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CHAPTER 10

On the joint distribution of variations of the Gini index and Welfare indices, *by P. D. Mergane, G. S. Lo and T. A.Kpanzou*

Pape Djiby Mergane †. Email : pdmergane@ufrsat.org.
Gane Samb Lo †. Email : gane-samb.lo@ugb.edu.sn.
Tchilabalo Abozou Kpanzou ††. Email : kpanzout@gmail.com.
† LERSTAD, Gaston Berger University, Saint Louis, Sénégal
†† Kara University, Kara, Togo

Abstract. The aim of this paper is to establish the asymptotic behavior of the mutual influence of the Gini index and the poverty measures by using the Gaussian fields described in Lo and Mergane (2013). (See the full abstract in next page).

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Full Abstract. The aim of this paper is to establish the asymptotic behavior of the mutual influence of the Gini index and the poverty measures by using the Gaussian fields described in Lo and Mergane (2013). The results are given as representation theorems using the Gaussian fields of the unidimensional or the bidimensional functional Brownian bridges. Such representations, when combined with those already available, lead to joint asymptotic distributions with other statistics of interest like growth, welfare and inequality indices and then, unveil interesting results related to the mutual influence between them. The results are also appropriate for studying whether a growth is fair or not, depending on the variation of the inequality measure. Datadriven applications are also available. Although the variances may seem complicated at a first sight, their computations which are needed to get confidence intervals of the indices, are possible with the help of R software. Beyond the current results, the provided representations are useful in connection with different ones of other statistics.

Résumé. *Le but de cet article est d'établir le comportement asymptotique de l'influence mutuelle de l'indice de Gini et des mesures de pauvreté en utilisant des champs gaussiens décrits dans Lo and Mergane (2013). Les résultats sont donnés sous la forme de théorèmes de représentations utilisant des champs gaussiens des ponts Browniens fonctionnels uni ou bi dimensionnels. De telles représentations, combinées à celles existantes, conduisent à des distributions asymptotiques conjointes avec d'autres statistiques d'intérêt comme la croissance, la richesse et les mesures d'inégalité, et ainsi dévoilent des résultats intéressants liés à leur influence mutuelle. Les résultats sont aussi appropriés pour étudier si la croissance est équitable ou non, en fonction de la variation de la mesure d'inégalité. Des applications à des données sont aussi faites. Malgré que les variances puissent sembler compliquées à première vue, leurs calculs, qui sont nécessaires pour l'obtention des intervalles de confiance, sont possibles grace à l'aide du language R. Au delà des résultats actuels, les représentations proposées sont utiles en rapport avec d'autres statistiques.*

1. Introduction and motivation

In this paper, the asymptotic behavior of the Gini inequality index (see Gini (1921)) is jointly studied with a general class of welfare indices within the frame of unified Gaussian fields both for in a one phase frame (fixed time) and in a two phase frame (variation between two periods). Beyond the results themselves, the obtained asymptotic representations allow future couplings of the studied statistics with other indices. These couplings will lead to joint asymptotic distributions, enabling interesting comparison and influence studies between indices.

We begin by a survey on the Gini index, based on historical and recent works, concerning its statistical properties, its asymptotic distributions and some of its generalizations. In a second step, we will explain the notion of Gaussian fields we mentioned before.

The Gini (1921) index has played and is playing an important role in the measurement of economic inequality since its development by Corrado Gini in the early 20th century. Besides, this index is also used in many other disciplines, including Biology (Graczyk (2007)), Astronomy (Lisker (2008)), Environment (Druckman and Jackson (2008); Groves-Kirkby *et al.* (2009)).

