CHAPTER 21

Strong limits related to the oscillation modulus of the empirical process based on the k-spacing process, *by G. S. Lo*

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Abstract. We extend strong laws related to the *iid* empirical process to the reduced empirical process based on the (non-overlapping) k-spacings generated by a sequence of *iid* $(0,1)$-uniform random variables, and on remarkable results on the increments of gamma probability laws of parameters k and one.

Keywords. Oscillation modulus; empirical processes; increments of functions; law of the iterated logarithm;order statistics.
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Foreword This paper was part of the Science Doctorate of the candidate (Cheikh Anta Diop University, 1991) that was not yet published in a peer-reviewed journal. A slightly different version was published in *Rapports Techniques, LSTA, Université Paris VI*, 48, 1986, under the same title.

Full Abstract. Recently, several strong limit theorems for the oscillation moduli of the empirical process have been given in the iid-case. We show that, with very slight differences, those strong results are also obtained for some representation of the reduced empirical process based on the (non-overlapping) k-spacings generated by a sequence of independent random variables (rv's) uniformly distributed on $(0,1)$. This yields weak limits for the mentioned process. Our study includes the case where the step k is unbounded. The results are mainly derived from several properties concerning the increments of gamma functions with parameters k and one.

1. Introduction and statement of the results

Consider $U_1, ..., U_n$ a sequence of independent rv's uniformly distributed on $(0,1)$, and let

$$U_{0,n} = 0 \leq U_{1,n} \leq ... \leq U_{n,n} \leq U_{n+1,n} = 1$$

be their order statistics. The rv's

$$D_{i,n}^k = U_{ki,n} - U_{(i-1)k,n}, 1 \leq i \leq \left[\frac{n+1}{k}\right] = N,$$

where $[x]$ denotes the integer part of x, are called the non-overlapping k-spacings. Throughout, we shall assume that N and k are given and that n is defined by $n = \inf\left\{j, \left[\frac{j+1}{k}\right] = N\right\}$ and then we will be able to study all our sequences as indexed by N since k will be either fixed or function of N.

The study of the properties of $D_{i,n}^k$ was introduced by Pyke (1972) and several related papers have appeared in recent years (see e.g. Deheuvels (1984)). One of the problem concerning the k-spacings is the study of the empirical process associated with $Nk\ D_{i,n}^k, 1 \leq i \leq N$.

In order to give a comprehensible definition of that process, we recall the following representation which can be found in Aly *et al.* (1984) in the case where $(n+1)/k$ is an integer :

(1.1)
$$\left\{ D_{i,n}^k, 1 \le i \le N \right\} =^d \left\{ \frac{Y_i}{S_{n+1}}, 1 \le i \le N \right\}$$

$$= : \left\{ \frac{\left(\sum_{j=(i-1)k+1}^{j=ik} E_i \right)}{S_{n+1,}}, 1 \le i \le N \right\},$$

where $=^d$ denotes the equality in distribution and S_n is the partial sum associated with $E_1, ..., E_n$, a sequence of independent and exponential rv's with mean one, i.e., $S_n = E_1 + ... + E_n$. Thus, it follows that, if $\frac{(n+1)}{k}$ is an integer, the limiting distribution function of $NkD_{i,n}^k$, for any i and k fixed, is

$$H_k(x) = \int_0^x \frac{t^{k-1}e^{-t}}{(k-1)!} dt, \ x \ge 0.$$

Therefore the empirical process (E.P.) associated with $NkD_{i,n}^k, 1 \le i \le N$, may be defined by

(1.2)
$$\beta_N(x) = N^{\frac{1}{2}} \left\{ F_N(x) - H_k(x) \right\}, 0 \le x \le +\infty,$$

where F_N is the empirical distribution function (*E.D.F.*) of $Nk \ D_{i,n}^k, 1 \le i \le N$, with

(1.3)
$$F_N(x) = \# \frac{\left\{ i, 1 \le i \le N, \ Nk \ D_{i,n}^k \le x \right\}}{N}, x \ge 0.$$

Straightforward manipulations from (1.1), (1.2) and (1.3) as given in Aly *et al.* (1984) show that even in the general case where $(N-1)k \le n+1 \le Nk$, the reduced process $\alpha_N(s) = \beta_N\left(H_k^{-1}(s) \right), 0 \le s \le 1$, satisfies

(1.4) $\left\{ \alpha_N(s), 0 \le s < 1 \right\} =^d \left\{ N^{\frac{1}{2}} \left\{ \xi_N\left(\delta_n H_k^{-1}(s) \right) - s \right\} + 0\left(N^{-\frac{1}{2}} \right), 0 \le s \le 1 \right\},$

where H_k^{-1} is the inverse function of H_k, ξ_N is the *E.D.F.* pertaining to $Y_1, ..., Y_n$ and $\delta_n = \frac{S_{n+1}}{Nk}$.

The aim of this paper is to give the behavior of the oscillation modulus of $\alpha_N(.)$ both where k is fixed and where $k \uparrow +\infty$. To this end we define

$$\wedge_N(a_N, R_N) = \sup_{0 \leq h \leq a_N} \sup_{0 \leq s \leq 1-h} |R_N(s+h) - R_N(s)|$$

and

$$k_N(a_N, R_N) = \frac{\wedge_N(a_N, R_N)}{\left(2a_N \log \log a_N^{-1}\right)^{\frac{1}{2}}},$$

for any sequence of functions $R_N(s), 0 \leq s \leq 1$ and for any sequence $(a_N)_{N \geq 1}, 0 < a_N < 1$. The properties of $\wedge_N(a_N, R_N)$, the oscillation modulus of R_N, have been first described by Csőrgö and Révèsz (1981) and Stute (1981) when R_N represents the *E.P.F.* pertaining to a sequence of independent and uniformly distributed rv's with

(S1) $$Na_N \to +\infty,$$

(S2) $$\frac{\left(\log a_N^{-1}\right)}{(Na_N)} \to 0$$

and

(S3) $$\frac{\left(\log a_N^{-1}\right)}{\log \log N \to +\infty}$$

as $N \to +\infty$. Later, Mason, Shorack and Wellner (MSW) (see Mason *et al.* (1983)) dealt with the same for several choices of (a_N) and give among the results an Erdös-Renyi law.

The chief achievement of this paper is the extension of those limit results to some sequence of process $\bar{\alpha}_N$ equal in distribution to α_N. In fact, the fundamental role is played here by the properties of the tails of the gamma function $H_k'(.)$, the derivative function of H_k. These properties are established in Section 2 through technical lemmas and the proofs of the following results are given in Section 3.

