CHAPTER 15

The $\bar{\partial} \partial$-problem for extendable currents defined on a half space of $\mathbb{C}^n$, by M. Eramane Bodian, W. Ndiaye and S. Sambou

Mamadou Eramane Bodian. Email : m.bodian2966@zig.univ.sn
Waly Ndiaye. Email : walyunivzig@yahoo.fr.
Salomon Sambou. Email : ssambou@univ-zig.sn
Université Assane Seck University de Ziguinchor, (SENÉGAL).

Abstract. We solve the $\bar{\partial} \partial$-problem for extendable currents defined on a half space of $\mathbb{C}^n$. ♦

Keywords. half-space; extensible currents; de Rham’s cohomology group.

AMS 2010 Mathematics Subject Classification. 32F32.

Cite the chapter as:
Bodian E. M., Ndiaye W. and Sambou S.(2018). The $\bar{\partial} \partial$-problem for extendable currents defined on a half space of $\mathbb{C}^n$. In A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour, 275 –281
Doi : 10.16929/sbs/2018.100-03-03

1. Introduction

In this paper, we solve the $\partial\bar{\partial}$-problem for extendable currents defined on $\Omega = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \text{Im}(z_n) < 0 \}$ that is an example of unbounded pseudo-convex domain as well as its complement. The de Rham cohomology group of the boundary $H^j(\partial \Omega)$ is trivial for $1 \leq j \leq 2n - 1$. In this context, we first solve the equation $dS = T$ where $T$ is an extendable current defined on $\Omega$ and we have:

**Theorem 40.** The de Rham cohomology group for extendable currents

\[ \tilde{H}^j(\Omega) = 0 \quad \text{for} \quad 1 \leq j \leq 2n - 1. \]


The domain $\Omega$ is fat i.e $\bar{\partial} \bar{\Omega} = \Omega$ therefore according to Martineau (1996) the extendable currents defined on $\Omega$ are topological dual of differential forms with compact support on $\bar{\Omega}$. for that we are led to solve the equation $df = g$ where $f$ et $g$ are differential forms with compact support on $\Omega$ and go to the extendable currents by duality. The first particularity lies on the resolution with prescribed support by the operator $d$ because if we solve with compact support in $\mathbb{C}^n$, then we can not as in Bodian et al. (2017b) correct by the solution with compact support. We use the results of Brinkschulte (2004) and Seeley (2002) to get a solution with compact support and then as the concave case, we use the same techniques to correct the solutions because the space of differential forms with compact support on $\bar{\Omega}$ is not a Frechet space but rather an inductive limit of Frechet spaces.

The second particularity compared to Bodian et al. (2016) and Bodian et al. (2017b) lies to resolution of the $\bar{\partial}$ with prescribed support because $\Omega$ being the unbounded Levi flat domain, we can not use the techniques of Sambou (2002a). Then we use the results of Brinkschulte (2004) to solve with prescribed support the equation $\bar{\partial}S = T$ in the unbounded domain $\Omega$ in order to establish:

**Theorem 41.** Let $\Omega = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \text{Im}(z_n) < 0 \} \subset \mathbb{C}^n$ be a domain, then for all extendable $(p, q)$-current $T$ defined on $\Omega$ and $d$-closed, there is $S$ an extendable $(p - 1, q - 1)$-current defined on $\Omega$ such that $\bar{\partial}S = T$ for $1 \leq p, q \leq n - 1$.

1.1. Notations.

we note by $\mathcal{D}_X^p(\Omega)$ the space of $p$-currents defined on $\Omega$ and extendable in $X$, $D^p(\Omega)$ the space of smooth differential $p$-forms defined in $X$ with compact
support in $\bar{\Omega}$. If $X$ is a complex manifold of dimension $n$, then we note by $\tilde{D}^{p,q}_X(\Omega)$ the space of extendable $(p,q)$-currents defined in $\Omega$ and $D^{p,q}(\bar{\Omega})$ the space of differential $(p,q)$-forms with compact support in $\bar{\Omega}$. We note by $\tilde{H}^p(\Omega)$ the de Rham cohomology group for extendable currents defined in $\Omega$, $\tilde{H}^{p,q}(\Omega)$ the Dolbeault cohomology group for extendable currents defined in $\Omega$. $H^p_\infty(X)$ is the cohomology group of de Rham for smooth differential $p$-forms defined in $X$, $H^p(X)$ is the de Rham cohomology group for smooth differential $p$-forms defined in $\Omega$ and finally $\Lambda^p(\Omega)$ the space of smooth differential $p$-forms in $\Omega$.