Various expressions for the Gini index are given by authors such as Davidson (2009), Dorfman (1979), Duclos and Araar (2006) . Extended Gini indices are also developed (see e.g., Weymark (1981); Yitzhaki (1983); Chakravarty (1998). Over the years, statistical inference for the Gini index has attracted many researchers. For example, Gastwirth J.L. (1972) discussed the estimation of the index from that of the Lorenz curve. Cowell and Flachaire (2007) have developed its influence function and looked at how influenced is its non-parametric estimator to extreme values. Moni (1991) also studied the Gini measure by means of the influence function. On their part, Qin *et al.* (2010) constructed empirical likelihood confidence intervals for the Gini coefficient and showed that these perform well, but only for large samples. In order to improve inference based on it, Sarno (1998) proposed, in a non-parametric setting, a new stabilizing transformation for the sample Gini coefficient.

Fakoor *et al.* (2011) considered non-parametric estimators of the Lorenz curve and Gini index based on a sample from the corresponding length-biased distribution, showed that such estimators are strongly consistent for the Gini index, and derived an asymptotic normality for that index. Davidson (2009) developed a reliable standard error for the plug-in estimator of the Gini index and derived an effective bias correction. Martínez-Camblor and Corral (2009) developed results on exact and an asymptotic distribution of the Gini coefficient. Asymptotic distribution of the S-Gini index is derived by Zitikis and Gastwirth (2002), who provided an explicit formula for the asymptotic variance. More on inference for the extended Gini indices can be found in, e.g., Xu (2000) and Barrett and Donald (2000).

But the Gini's index is one of a quite few number of inequality measures that are available in the literature. A considerable number of them has been gathered in a class named Theil-like family and studied jointly with welfare statistics. This study dit not concern the Gini's index nor the new Zenga (1984) inequality measure. Because of its great importance, a similar handling for the Gini's measure seems to be highly recommended alongside comparison investigations.

As mentioned above, a new approach, that is set to put the asymptotic results of indices related to welfare and inequality analysis in a unified frame of one Gaussian field, was attempted in Lo and Mergane (2013). In that paper, a large class of inequality measures named as the Theil-like family has been jointly studied with an other general class of poverty measures known under the name of General Poverty Index (*GPI*), both with respect to a spatial (horizontal) and a time (vertical) perspective. Such an approach leads to powerful tools when comparing different indicators or their variation over the space or the time scale. Since the joint asymptotic results are expressed with respect to one common Gaussian process, the method makes easy the comparison of the results for one particular index with those for different indices or statistics using the same frame. Our aim is to offer such representations for the Gini's index and to benefit from them, in order to have insightful relations with the GPI. These representations will be used later in a full study of all available inequality measures. In the coming Subsection 1.1, we will give a full description of the probability spaces holding the representations.

Our main results start from the complete description of the asymptotic representation of the Gini's index in a Gaussian field and in a residual Gaussian process β already introduced and studied in Lo (2010) for the fixed time scheme in Theorem 25. These results are extended to the two phase variation scheme in Theorem 26. Finally, their combination with available representations, yields successful descriptions of the mutual influence of the Gini's index and usual poverty indices including the Sen and Kakawani ones in Theorem 28. Unlike former works on the topic, we appeal to the Bahadur Representation Theorem (see Bahadur (1966)) as a tool for handling L-statistics in the lines of Lo (2010). Datadriven studies are included. But beyond this, the representations will serve in connection with similar ones for different indices of interest.

We will exclusively limit our study in the field of the welfare analyis and focus on the Gini's index and the General poverty measure. In future works, extentions of our current results will be extended to extension of the Gini's measures : the Generalized Gini, S-Gini, E-Gini. (see Barrett and Donald 2009 Barret (2009)).

Let us recall that we may and do measure poverty (or richness) with the help of poverty indices J based on the income variable X. To each income, a poverty line $Z > 0$ is associated. This poverty line is defined the minimum income under which an individual is declared as *poor*. Over two periods $s = 1$ and $t = 2$, we say that we have a gain against poverty when $\Delta J(s,t) = J(t) - J(s) \leq 0$, or simply a growth against poverty. But this variation is not enough to describe the situation of the population, one must be sure that, meanwhile, the income did not become more *unequally* distributed, that is the appropriate inequality coefficient I did not increase. One can achieve this by studying the ratio $R = \Delta J(s,t)/\Delta I(s,t)$, where $\Delta I(s,t) = I(t) - I(s)$ denotes the variation of the distribution of the income variable.