THEOREM 57. *Let k be fixed. Then, there exists a sequence of processes* $\alpha_N^-(s), 0 \leq s \leq 1, N = 1, 2, \ldots$ *such that*

(1.5) $$\forall N \geq 1, \{\alpha_N(s), 0 \leq s \leq 1\} \, d_= \{\alpha_N^-(s), 0 \leq s \leq 1\}.$$

(I) If $(a_N)_{N \geq 1}$ *is a sequence of non-decreasing numbers satisfying the Csörgo-Révèsz-Stute conditions (S1), (S2) and (S3), then*

$$\lim_{N \uparrow +\infty} k_N\left(a_N, \alpha_N^-\right) = 1, a.s.$$

(II) If

$$a_N = cN^{-1} \log N, c > 0, N \geq 1$$

then

$$\lim_{N \uparrow +\infty} k_N^\circ\left(a_N, \alpha_N^-\right) = \left(\frac{c}{2}\right)^{\frac{1}{2}} \left(\beta^+ - 1\right), \; a.s., \; where \; \beta^+ > 1$$

and

$$\beta^+ \left(\log \beta^+ - 1\right) = c^{-1} - 1.$$

(III)

$$If a_N = (\log N)^{-c}, c > 0,$$

then

$$c^{\frac{1}{2}} \leq \lim_{N \to +\infty} \inf k_N\left(a_N, \alpha_N^-\right) \leq \lim_{N \to +\infty} \sup k_N\left(a_N, \bar{\alpha}_N\right) \leq (1 + c)^{\frac{1}{2}}, a.s.,$$

(IV) If

$$a_N = c_N N^{-1} \log N, c_N \to 0$$

such that

$$(c_N \log N) = Na_N \to +\infty$$

and

$$(\log N)^{-1} \left(\log c_N^{-1}\right) \log \log N \to 0 \; as \; N \uparrow +\infty,$$

then

$$\lim_{N \uparrow +\infty} \sup \frac{N^{\frac{1}{2}} \log\left(\frac{1}{c_N}\right)}{\log N} \wedge_N \left(a_N, \bar{\alpha}_N\right) \leq 2, \textbf{\textit{a.s.}}$$

We also have

THEOREM 58. *If $k = k(N) \to +\infty$ such that for some $\delta > 2$ and for some N_0,*

$$0 < a_N \leq t_k(\delta) = k^{k(\delta-2)} \exp\left(\frac{-k^\delta}{2}\right), \ N \geq N_0,$$

then Parts (I), (II), (III) and (IV) remain true.

REMARK 20. *If each α_N is the spacings empirical process based on a sample depending on N, say $\chi(N)$, and if these samples $\chi(N)$, $N = 1, 2, ...$ are mutually independent (this statistical situation is quite conceivable, for instance when checking homogeneity) the strong limit results of Theorem 57 are also valid for α_N. One might seek other conditions to get the same extensions. Here, we restrict ourselves to weak extensions in the following*

COROLLARY 18. *Let k be either fixed or $k \to +\infty$. Let $(a_N)_{N \geq 1}$ be a sequence of positive numbers such that $0 \leq a_N \leq t_k(\delta)$ when $N \geq N_0$, for some N_0 and $\delta > 2$. Then:*

(I) Under the assumptions of Part I of Theorem 57, we have

$$\lim_{N \to +\infty} k_N(a_N, \alpha_N) = 1 \text{ in probability.}$$

(II) Under the assumptions of Part II of Theorem 57, we have

$$\lim_{N \to +\infty} k_N(a_N, \alpha_N) = \left(\frac{c}{2}\right)^{\frac{1}{2}} \left(\beta^+ - 1\right) \text{ in probability.}$$

(III) Under the assumptions of Part III of Theorem 57, we have

$$\lim_{N \to +\infty} k_N(a_N, \alpha_N) = c^{\frac{1}{2}}, \text{ in probability.}$$

(IV) Under the assumptions of Part IV of Theorem 57, we have

$$\lim_{N \to +\infty} P\left(\frac{N^{\frac{1}{2}} \log\left(\frac{1}{c_N}\right)}{\log N} \right) \wedge_N (a_N, \alpha_N) > (2 + \varepsilon) = 0 \text{ for all } \varepsilon > 0.$$

REMARK 21. *It appears from Theorems 57 and 58 that the oscillation modulus of $\bar{\alpha}_N$ and that of the uniform empirical process are almost the same. In LO (1986), we prove that the exact strong bounds in (I) and (II) remain valid for α_N when a_N satisfies further conditions.*

REMARK 22. *One might think that deriving the result of our corollary by using invariance principles (as given in Aly et al. (1984) and LO (1986)) and well-known results for the Brownian bridge would be easier (at least for some sequences a_N). This is not true at all (see Remark 23 below).*

2. Technical lemmas

It will follow from Lemma 1 of section 3 that the increments of $\bar{\alpha}_N$ behave as the increments of $\gamma_N(\psi(.))$ and those of $\phi(.)$ where $\gamma_N(.)$ is the E.P. pertaining to

$$U_1, ..., U_N, \psi(s) = H_k\left(\mu_n H_k^{-1}(s)\right), 0 \leq s \leq 1, \mu_n d_= \delta_n,$$

$$n = 1, 2, ..., \phi(s) = H_k'\left(H_k^{-1}(s)\right) H_k^{-1}(s), 0 \leq s \leq 1,$$

with $H_k'(x) = \frac{dH_k(x)}{dx}$, for all positive x. Then, since $k_N(., \gamma N)$ is known, our study is reduced to describing the increments of $\psi(.)$ and that of $\phi(.)$, what we do in this paragraph.

LEMMA 37. *Let k be fixed and $a = a_N$ be a sequence of positive numbers satisfying*

(Q1) $$\left(n^{-1} \log \log n\right)^{\frac{1}{2}} \log\left(\frac{1}{a}\right) \to 0 \text{ as } N \to +\infty \text{ and } a \to 0,$$

then as $N \to ++\infty$, we have the following properties

(2.1) $$\sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} |\psi(s+h) - \psi(s)| = a(1 + o(1)) \text{ a.s.},$$

uniformly in s,

(2.2) $$0 \leq s \leq 1 - a, |\psi(s+a) - \psi(s)| = a(1 + q(a)),$$

where $q(a) \to 0$, a.s., as $a \to 0$.