2. Resolution of the equation $dS = T$ for a half space $\Omega$ of $\mathbb{R}^{n+1}$

we consider

$$\Omega = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} < 0\} \subset \mathbb{R}^{n+1}$$

a convex domain, its boundary $\partial \Omega = \mathbb{R}^n \times \{0\} \simeq \mathbb{R}^n$ and the interior of its complement

$$\mathcal{C} = \mathbb{R}^{n+1} \setminus \bar{\Omega} = \mathbb{R}^n \times \{x_{n+1} > 0\}.$$ 

$\Omega$ is convex and unbounded and so is its complement $\mathcal{C}$. So we have $H^j(\Omega) = 0$ and $H^j(\partial\Omega) = 0$ for $j \geq 1$. Then the principal result of this part is :

Theorem 40. The de Rham cohomology group for extendable currents

$$\tilde{H}^j(\Omega) = 0 \quad \text{pour} \quad 1 \leq j \leq n.$$ 

For giving the proof we need the following lemma :

Lemma 28.

$$D^p(\bar{\Omega}) \cap \ker d = d(D^{p-1}(\bar{\Omega}))$$

for $1 \leq p \leq n$.

Proof. Let $f \in D^p(\bar{\Omega}) \cap \ker d$, then there is $\Omega'$ a ball of center $z_0$ and radius $R$ such that for $f \in D^p(\Omega') \cap \ker d$, $0 < p \leq n$, there is $g \in D^{p-1}(\Omega')$ with $dg = f$. This implies that $dg|_B = 0$ where $B = \Omega' \cap (\mathbb{R}^{n+1} \setminus \Omega)$. If $p = 1$, then $g$ is a constant with compact support so $g = 0$ in $B$.

If $1 < p \leq n$, then $g|_B$ is a differential $(p-1)$-form $d$-closed then it exists a differential smooth $(p-2)$-form $h$ in $\bar{B}$ such that $dh = g|_B$. Let $\tilde{h}$ a smooth extension with compact support of $h$ in $\Omega'$ (we can use the extension operator of Seeley Seeley (2002)), $u = g - d\tilde{h}$ is a smooth differential $(p-1)$-form in $\mathbb{R}^{n+1} \setminus \Omega$ with compact support in $\bar{\Omega}$ and $du = f$. $\square$
Proof (Theorem 40). According to Martineau (1996), since \( \bar{\Omega} = \Omega \), the currents defined in \( \Omega \) and extendable in \( \mathbb{R}^{n+1} \) are the elements of \( (D^p(\bar{\Omega}))' \) topological dual of smooth differential \( p \)-forms in \( \mathbb{R}^{n+1} \) with compact support in \( \bar{\Omega} \). However \( \bar{\Omega} \) being unbounded, \( D^p(\bar{\Omega}) \) is an inductive limit of Fréchet spaces.

We consider a compact \( K \subset \bar{\Omega} \) of \( \mathbb{R}^{n+1} \) and \( D^p(K) \) the space of \( p \)-forms in \( \mathbb{R}^{n+1} \) with compact support in \( K \). We set

\[
L^K_T : d(D^p(\Omega) \cap D^p(K) \cap \ker d) \rightarrow \mathbb{C}
\]
\[
\bar{\partial} \varphi \mapsto \langle T, \varphi \rangle
\]

a continuous linear application, and then \( L^K_T \) extend as a continuous linear operator:

\[
\tilde{L}^K_T : D^{p+1}(\bar{\Omega}) \cap D^{p+1}(K) \rightarrow \mathbb{C}.
\]

It is an extendable current and

\[
d\tilde{L}^K_T = (-1)^{n-p+1}T \text{ on } \bar{\omega}.
\]

We consider a family \( (K_n)_{n \in \mathbb{N}} \) of compacts set of \( \bar{\Omega} \) then we can find in \( K_n \), a current \( S_n \) extendable such that \( dS_n = T \) in \( K_n \) with \( K_n \subset \bar{K}_{n+1} \).

\( S_{n+1} - S_n \) is \( d \)-closed and \( S_{n+1} - S_n = dv_n \) in \( \bar{K}_{n+1} \).

Let \( \chi \) be a smooth function on \( \mathbb{R}^{n+1} \) with compact support in \( K_{n+1} \) such that \( \chi = 1 \) in a neighborhood of \( K_n \) contained in \( K_{n+1} \) and

\[
S_{n+1} - d(\chi v_n) = S_n + d(1 - \chi)v_n \text{ on } \bar{K}_n.
\]

Let us put \( U_{n+1} = S_{n+1} - d(\chi v_n) \) and \( U_n = S_n + d(1 - \chi)v_n \).