To make the ideas more precise, let us suppose that we are monitoring the poverty scene on some population over the period time $[1, 2]$ and let $Y = (X^1, X^2)$ be the income variable of that population at periods 1 and 2. Let us consider one sample of $n \geq 1$ individuals or households, and observe the income couple $Y_j = (X_j^{(2)}, X_j^{(2)})$, $j = 1, ..., n$. For each period $i \in \{1, 2\}$, we also denoted by $X_{1,n}^{(i)} \leq X_{2,n}^{(i)} \leq \cdots X_{n,n}^{(i)}$ the order statistics. We assume that X^i is strictly positive, and we compute the poverty measure $J_n(i)$ and the inequality measure $I_n(i)$.

For poverty, we consider the Generalized Poverty Index (GPI) introduced by *et al.* Lo *et al.* (2006) and Lo (2013) as an attempt to gather a large class of poverty measures reviewed in Zheng (1997) defined as follows for period i,

$$(1.1)\quad J_n(i) = \frac{A(Q_n(i), n, Z(i))}{nB(Q_n(i))} \sum_{j=1}^{Q_n(i)} w(\mu_1 n + \mu_2 Q_n(i) - \mu_3 j + \mu_4)\ d\left(\frac{Z(i) - X_{j,n}^i}{Z(i)}\right)$$

where $B(Q_n(.)) = \sum_{j=1}^{n} w(j)$, $Z(i)$ is the poverty line at time $t = i$, $Q_n(.)$ is the number of poor, μ_1, μ_2, μ_3 and μ_4 are constants, $A(u, v, s)$, $w(t)$, and $d(y)$ are measurable functions of $(u, v, s) \in \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+^*$, $t \in \mathbb{R}_+^*$, and $x \in (0, 1)$. By particularizing the functions A and w and by giving fixed values to the $\mu_i' s$, we may find almost all the available indices, as we will do it later on. *In the sequel, (1.1) will be called a poverty index (indices in the plural) or simply a poverty measure according to the economists' terminology..*

This class includes the most popular indices such as those of Sen (1976), Kakwani (1980), Shorrocks (1995), Clark *et al.* (1981), Foster *et al.* (1984), etc., see (Lo (2013)) for a review of the GPI. From the works of many authors (Sall and Lo (2007), Sall and Lo (2010) for instance), $J_n(i)$ is an asymptotically sufficient estimate of the exact poverty measure

$$J(i) = \int_0^{Z(i)} L\left(x, F_{(2),i}\right) d\left(\frac{Z(i) - x}{Z(i)}\right) dF_{(2),i}(x) \tag{1.2}$$

where $F_{(2),i}$ is the distribution function of $X^{(i)}$ $(i = 1, 2)$, and L is some weight function.

As for the inequality measure, we only use the Gini index (GI) which is based on the Lorenz curve (1905). And, for a given date $i \in \{1, 2\}$, we denote by

$$GI_n(i) = \frac{1}{\mu_n(i)} \frac{1}{n} \sum_{j=1}^{n} \left(\frac{2j-1}{n} - 1\right) X_{j,n}^{(i)} \tag{1.3}$$

the empirical measure of the Gini index (see *Greselin et al.*, Greselin (2009)), and its continuous form is defined as follows

$$GI(i) = \frac{1}{\mu(i)} \int_0^{+\infty} F_{(2),i}^{-1}(x) \left(2F_{(2),i}(x) - 1\right) dF_{(2),i}(x), \tag{1.4}$$

where $\mu(i) = \mathbb{E}(X^i)$ is the mathematical expectation of X^i and $F_{(2),i}^{-1}$ denotes the generalized inverse of the *cdf* $F_{(2),i}$.