Proof of lemma 37. We need several properties of gamma functions. First note that for a fixed k,

(2.3) $$s = 1 - H_k(s) = \frac{e^{-x} x^{k-1}}{(k-1)!}\left\{1 + \frac{k-1}{x} + \frac{k-2}{x^2} + ... + \frac{(k-a)!}{x^{k-1}}\right\},$$

and

$$x = H_k^{-1}(1-s) = \log\left(\frac{1}{s}\right) - \log(k-1)! + (k-1)\log x$$

(2.4) $$+ \log\left(1 + \frac{k-1}{x} + ... + \frac{(k-1)!}{x^{k-1}}\right)$$

from $k - 1$ integrations by parts. Next for a fixed k or for $k \to +\infty$, we have, as $x \downarrow 0$,

$$(2.5) \qquad s = H_k(x) = \frac{x^k}{k!} (1 + 0(x)),$$

and

$$(2.6) \qquad x = H_k^{-1}(s) = (k!)^{\frac{1}{k}} \left(1 + O\left(\frac{x}{k}\right)\right),$$

where for any function $g(.)$, $g(x) = O(y)$ as $x \downarrow 0$ means that $\lim_{x \downarrow 0} \sup \left|\frac{g(x)}{y}\right| < +\infty$. To see this, use the following inequalities:

$$0 \le t \le x \Rightarrow e^{-x} \le e^{-t} \le 1,$$

to obtain that $e^{-x} \frac{x^k}{k!} \le H_k(x) \le \frac{x^k}{k!}$ and the results follow. Now, we are able to prove lemma A1.

Let us continue the proof of Lemma 37. Define

$$(2.7) \qquad \Psi_h(s) = \psi(s + h) - \psi(s), 0 \le s \le 1 - h, 0 \le h \le a, h = 1, 2, ...$$

Straightforward computations give

$$(2.8) \qquad \frac{d\Psi_h(s)}{ds} = \mu_n^{k-1} \left\{\exp\left((\mu_n - 1) H_k^{-1}(s + h)\right) - \exp\left((\mu_n - 1) H_k^{-1}(s)\right)\right\}.$$

Thus, for each elementary event ω of the probability space, for each N (that is to say for each n) and for each h, $\Psi_{h.}(.)$ is non-decreasing of non-increasing according to the sign of $\mu_n(\omega) - 1$. Thus we have

$$(2.9) \qquad \sup_{0 \le s \le 1 - h} |\Psi_h(s)| = \max\{\Psi_h(0), |\Psi_h(1 - h)|\}.$$

Computation of $\Psi_h(1 - h)$. By using (2.3) and (2.4), with $h = 1 - H_k(x)$, we have

$$\mu_n H_k^{-1}(1 - h) = \mu_n \log\left(\frac{1}{h}\right) - \mu_n \log(k - 1)! + \mu_n(k - 1) \log x$$

$$+\mu_n \log \left(1 + ... + \frac{(k-1)!}{x^{k-1}} \right)$$

Now recall that $\Psi_h (1-h) = 1 - H_k \left(\mu_n H_k^{-1} (1-h) \right)$ and, using (2.3), get

$$(2.10) \qquad \Psi_h (1-h) = \mu_n^{k-1} \frac{\left(\log \left(\frac{1}{h} \right) \right)^{k-1} h^{\mu_n} \left((k-1)! \right)^{\mu_n}}{(k-1)!} x^{-(k-1)\mu_n} \left(1 + q_1 (h) \right),$$

where there exists A_k and B_k depending only on k (k being fixed) such that

$$(2.11) \qquad |q_1 (h)| \le A_k x^{-1} \log x + B_k x^{-1}, \; as \; h \to 0 \; (i.e. \; as \; x \to +\infty).$$

These constants A_k and B_k are provided by the approximation

$$\left| \frac{\log \left(\frac{1}{h} \right)}{x} - 1 \right| \le A_k x^{-1} \log x + B_k x^{-1},$$

as $h \to 0$, $x = H_k^{-1} (1-h) \to +\infty$. And (2.11) leads to

$$(2.12) \qquad \Psi_h (1-h)$$
$$= h^{MU_n} \left(\log \left(\frac{1}{h} \right) \right)^{(k-1)(1-\mu_n)} \left(1 + q_2 (h) \right)^{-(k-1)\mu_n+1} \left(1 + q_3 (N) \right),$$

where $q_2 (.)$ satisfies(2.11) with the same constants A_k and B_k and $q_3 (N) = o(1)$, a.s., independently of h, $0 \le h \le 1$, as $N \to +\infty$. Since the functions $x^{-1} \log x$ and x^{-1} are non-increasing as $x \to +\infty$, it follows from (2.10), (2.11) and (2.12) that

$$(2.13) \qquad \forall 0 \le h \le a, \Psi_h (1-h) = (1 + q(a)) h^{\mu_n} \left(\log \left(\frac{1}{h} \right) \right)^{(k-1)\mu_n},$$

where q(a) $\to 0$ and N$\to +\infty$. By convention, we shall write $g(h) = q(a)$ for $0 \le h \le a$, for all h, $0 \le h \le a$, $g(h) = o(1)$ where the $o(1)$ depends only on a, as $a \to 0$.

Computation of $\Psi_n(0)$.

We have $\Psi_h(0) = H_k\left(\mu_n H_k^{-1}\right)$. Then by using (2.5)-(2.6), we obtain

$$(2.14) \qquad \mu_n H_k^{-1}(h) = \mu_n (k!)^{\frac{1}{k}} h^{\frac{1}{k}} (1 + q(a)), 0 \leq h \leq a.$$

Use again (2.5)-(2.6) and get $H_k\left(\mu_n H_k^{-1}(h)\right) = h\mu_n^k (1 + q(a))$, a.s., $0 \leq h \leq a \to 0$, since k is fixed and $\mu_n \to 1$, a.s., as $N \to +\infty$. Then,

$$(2.15) \quad \sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} \Psi_h(s) = (1 + q(a)) \sup_{0 \leq h \leq a} \max\left\{h, h^{\mu_n}\left(\log h^{-1}\right)^{(k-1)(1-\mu_n)}\right\},$$

a.s., and $N \to +\infty$. But,

$$(2.16) \;\; \forall N \geq 1, \frac{d\left\{h^{\mu_n}\left(\log h^{-1}\right)^{(k-1)(1-\mu_n)}\right\}}{dh} = h^{\mu_n}\left(\log h^{-1}\right)^{(k-1)(1-\mu_n)}$$

$$(2.17) \qquad\qquad\qquad\qquad \times \left\{\mu_n - \frac{(k-1)(1-\mu_n)}{\log h^{-1}}\right\}.$$