We have \( dU_{n+1} = dU_n = T \) in \( \bar{K}_n \) and \( U_{n+1} = U_n \) in \( K_n \). We set

\[
S = \lim_n U_{n+1}.
\]

This is an extendable current in \( \Omega \) and verifies \( dS = T \).
3. Resolution of the $\partial \bar{\partial}$ for extendable currents in a half space of type $\{\text{Im}(z_n) < 0\} \subset \mathbb{C}^n$

We give the following fundamental result of $\partial\bar{\partial}$-problem with prescribed support:

**Theorem 42.** Let $\Omega$ be a domain and $f \in D^{p,q}(\overline{\Omega}) \cap \text{ker } \partial$. Then it exists $g \in D^{p,q-1}(\overline{\Omega})$ such that $\partial \bar{\partial} g = f$ for $1 \leq q \leq n - 1$.

**Proof.** This is a consequence of the result of a resolution of the $\bar{\partial}$ with prescribed support (Theorem 4.2 in Brinkschulte (2004)). If the support of $f$ is compact in $\Omega$, then we choose pseudo-convex domain $\Omega'$ in $\Omega$ which contains the support of $f$. According to Theorem 4.2 in Brinkschulte (2004), there is $g \in D^{p,q-1}(\overline{\Omega'})$ such that $\partial \bar{\partial} g = f$.

If now $\text{supp}(f) \cap b\Omega \neq \emptyset$, since $f$ has compact support and $b\Omega$ is Levi flat, we can find $K \subset \Omega$ a compact with pseudo-convex interior and smooth boundary which contains the support of $f$. According to Theorem 4.2 in Brinkschulte (2004), it exists $h$ a differential $(p, q - 1)$-form with support in $K$ such that $dh = f$. We extend $h$ by 0 in $\mathbb{C}^n \setminus K$ and we have the desired solution. So for all $f \in D^{p,q}(\overline{\Omega}) \cap \text{ker } \partial$, it exists $g \in D^{p,q-1}(\overline{\Omega})$ such that $\partial \bar{\partial} g = f$. \hfill $\square$

By classical duality (refer theorem 40), we have the following result:

**Theorem 43.**

Let $\Omega = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \text{Im}(z_n) < 0\}$ and $T$ be an extendable current of bi-degree $(p, q)$ $\partial\bar{\partial}$-closed in $\Omega$. Then there is an extendable current $S$ defined in $\Omega$ such that $\partial \bar{\partial} S = T$ for $1 \leq p \leq n$ and $1 \leq q \leq n - 1$.

We are going to establish the following result.

**Theorem 41.** Let $\Omega = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \text{Im}(z_n) < 0\} \subset \mathbb{C}^n$ be a domain, then for all extendable $(p, q)$-current defined in $\Omega$ and $d$-closed, it exists $S$ a extendable $(p - 1, q - 1)$-current defined in $\Omega$ such that $\partial \bar{\partial} S = T$ with $1 \leq p, q \leq n - 1$.

**Proof.** Let $T$ a $(p, q)$-current, $1 \leq p \leq n - 1$ and $1 \leq q \leq n - 1$, $d$-closed defined in $\Omega$ and extendable in $\mathbb{C}^n$ with $1 \leq p + q \leq 2n - 2$. Since the theorem 40 assures us that $H^{p+q}(\Omega) = 0$, it exists a extendable current $\mu$ defined in $\Omega$ such that $d\mu = T$. $\mu$ is an extendable $(p + q - 1)$-current, it breaks down into $(p - 1, q)$-current $\mu_1$ and into $(p, q - 1)$-current $\mu_2$. We have

$$d\mu = d(\mu_1 + \mu_2) = d\mu_1 + d\mu_2 = T.$$
Since $d = \partial + \bar{\partial}$, we have for bi-degree reasons, $\partial \mu_2 = 0$ and $\bar{\partial} \mu_1 = 0$. We get by theorem 43 $\mu_1 = \partial u_1$ and $\mu_2 = \bar{\partial} u_2$ where $u_1$ and $u_2$ are extendable currents defined in $\Omega$. So we have:

$$
T = \partial \mu_2 + \bar{\partial} \mu_1 \\
= \partial \bar{\partial} u_2 + \bar{\partial} \partial u_1 \\
= \partial \bar{\partial} (u_2 - u_1)
$$

We set $S = u_2 - u_1$, then $S$ is an extendable $(p - 1, q - 1)$-current defined in $\Omega$ such that $\partial \bar{\partial} S = T$. □
Bibliography