The motivations stated above lead to the study of the behavior of

$$(\Delta J_n(s, t), \Delta GI_n(s, t)),$$

defined for two periods $s < t$, as an estimate of the unknown value of

$$(\Delta J(s,t), \Delta GI(s,t)).$$

Precisely confidence intervals of

$$R(s,t) = \frac{\Delta J(s,t)}{\Delta GI(s,t)}$$

will be an appropriate set of tools for the study of the mutual influence of the Gini index and the poverty measures.

To achieve our goal we need a coherent asymptotic theory allowing the handling of longitudinal data as it is the case here and a stochastic process approach leading to asymptotic sub-results with the help of the continuity mapping theorem.

The rest of the paper is structured as follows. In the rest of this Section 1, we describe the probability space on which the asymptotic representations will take place. In Section 2 we provide a study on the asymptotic behavior of the Gini index. Then in Section 3, a complete study of the variation of this index between two given dates is provided. And next, Section 4, we treat the mutual influence of the latter on the Generalized Poverty Index (GPI) introduced by *et al.* Lo *et al.* (2006) and Lo (2013). The paper ends with some final comments in Section 5.

The notation used in the paper may be seen as complicated, but knowing the following simple facts may help in making them very comprehensive. The subscript (1) means that we are working un on dimension, where the randoms variables do not have a superscript. In dimension 2, we always have the subscript (2) to main functions : *cdf*'s, copulas, empirical process,etc. When followed by i, like $F_{(2),i}$, it refers to a margin. For example $F_{(2),1}$ is the first marginal *cdf* of $F_{(2)}$. Still in dimension 2, any superscript $i = 1,2$ refers to the first coordinate of a couple.

1.1. Notations and Probability Space.

In this Subsection, we complete the notations we already gave and precise our probability space.

Univariate frame. We are going to describe the general Gaussian field in which we present our results. Indeed, we use a unified approach when dealing with the asymptotic theories of the welfare statistics. It is based on the Functional Empirical Process (*fep*) and its Functional Brownian Bridge (*fbb*) limit. It is laid out as follows.

When we deal with the asymptotic properties of one statistic or index at a fixed time, we suppose that we have a non-negative random variable of interest which may be the income or the expense X whose probability law on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, the Borel measurable space on $\mathbb{R}$, is denoted by $\mathbb{P}_X$. We consider the space $\mathcal{F}_{(1)}$ of measurable real-valued functions f defined on $\mathbb{R}$ such that

$$V_X(f) = \int (f - \mathbb{E}_X(f))^2 d\mathbb{P}_X = \mathbb{E}(f(X) - \mathbb{E}(f(X))^2 < +\infty,$$

where

$$\mathbb{E}_X(f) = \mathbb{E}f(X).$$

On this functional space $\mathcal{F}_{(1)}$, which is endowed with the L_2-norm

$$\|f\|_2 = \left(\int f^2 d\mathbb{P}_X\right)^{1/2},$$

we define the Gaussian process $\{\mathbb{G}_{(1)}(f), f \in \mathcal{F}_{(1)}\}$, which is characterized by its variance-covariance function

$$\Gamma_{(1)}(f,g) = \int^2 (f - \mathbb{E}_X(f))(g - \mathbb{E}_X(g)) d\mathbb{P}_X, (f,g) \in \mathcal{F}_{(1)}^2.$$

This Gaussian process is the asymptotic weak limit of the sequence of functional empirical processes (fep) defined as follows. Let $X_1, X_2, ...$ be a sequence of independent copies of X. For each $n \geq 1$, we define the functional empirical process associated with X by

$$\mathbb{G}_{n,(1)}(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (f(X_i) - \mathbb{E}f(X_i)), f \in \mathcal{F}_{(1)},$$

and denote the integration with respect to the empirical measure by

$$\mathbb{P}_{n,(1)}(f) = \frac{1}{n} \sum_{i=1}^{n} (f(X_i), \ f \in \mathcal{F}_{(1)},$$

Denote by $\ell^\infty(T)$ the space of real-valued bounded functions defined on $T = \mathbb{R}$ equipped with its uniform topology. In the terminology of the weak convergence theory, the sequence of objects $\mathbb{G}_{n,(1)}$ weakly converges to $\mathbb{G}_{(1)}$ in $\ell^{\infty}(\mathbb{R})$, as stochastic processes indexed by $\mathcal{F}_{(1)}$, whenever it is a Donsker class. The details of this highly elaborated theory may be found in Billingsley (1968), Pollard (1984), van der Vaart and Wellner (1996) and similar sources.

we only need the convergence in finite distributions which is a simple consequence of the multivariate central limit theorem, as described in Chapter 3 in Lo *et al.* (2016).