Thus $h^{\mu_n}\left(\log h^{-1}\right)^{(k-1)(\mu_n-1)}$ is non-decreasing when n sufficiently large since $k(1-\mu_n) \to 0$, a.s., as $N \to +\infty$ by the strong law of large numbers (k being fixed).Then,

$$(2.18) \qquad 0 \leq h \leq a \Rightarrow h^{\mu_n}\left(\log h^{-1}\right)^{(k-1)(1-\mu_n)} \leq a^{\mu_n}\left(\log a^{-1}\right)^{(k-1)(1-\mu_n)},$$

a.s., as $N \to +\infty$. Furthermore,

$$(2.19) \qquad \left(\log a^{-1}\right)^{(k-1)(1-\mu_n)} = \exp\left((k-1)(1-\mu_n)\log\log a^{-1}\right) = 1 + o(1),$$

a.s., whenever $(1-\mu_n)\log\log a^{-1} \to 0$, a.s. But this is implied by (Q1). Indeed, we have by the law of the iterated logarithm (the loglog law) that

$$(2.20) \qquad \lim_{N \to +\infty} \sup \left(2n^{-1}\log\log n\right)^{\frac{1}{2}} |\mu_n - 1| \leq 1, \; a.s.$$

This together with (Q1) imply that $(1-\mu_n)\log\log a^{-1} \to 0$, a.s., as $N \to +\infty$. In fact, the loglog law holds for δ_n, that is

(2.21)
$$\lim_{N \to +\infty} \sup \left(2n^{-1} \log\log n\right)^{\frac{1}{2}} |\delta_n - 1| \leq 1, a.s.$$

But (2.21) may be obtained from (see LO (1986), Appendix)

$$\sum_{p \geq 0} P \left(\bigcup_{n_p}^{n_{p+1}-1} \left\{ \left(2n^{-1} \log\log n\right) |\delta_n - 1| \geq 1 + \frac{\varepsilon}{2} \right\} \right) < +\infty,$$

where (n_p) is an increasing and unbounded sequence of positive integers and $\varepsilon > 0$ is arbitrary. This and the equality in distribution of δ_n and μ_n for each N imply (2.20). The same loglog-law shows that (Q1) implies that

(2.22)
$$a^{1-\mu_n} = \exp\left((1 - \mu_n) \log a^{-1}\right) = 1 + o(1) \ \ as, \ N \to +\infty.$$

We finally get from (2.13), (2.18), (2.19) and (2.22) that

(2.23)
$$\sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} |\Psi_h(s)| = a(1 + o(1)) \, a.s., ad N \to +\infty,$$

which proves Part (2.1) of Lemma 37. To prove part (2.2), it suffices to remark that we may have through (2.8) that

$$\min\left(\phi_a(0), \phi_a(1-a)\right) \leq \phi_a(s) \leq \max\left(\phi_a(0), \phi_a(1-a)\right), 0 \leq s \leq -a,$$

and the part in question follows since the first part implies that $\phi_a(0) = a(1 + o(1))$, a.s. and $\phi_a(1-a) = a(1 + o(1))$, a.s., as $N \to +\infty$.

LEMMA 38. *Let k be fixed, then we have as $a \to 0$, $N \to +\infty$,*

$$\sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} |\phi(s) - \phi(s+h)| = \left(a \log a^{-1}\right)(1 + o(1)).$$

Proof of Lemma 38. Consider $\Phi_h(s) = \phi(s+h) - \phi(s)$, $0 \leq s \leq 1-h$. Direct considerations yield that

$$\frac{d\Phi_h(s)}{ds} = H_k^{-1}(s) - H_k^{-1}(s+h), 0 \leq s \leq 1-h.$$

Then for each h, $\Phi_h (.)$ is non-increasing and thus,

$$\sup_{0 \leq s \leq 1-h} |\Phi_h (s)| = \max \{|\Phi_h (0)|, \ |\Phi_h (1 - h)|\}.$$

But, by (2.5)-(2.6),

$$\Phi_h (0) = H'_k \left(H_k^{-1} (h)\right) H_k^{-1} (h) = kh \left(1 + q (a)\right),$$

$0 \leq h \leq a \to 0$. Here we omit the details concerning the uniform approximations which provide $q (.)$. These details are very similar to those of the computation of $\Phi_h (0)$. By the considerations that were previously used for getting (2.10) from (1.1), we have

$$\Phi_h (1 - h) = H'_k \left(H_k^{-1} (1 - h)\right) H_k^{-1} (1 - h) = h \log h^{-1} \left(1 + q (a)\right), 0 \leq h \leq a \to 0.$$

Notice that $H'_k \left(H_k^{-1} (1 - h)\right)$ yields something like (2.10) while $H_k^{-1} (1 - h)$ yields $(\log h^{-1}) (1 + q (a))$, $0 \leq h \leq a \to 0$. We obtain

$$\sup_{0 \leq h \leq a} \sup_{0 \leq s \leq 1-h} |\Phi_h (s)| = 1 + q (a) \sup_{0 \leq h \leq a} \max \left(kh, \ h \log h^{-1}\right) = \left(1 + q (a) \left(a \log a^{-1}\right)\right),$$

$a \to 0$, since k is fixed here. Hence Lemma 38 is proved.

Now, we concentrate on the case where $k \to +\infty$. First, we give the following

PROPOSITION 22. *Let*

$$0 \leq s \leq t_k (\delta) = k^{k(\delta-2)} \exp \left(-\frac{1}{2} k^\delta\right),$$

Then, as $k \to +\infty$, we have

(2.24) $$x = H_k^{-1} (1 - s) = \left(\log s^{-1}\right) (1 + q_4 (s)),$$

where there exist A and k_0 such that $|q_4 (s)| \leq A \frac{\log k}{k^{\delta-1}}$ for all $0 \leq s \leq t_k (\delta)$, $k \geq k_0$.

Proof Integrating by parts, we get

$$\forall x \geq 0, \frac{x^{k-1}e^{-x}}{(k-1)!} \leq 1 - H_k(x) \leq \frac{x^{k-1}e^{-x}}{(k-1)!} + \frac{k}{x}(1 - H_k(x))$$

Then,

(2.25) $$\frac{x^{k-1}e^{-x}}{(k-1)!} \leq 1 - H_k(x) \leq \left\{1 - \frac{k}{x}\right\}^{-1} \frac{x^{k-1}e^{-x}}{(k-1)!}, \forall x \geq 0.$$

We are able to see that the expansion of $H_k(x)$ is then possible if $k/x \to 0$. Now, let $0 \leq s \leq s_k = 1 - H_k(k^\delta)$. Apply (2.25) and get

$$0 \leq s \leq s_k \Rightarrow s = 1 - H_k(x) = \frac{x^{k-1}e^{-x}}{(k-1)!}\{1 + q_5(x)\},$$

with $|q_5(x)| \leq \left(1 - k^{-(\delta-1)}\right)k^{-(\delta-1)}$ for all $0 \leq s \leq s_k$. But

$$s_k = \frac{k^{\delta(k-1)}e^{-k}}{(k-1)!}\left(1 + 0\left(k^{-(\delta-1)}\right)\right).$$

