon/2\right) \leq \mathbb{P}\left(S_{Nk}/Nk > \varepsilon/4\right) + \mathbb{P}\left(S_k/\left(Nk\right)^2 > \varepsilon/4\right).$$

Hence since $Nk \to +\infty$, $N^2 k \to +\infty$ as $N \to +\infty$,

(4.13)
$$\sum_N \mathbb{P}\left(\left|\mu_N - S_{n+1}/n+1\right| > \varepsilon/2\right) < +\infty.$$

Thus (4.12) and (4.13) together imply (2.8).

Proof of (2.9). We have

$$\frac{S_{n+1} - Nk}{\left(2Nk \log\log Nk\right)^{\frac{1}{2}}} = \frac{S_{n+1} - S_{Nk}}{\left(2Nk \log\log Nk\right)^{\frac{1}{2}}} + \frac{S_{n+1} - Nk}{\left(2Nk \log\log Nk\right)^{\frac{1}{2}}} =: S'_N + S''_N.$$

First, since $0 \leq (n+1) - Nk \leq k$,

$$\mathbb{P}\left(S'_N > \varepsilon/2\right) \leq \mathbb{P}\left(S_k > \varepsilon\left(2Nk\log\log Nk\right)^{\frac{1}{2}}/2\right)$$

$$\leq 1 - H_k\left(k^{\frac{1}{2}}N^{\frac{1}{2}}\right) \leq const.\exp\left(-\frac{1}{4}k^{\frac{1}{2}}N^{\frac{1}{2}}\right)$$

as $k/N \to 0$, $N \to +\infty$ (see Statement (2.20)). Thus

$$\sum_N \mathbb{P}\left(S'_N > \varepsilon/2\right) < +\infty. \tag{4.14}$$

Now, let

$$p = p(N) = \inf\{j, N > N_j\}$$

and

$$q(N) = \inf\left\{j, k(N) > N_j = \left[(1+\rho)^j\right],\ j = 1, 2, ...\right\}$$

Then $N_{p-1} \leq N \leq N_p, N_{p-1}N_{q-1} \leq NK \leq N_pN_q, \log\log N_pN_q = (\log\log N_p)(1+o(1))$, as $N/k \to +\infty, N \to +\infty, N_{p+1}/N_p \to 1+\rho$, as $N \to +\infty$. Thus (see Loève (1974), p.259-262).

$$\mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}}\{S''_N \geq 1+\varepsilon/2\}\right) \leq A_4\mathbb{P}\left(S_{N_pN_q} > 1+\delta(\varepsilon,\rho)\left(2N_p loglog N_p\right)^{\frac{1}{2}}\right)$$

$$\leq A_5 p^{-(1+\delta(\varepsilon,\rho))}$$

as $p \to +\infty$, for ρ small enough, $\delta(\varepsilon,\rho) > 0$. The same holds for $-S''_N$. Thus

$$\sum_p \mathbb{P}\left(\bigcup_{N=N_p}^{N=N_{p+1-1}}\left(|S''_N| > 1+\varepsilon/2\right)\right) < +\infty. \tag{4.15}$$

Finally (4.14) and (4.15) together imply (2.9).

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