We will use the Renyi's representation of the random variable X_i's of interest by means (*cdf*) $F_{(1)}$ as follows

$$X =_d F_{(1)}^{-1}(U),$$

where U is a uniform random variable on $(0,1)$, $=_d$ stands for the equality in distribution and $F_{(1)}^{-1}$ is the generalized inverse of $F_{(1)}$, defined by

$$F_{(1)}^{-1}(s) = \inf\{x, F_{(1)}(x) \geq s\},\ \ s \in (0,1).$$

Based on these representations, we may and do assume that we are on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ holding a sequence of independent $(0,1)$-uniform random variables U_1, U_2, ..., and the sequence of independent observations of X are given by

$$X_1 = F_{(1)}^{-1}(U_1),\quad X_2 = F_{(1)}^{-1}(U_2),\quad etc. \tag{1.5}$$

For each $n \geq 1$, the order statistics of $U_1, ..., U_n$ and of $X_1, ..., X_n$ are denoted respectively by $U_{1,n} \leq \cdots \leq U_{n,n}$ and $X_{1,n} \leq \cdots \leq X_{n,n}$.

To the sequences of $(U_n)_{n\geq 1}$, we also associate the sequence of real empirical functions

$$\mathbb{U}_{n,(1)}(s) = \frac{1}{n}\#\{i, 1 \leq i \leq n,\ U_i \leq s\},\ s \in (0,1)\ n \geq 1 \tag{1.6}$$

and the the sequence of real uniform quantile functions

$$\mathbb{V}_{n,(1)}(s) = U_{1,n}1_{(0\leq s\leq 1/n)} + \sum_{j=1}^{n} U_{j,n}1_{((i-1)/n \leq s \leq (i/n))},\ s \in (0,1),\ n \geq 1 \tag{1.7}$$

and next, the sequence of real uniform empirical processes

$$\alpha_{n,(1)}(s) = \sqrt{n}(\mathbb{U}_{n,(1)} - s), \ s \in (0,1) \ n \geq 1 \tag{1.8}$$

and the sequence of real uniform quantile processes

$$\gamma_{n,(1)}(s) = \sqrt{n}(s - \mathbb{V}_{n,(1)}), \ s \in (0,1) \ n \geq 1. \tag{1.9}$$

The same can be done for the sequence $(X_n)_{n\geq 1}$, and we obtain the associated sequence of real empirical procecesses a

$$\mathbb{G}_{n,r,(1)}(x) = \sqrt{n}\left(\mathbb{F}_{n,(1)}(x) - F_{(1)}(x)\right), \ x \in \mathbb{R}, \ n \geq 1 \tag{1.10}$$

where

$$\mathbb{F}_{n,(1)}(x) = \frac{1}{n}\#\{i, 1 \leq i \leq n, \ X_i \leq x\}, \ x \in \mathbb{R} \ n \geq 1 \tag{1.11}$$

is the associated sequence of empirical functions. We also have the associated sequence of quantile processes

$$\mathbb{Q}_{n,(1)}(x) = \sqrt{n}\left(\mathbb{F}^{-1}_{(n),(1)}(s) - F^{-1}(s)\right), \ s \in (0,1), \ n \geq 1 \tag{1.12}$$

where, for $n \geq 1$,

$$\mathbb{F}^{-1}_{n,(1)}(s) = X_{1,n}1_{(0\leq s \leq 1/n)} + \sum_{j=1}^{n} X_{j,n}1_{((i-1)/n \leq s \leq (i/n))}, \ s \in (0,1), \tag{1.13}$$

is the associated sequence of quantile processes.