Then by Sterling's formula and some straightforward calculations, it is possible to find a k_1 such that $t_k(\delta) = k^{k(\delta-2)}\exp\left(-\frac{1}{2}k^\delta\right) \leq s_k$ for all $k \leq k_1$. Then for $0 \leq s \leq s_k, k \geq k_1$,

(2.26) $$x = H_k^{-1}(1 - s) = \left(\log s^{-1}\right) - \log(k-1)! + (k-1)\log x + o(q_5(s)).$$

Now since

$$0 \leq s \leq s_k \Rightarrow \left|\frac{(k-1)\log x}{x}\right| \leq \frac{\log k}{k^{\delta-1}} = 0\left(k^{-(\delta-1)}\log k\right),$$

$$0 \leq s \leq s_k \Rightarrow \left|\frac{\log(k-1)!}{x}\right| \leq \frac{\log k!}{k^\delta} = 0\left(k^{-(\delta-1)}\log k\right),$$

by Sterling's formula. Thus, these two facts and (2.26) together imply that

$$\forall s \leq s_k, \log s^{-1} = x\left(1 + 0\left(k^{-(\delta-1)}\log k\right) = H_k^{-1}(1-s)\left(1 + 0\left(k^{-\delta+1}\log k\right)\right),$$

which was to be proved. We finally give two lemmas which correspond to Lemmas 37 and 38 in the case of infinite steps k. $\square$

LEMMA 39. *Let k satisfy, as $N \to +\infty$,*

(K)
$$(K) \, kN^{-1} (\log \log n) \to 0$$

and

(Q2)
$$k = k(N) \to +\infty$$

Then the following assertions hold.

(2.27)
$$\sup_{0 \le h \le a} \sup_{0 \le s \le 1-h} |\psi(s+h) - \psi(s)| = a(1 + o(1)), \text{ **a.s., as** } N \to +\infty.$$

and, as $N \to +\infty$,

(2.28) $|\psi(s+h) - \psi(s)| = a(1 + q(a)), \text{ for } 0 \le s \le 1 - a \text{ , with } q(a) \to 0 \text{ a.s.}$

Proof of of Lemma 39. As in Lemma 37, we have

(2.29)
$$\sup_{0 \le s \le 1-h} |\psi_h(s)| = \max\{|\psi_h(0)|, |\psi_h(1-h)|\}.$$

First we treat $\psi_h(0) = H_k\left(\mu_n H_k^{-1}(h)\right)$. Equations **(2.5)**-**(2.6)** yield

$$\mu_n H_k^{-1}(h) = (k!)^{\frac{1}{k}}\left(1 + o\left(H_k^{-1}\frac{(a)}{k}\right)\right)\mu_n, 0 \le h \le a.$$

Now, we note that $0 \le s \le a$ implies that $0 \le H_k^{-1}(h) \le C_1 a^{\frac{1}{k}}(k!)^{\frac{1}{k}}$ for small values of a, C_1 being a constant. Sterling's formula then implies for large values of k,

$$0 \le s \le t_k, 0 \le H_k^{-1}(h) \le Const. \, k^{\delta-1} \exp\left(-\frac{1}{2}k^{\delta-1}\right).$$

Then $H_k^{-1}(h) \to 0$ and we are able to use **(2.5)**-2.6 to get

$$\forall 0 \le h \le a \Rightarrow H_k\left(\mu_n H_k^{-1}(h)\right) = \mu_n^k h\left(1 + o(H_k^{-1}(a))\right).$$

The $loglog - law$ implies that

$$k\left(1-\mu_n\right) = 0\left(k\left(2n^{-1}\log\log n\right)^2\right), a.s.$$

Thus, whenever (K) is satisfied, one has

$$\mu_n^k = \exp(-k\left(1-\mu_n\right)\left(1+o\left(1\right)\right)) \to 1, \textbf{a.s.}$$

Hence

(2.30) $\forall 0 \le s \le a \le t_k\left(\delta\right), \psi_h\left(0\right) = h\left(1+q\left(a\right)\right) \le a\left(1+q\left(a\right)\right) = \psi_a\left(0\right).$

We now treat $\psi_h\left(1-h\right)$. By the proposition, we get
(2.31)
$$\forall 0 \le h \le a \le t_k\left(\delta\right), \ X = \mu_n H_k^{-1}\left(1-h\right) = \mu_n\left(\log h^{-1}\right)\left(1+0\left(\frac{\log k}{k^{\delta-1}}\right)\right), a.s.$$

Since $\frac{X}{k} = 0\left(k^{-(\delta-1)}\right), a.s.$, one has

$$1 - H_k\left(X\right) = \mu_n^{k-1}\frac{\left(\log h^{-1}\right)^{k-1}}{\left(k-1\right)!}$$

(2.32) $$\times \ \left(1+0\left(\frac{\log k}{k^{\delta-2}}\right)\right)h^{\mu_n}x^{-(k-1)\mu_n}\left(\left(k-1\right)!\right)^{\mu_n}\left(1+q_4\left(h\right)\right), a.s.,$$

as $N \to +\infty$. Replace x by $\log h^{-1}$ in (2.32). On account of (2.31) and of the fact that $\left(1+0\left(k^{1-\delta}\log k\right)^{k-1}\right) = \left(1+0\left(k^{2-\delta}\log k\right)\right)$, we get

$$\psi_h\left(1-h\right) = \left(\left(k-1\right)!\right)^{1-\mu_n}\left(\log h^{-1}\right)^{(k-1)(1-\mu_n)}h^{\mu_n}\left(1+q\left(a\right)\right).$$

Finally, by taking (K) and $(Q2)$ into account, we find ourselves in the same situation as in the proof of Lemma 37 (see Statement (2.12)). But in order to have the same conclusion, i.e.,

(2.33) $$\sup_{0 \le h \le a} \psi_h\left(1-h\right) = a\left(1+q\left(a\right)\right), a.s., \text{ as } N \to +\infty,$$

we have to check that

$$\left(\left(k-1\right)!\right)^{1-\mu_n} = \exp\left(\left(1-\mu_n\right)\log\left(k-1\right)!\right) =: \rho_n \to 1, a.s., as N \to +\infty.$$

But the $loglog - law$ and Sterling's formula together show that

$$\rho_n = \exp\left(0\left(\left(\frac{k^{\frac{1}{4}}\left(\log\log n\right)^{\frac{1}{2}}}{N^{\frac{1}{2}}}\log a^{-1}\right)\left(\frac{\log k}{\log a^{-1}}\right)\right)\right).$$

Obviously the condition $0 < a \le t_k\left(\delta\right)$ implies that $(logk)/(\log a^{-1}) \to 0$ as $N \to +\infty$, and as $k \to +\infty$. This fact combined with (Q2) clearly shows that $\rho_n \to 1$ as $N \to +\infty$. Now, by putting together (2.29), (2.30) and (2.33), we get

(2.34)
$$\left(0 < a \le t_k\left(\delta\right), \delta > 2\right)$$
$$\Rightarrow \sup_{0 \le 0 \le a} \sup_{0 \le s \le 1-h} |\psi_h\left(s\right)| = a\left(1 + q\left(a\right)\right), a.s., \text{ as } N \to +\infty.$$

LEMMA 40. *Let* $0 < a \le t_k\left(\delta\right), \delta > 2$. *Then as* $k \to +\infty$, *we have, as* $N \to +\infty$ *and as* $a \to 0$,

$$\sup_{0 \le h \le a} \sup_{0 \le s \le 1-h} |\phi\left(s+h\right) - \phi\left(s\right)| = \left(a\log a^{-1}\right)\left(1 + q\left(a\right)\right), q\left(a\right) \to 0$$

Proof of of Lemma 40. If we proceed as in Lemma A2 and as in Lemma A3, we get

$$\sup_{0 \le h \le a} \sup_{0 \le s \le 1-h} |\Phi_h\left(s\right)| = \max\left(ka, \left(a\log a^{-1}\right)\right)\left(1 + q\left(a\right)\right), \text{ as } N \to +\infty, \ a \to 0$$

From there, the conclusion is obtained by noticing that the condition $0 < a \le t_k\left(\delta\right)$ implies that $\frac{\left(\log a^{-1}\right)}{k} \to +\infty$ as $k \to +\infty$.

3. Proofs of the results

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Throughout, we shall use the following representation which follows from LO (1986) (see e.g. the study of $R_{N1}\left(x\right)$).

LEMMA 41. *Let k be fixed of* $k \to +\infty$ *as* $N \to +\infty$, *then*

$$\left\{\alpha_N\left(s\right), 0 \le s \le 1\right\} d_= \left\{\gamma_N\left(\psi\left(s\right)\right) + N^{\frac{1}{2}}\left\{H_k\left(\mu_n H_k^{-1}\left(s\right)\right) - s\right\}, 0 \le s \le 1\right\}$$

$$=: \left\{\bar{\alpha}_N\left(s\right), 0 \le s \le 1, a.s.\right\}$$

Lemma 41 will be systematically used. Then, if a_N satisfies

$$(3.1) \qquad \lim_{N \to +\infty} \frac{(\log \log n)^2}{N a_N \log a_N^{-1}} = 0,$$

we will be able to focus our attention on $\gamma_N \left(\psi \left(s \right) \right) + N^{\frac{1}{2}} \left(\mu_n - 1 \right) \phi \left(s \right)$ in the following way

$$(3.2) \qquad \frac{\bar{\alpha}_N \left(s \right)}{b_N} = \frac{\gamma_N \left(\psi \left(s \right) \right)}{b_N} + \frac{N^{\frac{1}{2}} \left(\mu_n - 1 \right) \phi \left(s \right)}{b_N} + b_N^{-1} 0 \left(N^{\frac{1}{2}} \log \log n \right), a.s.$$

$$=: A_{N1} \left(s \right) + A_{N2} \left(s \right) + A_{N3} \left(s \right).$$

with $b_N = \left(2 a_N \log \log a_N^{-1} \right)^{\frac{1}{2}} = b \left(a_N \right)$. It follows that if $(Q3)$ holds we have $A_{N3} \left(s \right) = o \left(1 \right), a.s.$, uniformly with respect to s, $0 \le s \le 1$.

Proof of Part I of Theorem 57. By (2.34), we have

$$(3.3) \qquad k_N \left(a_N, \bar{\alpha}_N \right) \le k_N \left(a_N, \ A_{N1} \right) + k_N \left(a_N, \ A_{N2} \right) + k_N \left(a_N, \ A_{N3} \right),$$

and by Lemma 39, we have for a fixed k,

$$k_N \left(a_N, \ A_{N2} \right) \le N^{\frac{1}{2}} \left| 1 - \mu_n \right| b \left(a_N \right) \left(1 + o \left(1 \right) \right), a.s.,$$

as $N \uparrow +\infty$. Thus the $loglog - law$ implies that

$$(3.4) \qquad \lim_{N \to +\infty} k_N \left(a_N, \ A_{N2} \right) = o \left(1 \right), a.s.,$$

whenever

$$(Q4) \qquad k^{-\frac{1}{2}} \left(2 \log \log n \right)^{\frac{1}{2}} b \left(a_N \right) \to 0 \text{ as } N \to +\infty$$

is satisfied. On the other hand, Lemma 37 and Theorem 0.2 of Stute (1981) together yield that

$$(3.5) \qquad k_N \left(a_N, A_{N1} \right) \le k_N \left(a_N, \ \gamma_N \right) \left(1 + o \left(1 \right) \right) = 1 + o \left(1 \right), a.s., \text{ as } N \to +\infty$$

Then if $(Q1)$, $(Q3)$, $(Q4)$, $(S1)$ and $(S3)$ are satisfied, we get

$$\lim_{N \to +\infty} k_N \left(a_N, \ \bar{\alpha}_N \right) \le 1, a.s.$$

Now let
$$\theta_N \left(a_N, \ R_N \right) = \sup_{0 \leq s \leq 1 - a_N} \left\{ R_N \left(s + a_N \right) - R_N \left(s \right) \right\}.$$

By Lemma 38, we have for large N that

(3.6) $\quad b_N^{-1} \sup_{0 \leq s \leq 1 - a_N} \left| A_{N2} \left(s + a_N \right) - A_{N2} \left(s \right) \right| \leq N^{\frac{1}{2}} \left| 1 - \mu_n \right| b \left(a_N \right) \left(1 + o \left(1 \right) \right), a.s.$

Thus if $(Q3)$ and $(Q4)$ are satisfied, we get

(3.7) $\qquad \theta_N \left(a_N, \ \bar{\alpha}_N \right) \leq \theta_N \left(a_N, \ A_{N1} \right) + o \left(1 \right), a.s., \ \text{as } N \to +\infty.$

Furthermore it may be derived from Theorem 0.2 of Stute (1981) that $(S1)$, $(S2)$ and $(S3)$ yield, as $N \to +\infty$,

(3.8) $\qquad b_N^{-1} \sup_{0 \leq s \leq 1 - a_N} \left| \gamma_N \left(\psi \left(s \right) + o \left(a_N \right) \right) - \gamma_N \left(\psi \left(s \right) + a_N \right) \right| = o \left(1 \right), \ a.s.,$