By passing, we recall that $\mathbb{F}^{-1}_{n,(1)}$ is actually the generalized inverse of $\mathbb{F}_{(n),(1)}$ and for the uniform sequence, we have

$$\mathbb{V}_{n,(1)} = \mathbb{U}^{-1}_{n,(1)} \tag{1.14}$$

In virtue of Representation (1.5), we have the following remarkable relations

$$\mathbb{G}_{n,r,(1)}(x) = \alpha_{n,(1)}(F_{(1)}(x)), \ x \in \mathbb{R} \tag{1.15}$$

and

$$\mathbb{Q}_{n,(1)}(x) = \sqrt{n}\left(F_{(1)}^{-1}(\mathbb{V}_{n,(1)}(s)) - F_{(1)}^{-1}(s)\right) \ s \in (0,1), \ n \geq 1, \tag{1.16}$$

We also have the following relations between the empirical functions and quantile functions

$$\mathbb{F}_{n,(1)}(x) = \mathbb{U}_{n,(1)}(F_{(1)}(x)), \ x \in \mathbb{R} \tag{1.17}$$

and

$$\mathbb{F}_{n,(1)}^{-1}(s) = F_{(1)}^{-1}(\mathbb{V}_{(n),(1)}(s)), \ s \in (0,1), \ n \geq 1. \tag{1.18}$$

As well, the real and functional empirical processes are related as follows : for $n \geq 1$,

$$\mathbb{G}_{n,r,(1)}(x) = \mathbb{G}_{n,(1)}(f_x^*), \ \alpha_{n,(1)}(s) = \mathbb{G}_{n,(1)}(f_s), \ s \in (0,1) \ x \in \mathbb{R}, \tag{1.19}$$

where for any $x \in \mathbb{R}$, $f_x^* = 1_{]-\infty,x]}$ is the indicator function of $]-\infty, x]$ and for $s \in (0,1)$, $f_s = 1_{[0,s]}$.

To finish the description, a result of Kiefer-Bahadur (See Bahadur (1966)) that says that the addition of the sequences of uniform empirical processes and quantiles processes (1.8) and (1.9) is asymptotically, and uniformly on $[0,1]$, zero in probability, that is

$$\sup_{s\in[0,1]} \left|\alpha_{n,(1)}(s) + \gamma_{n,(1)}(s)\right| = o_{\mathbb{P}}(1) \text{ as } n \to +\infty. \tag{1.20}$$

This result is a powerful tool to handle the rank statistics when our studied statistics are L-statistics.

Bivariate frame. As to the bivariate case, we use the Sklar's theorem (See Sklar (1959)). Let us begin to define a copula in $\mathbb{R}^2$ as bivariare probability distribution function $C(u,v)$, $(u,v) \in \mathbb{R}^2$ with support $[0,1]^2$ and with $[0,1]$-uniform margins, that is

$$C(u,v)=0 \text{ for } (u,v)\in]-\infty,0[\times\mathbb{R} \text{ for } (u,v)\in\mathbb{R}\times]-\infty,0[.$$

Let us denote by $F_{(2)}$ the bivariate distribution function of our random couple $Y=(X^{(1)},X^{(2)})$ and by $F_{(21)}$ and $F_{(22)}$ its margins, which are the *cdf* of $X^{(1)}$ and $X^{(2)}$ respectively. The Sklar's theorem (Sklar (1959)) says that there exists a copula $C_{(2)}$ such that we have

$$F_{(2)}(x,y)=C_{(2)}(F_{(21)}(x),F_{(22)}(y)), \text{ for any } (x,y)\in\mathbb{R}^2. \tag{1.21}$$