It follows from (3.7) and (3.8) that $(S1-2-3)$ and $(Q1-3-4)$ together imply

$$b_N^{-1} \theta_N \left(a_N, \ \bar{\alpha}_N \right) \geq b_N^{-1} \left\{ \sup_{0 \leq s \leq 1 - a_N} \gamma_N \left(\psi \left(s \right) + a_N \right) - \gamma_N \left(\psi \left(s \right) \right) \right\} + o \left(1 \right) \ a.s,$$

as $N \to +\infty$. Since $\psi : \left(0, \ 1 - a_N \right) \to \left(0, \ \psi \left(1 - a_N \right) \right)$, is a bijection and since $\psi \left(1 - a_N \right) = 1 - a_N \left(1 + o \left(1 \right) \right), a.s.$, we may use Lemma 37 (formulas (2.8) and (2.9) when $(Q1)$ holds to find for any $\varepsilon > 0$, for any elementary event ω, an $N_o \left(\omega \right)$ such that

$$N > N_o \Rightarrow b_N^{-1} \theta_N \left(a_N, \ \bar{\alpha}_N \right) \geq \sup_{0 \leq s \leq 1 - a_N (1 + \varepsilon)} \left(\gamma_N \left(s + a_N \right) - \gamma_N \left(s \right) \right) + o \left(1 \right), a.s.$$

Once again, we use the Theorem 02 of Stute (1981) to see that, under (S1-2-3), we have

$$\lim_{N \to +\infty} \sup_{0 \leq s \leq 1 - a_N (1 + \varepsilon)} \left\{ \frac{\left| \gamma_N \left(s + a_N + \varepsilon a_N \right) - \gamma_N \left(s + a_N \right) \right|}{b_N} \right\} \leq \varepsilon^{\frac{1}{2}}, a.s.$$

Thus, under (Q1-3-4) and for large values of N, we get

(3.9) $\qquad b_N^{-1} \theta_N \left(a_N, \ \bar{\alpha}_N \right) \geq b_N^{-1} \theta_N \left(a_N, \ \gamma_N \right) - \left(1 + \frac{1}{2} \right) \varepsilon^{\frac{1}{2}}, a.s.,$

Hence Lemma 2.13 in Stute (1981) and (3.9) together yield

(3.10) $\qquad \forall \varepsilon > 0, \lim_{N \to +\infty} \inf \theta_N \left(a_N, \ \bar{\alpha}_N \right) \geq (1 - \varepsilon)^{\frac{1}{2}} - \left(1 + \frac{1}{2} \right), a.s.$

Finally (3.5) and (3.10) together ensure that

$$\lim_{N \to +\infty} k_N \left(a_N, \ \bar{\alpha}_N \right) = 1, a.s.,$$

whenever $(Q1 - 3 - 4)$ and $(S1 - 2 - 3)$ hold. But since $\log n \sim \log N$ (k being fixed), one has

(3.11) $\qquad \left(\dfrac{\log \log n}{n} \right)^{\frac{1}{2}} \log a_N^{-1} \sim k^{\frac{1}{2}} \left(\dfrac{\log \log N}{\log a_N^{-1}} \right)^{\frac{1}{2}} \dfrac{\left(a_N^{\frac{1}{2}} \log a_N^{-1} \right)^{\frac{1}{2}}}{\left(N^{\frac{1}{2}} a_N^{\frac{1}{2}} \right)},$

(3.12) $\qquad \left(\dfrac{\log \log N}{N a_N \log a_N^{-1}} \right)^2 \log a_N^{-1} \leq \dfrac{(\log \log N)^2}{\left(\log a_N^{-1} \right)^2},$

(3.13) $\qquad (2 \log \log n) \left(a_N \log a_N^{-1} \right) \sim \left(\dfrac{\log \log N}{\log a_N^{-1}} \right) \left(a_N^{\frac{1}{2}} \log a_N^{-1} \right)^{\frac{3}{2}},$

for large N. (2.1), (2.2) and (3.13) show that (S1) and (S2) imply (Q1-2-3) and this completes the proof of part I of Theorem 57.

Proof of Part II of Theorem 57. The proof is the same as that of the first part. We only notice that if $a_N = c N^{-1} \log N$, $c > 0$, $(Q1 - 3 - 4)$ are satisfied for a fixed k. To get Part II of Theorem 57, we use Theorem 1 (Part I) of Mason *et al.* (1983) for the inequality " $\leq$ " and the Erdös-Renyi law for the increments of the uniform empirical process due to Mason *et al.* (1983), and Kómlos *et al.* (1975) for the inequality " $\geq$ ". Similarly to the first case, we get an analogue to (3.9). That is, for any $\varepsilon > 0$, for any elementary event ω, we can find an $N_1(\omega)$ such that

$$N > N_1 \Rightarrow k_N \left(a_N, \ \bar{\alpha}_N \right) \geq \sup_{0 \leq s \leq 1 - a(1+\varepsilon)} \left| \dfrac{\gamma_N \left(s + a \left(1 + \varepsilon \right) \right) - \gamma_N (s)}{b_N} \right|$$

(3.14) $\qquad\qquad\qquad\qquad - 2\varepsilon^{\frac{1}{2}} h \left(c\varepsilon \right) + \varepsilon^{\frac{1}{2}},$

where for any s, $h(s) = \left(\frac{s}{2}\right)^{\frac{1}{2}}(\beta^+(s) - 1)$ and $\beta^+(s)$ is the unique solution of the equation $x(\log x - 1) + 1 = s^{-1}$ such that $\beta^+(s) \geq 1$. Now, since for any $f(.)$, $g(.)$, K,

$$\sup_{s\in K}\max(f(s), g(s)) = \max\left(\sup_{s\in K} f(s), \sup_{s\in K} g(s)\right) \text{ and } |x| = \max(x, -x)$$

and since (see e.g. the third formula that follows Statement 11 in Mason *et al.* (1983))

$$\forall \varepsilon > 0, \lim_{N\to+\infty} \inf \frac{|\gamma_N(s + a(1+\varepsilon)) - \gamma_N(s)|}{b_N} = (1+\varepsilon)^{\frac{1}{2}} h((1+\varepsilon)c), \, a.s.,$$

then (3.14) implies that

$$\forall \varepsilon > 0, \lim_{N\to+\infty} \inf k_N(a_N, \, \bar{\alpha}_N) \geq (1+\varepsilon)^{\frac{1}{2}} h((1+\varepsilon)c) - 2^{\frac{1}{2}} h(c\varepsilon) - \varepsilon^{\frac{1}{2}}, a.s.$$

Thus it suffices to prove that : (2.1) for each fixed c, $h((1+\varepsilon)c) \to h(c)$ as $\varepsilon \to 0$, and : (2.2) for each fixed c, $\varepsilon^{\frac{1}{2}} h(c\varepsilon) \to 0$ as $\varepsilon \to 0$. But these two points may be directly obtained by simple considerations.

Proof of part III of Theorem 57.

The proof is very similar to that of Part I of Theorem 57. If suffices to remark that part III of Theorem 1 in Mason *et al.* (1983) holds in the general case where $a_N = \alpha(\log N)^{-c}, c > 0, \alpha > 0$.

Proof of Part IV of Theorem 57.

Here $a_N = c_N N^{-1} \log N, c_N \to 0$ as $N \to +\infty$. Let $d_N = N^{\frac{1}{2}}(\log N)^{-1} \log c_N^{-1}$. On the one hand, we have

(A) $\left(N^{-1}\log\log N\right)^{\frac{1}{2}} \log a_N^{-1} \sim \left(N^{-1}\log\log N\right)^{\frac{1}{2}} \log c_N^{-1} + N^{-\frac{1}{2}}(\log\log N)^{\frac{1}{2}} \log N,$

$$\frac{(\log\log n)^{\frac{1}{2}}}{c_N \log N}\left(\log a_N^{-1}\right)^{-1} \sim \frac{(\log\log N)^{\frac{1}{2}}}{\left(\log c_N^{-1}\right) c_N \log N + cN(\log N)^2(1 + o(1))}$$

(B)
$$= \frac{(\log \log N)^{\frac{1}{2}}}{\log N} (c_N \log N)^{-1} \left(1 + \frac{\log c_N^{-1}}{\log N} + o(1)\right),$$

(C)
$$\left((\log \log n)^{\frac{1}{2}} \left(a_N \log a_N^{-1}\right)^{\frac{1}{2}}\right)^2$$

(3.15)
$$= c_N \left(\log c_N^{-1}\right) N^{-1} (\log N) (\log \log N) + c_N N^{-1} (\log N)^2 (\log \log n) (1 + o(1)).$$

Obviously (A), (B) and (C) together imply that the conditions of Part IV of Theorem 57, namely, as $N \to +\infty$,

(W1)
$$c_N \to 0,$$

(W2)
$$c_N \log N \to +\infty$$

and

(W3)
$$(\log N)^{-1} \left(\log c_N^{-1}\right) (\log \log N) \to 0.$$

In turn these facts imply the conditions (Q1-3-4). On the other hand, we have
$$d_N \wedge_N (a_N, \ \bar{\alpha}_N) \leq d_N \wedge_N (a_N, \ A_{N1})$$

$$+0\left(\frac{\left(a_N \log a_N^{-1}\right) (\log \log n)^{\frac{1}{2}} N^{\frac{1}{2}} \log c_N^{-1}}{k^{\frac{1}{2}} \log N}\right) + 0\left(\frac{\log \log n}{\log N} \log c_N^{-1}\right),$$

a.s., as $N \to +\infty$, where we have used Lemma 38 and (3.2). Further, as $N \to +\infty$,

(Q5)
$$\frac{\left(a_N \log a_N^{-1}\right) (\log \log n)^{\frac{1}{2}} N^{\frac{1}{2}} \log c_N^{-1}}{k^{\frac{1}{2}} \log N} \sim \frac{\left(\log a_N^{-1}\right) (\log \log N) (\log N)^3}{(\log \ N) \left(k^{\frac{1}{2}} \log N\right) N} c_N \to 0,$$

by the definition of a_N and by $(W1)$ and $(W2)$. Thus, as $N \to +\infty$, we have
$$d_N \wedge_N (a_N, \ \bar{\alpha}_N) \leq d_N \wedge_N (a_N, \ A_{N1}) + o(1), a.s.$$

At this step, we apply Part II of Theorem 1 of Mason *et al.* (1983) by using Lemma 37 which is true on account of $(Q1)$.

Proof of Theorem 58.

We shall omit details of the proofs of the different parts that are the same as those of the parts of Theorem 57. The only problem concerns the bounds depending on k. However, this problem is solved by Lemmas 39 and 40. Hence we only provide the following remarks.

(R1) In our different choices of (a_N), we have that $Na_N \to +\infty$, as $N \to +\infty$.

(R2) If $a_N \leq t_k(\delta), \delta > 2$, then for any $y > 0$, there exists k_y such that

$$\forall k > k_y, \ k^{-y} N \leq (Nt_k)^{-1} \leq (Na_N)^{-1} \to 0 \text{ as } N \to +\infty.$$

(R3) $(\log \log n) = (\log \log N)(1 + o(1))$ **and** $\log n = (\log N)(1 + o(1))$, **as** $N \to +\infty$.

With these remarks, it is easily seen, as in the proof of Theorem 57 that the conditions $(K), (Q2), (Q3), (Q4)$ and $(Q5)$ are satisfied at the same time with the specific assumptions of each part of Theorem 58 as follows.

(a) $(Q2)$ and (K) are always satisfied if $a_N = t_k$. Indeed,

$$(3.16) \qquad kN^{-1}\log\log n \sim \left(kN^{\frac{1}{2}}\right)\left(N^{-\frac{1}{2}}\log\log N\right) \to 0 \text{ as } N \to +\infty,$$

by $(R2)$ and (K) and

$$(3.17) \quad k(\log\log n)\left(\log a_N^{-1}\right) \sim \left(N^{-\frac{1}{4}}k\log a_N^{-1}\right)\left(N^{-\frac{1}{4}}\log\log N\right)(Na_N)^{-\frac{1}{4}} \to,$$

by $(R2) - (R3)$ and $(Q2)$.

(b) In Parts I, II and III of Theorem 58, the implication $\{(S1), (S3)\} \Rightarrow \{(Q3), (Q4)\}$ is true whenever $\log\log n \sim \log\log N$ (see the lines that follow Formula (3.10)) and $(Na_N) \to +\infty$ as $N \to +\infty$, which are derived from $(R1), (R2)$ and $(R3)$.

(c) In Part IV, $(Q5)$ is true independently of the behavior of k.

Thus we may use Lemmas 39 and 40 instead of Lemmas 37 and 38 in the proofs of Theorem 57 to get the results of Theorem 58 in the same way.

Proof of The Corollary.

This is a direct consequence of Theorems 57 and 58 and of Lemma 41. For Part III, the methods used in Part I of Theorem 57 must be repeated.

REMARK 23. *By letting* $(N \log \log N)^{\frac{1}{4}} (\log N)^{\frac{1}{2}} \left(2a_N \log a_N^{-1}\right)^{-\frac{1}{2}} \to 0$, *it would be possible to derive part I, III and IV of the Theorem 57 from invariance principles such as in Aly et al. (1984) or LO (1986). But the necessary amount of work would be unchanged relatively to our method.*

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