This copula is unique if the marginal *cdf*'s are continuous. In this paper, we will suppose that the marginal *cdf*'s are continuous and then $C_{(2)}$ is unique andfixed for once. By the Kolmogorov Theorem, there exists a probability space $(\Omega,\mathcal{A},\mathbb{P})$ holding a sequence of independent random couples $(U_i^{(1)},U_n^{(2)})$, $n\geq 1$, of common bivariate distribution function $C_{(2)}$. On that space the random couples $(F_{(21)}^{-1}(U_n^{(1)}),\ F_{(22)}^{-1}(U_n^{(2)}))$ are independent and have a common bivariate distribution function equal to $C_{(2)}$, since

$$\begin{aligned}
&\mathbb{P}(F_{(21)}^{-1}(U_i^{(1)})\leq x_1,\ F_{(22)}^{-1}(U_i^{(2)})\leq x_2)\\
=\ &\mathbb{P}(U_i^{(1)}\leq F_{(21)}(x_1),\ U_i^{(2)}\leq F_{(22)}(x_2))\\
=\ &C_{(2)}(F_{(21)}(x_1),\ F_{(22)}(x_2))\\
=\ &F_{(2)}(x_1,\ x_2),
\end{aligned}$$

by (1.21), and where we applied the general formula for generalized inverses functions for a *cdf* :

$$F^{-1}(s)\leq y \Leftrightarrow s\leq F(x), \text{ for } (s,x)\in[0,1]\times\mathbb{R}.$$

For more on interesting properties of generalized inverses of monotone functions, see Lo *et al.* (2016), Chapter 4.

Based on this remark, we place ourselves on the probability space holding the sequence of independent random couples $(U^{(1)},U^{(2)})$, $(U_n^{(1)},U_n^{(2)})$, $n\geq 2$, with common distribution function $C_{(2)}$, and the observations from $Y=(X^{(1)},X^{(2)})=(F_{(2),1}^{-1}(U^{(1)}),F_{(2),2}^{-1}(U^{(2)}))$, are generated as follows :

$$Y_n=(F_{(21)}^{-1}(U_n^{(1)}),F_{(22)}^{-1}(U_n^{(2)})),\ n\geq 1. \tag{1.22}$$

In this setting, we rather use the the bidimensional functional empirical process based on $\left\{\left((U_i^{(1)}, U_i^{(2)})\right)\right\}_{i=1,\dots,n}$ and defined by

$$\mathbb{T}_{n,(2)}(f) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(f\left((U_i^{(1)}, U_{(i)}^{2})\right) - \mathbb{P}_{\left((U^{(1)}, U^{(2)})\right)}(f) \right), \tag{1.23}$$

whenever f is a function of $(u,v) \in [0,1]^2$ such that $\mathbb{E}(f(U^{(1)}, U^{(2)})^2)$.

For any Donsker class $\mathcal{F}_{(2)}([0,1]^2)$, the stochastic process $\mathbb{T}_{n,(2)}$ converges to a Gaussian process $\mathbb{T}$ with variance-covariance function, for $(f,g) \in \mathcal{F}_{(2)}^2([0,1]^2)$, denoted by $\Gamma^*(f,g)$, is given by

$$\int_{[0,1]^2} \left(f(u,v) - \mathbb{P}_{\left(U^{(1)}, U^{(2)}\right)}(f)\right) \left(g(u,v) - \mathbb{P}_{\left(U^{(1)}, U^{(2)}\right)}(g)\right) dC(u,v) \tag{1.24}$$

where

$$\mathbb{P}_{\left(U^{(1)}, U^{(2)}\right)}(f) = \mathbb{E}\left(f\left(U^{(1)}, U^{(2)}\right)\right) = \int_{[0,1]^2} f(u,v)\, dC(u,v).$$

Another form of the variance-covariance function 1.24 is also

$$\gamma_{(2)}(f,g) = \int_{[0,1]^2} f(s)g(t)\left(C(s,t) - s\,t\right) ds\, dt \tag{1.25}$$

By deciding to use the functional empirical process based on the observations provided by the Copula C, the functional empirical process based on the incomes and defined by

$$\mathbb{G}_{n,(2)}(g) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(g\left((X_j^{(1)}, X_{(j)}^{2})\right) - \mathbb{P}_{\left((X^{(1)}, X^{(2)})\right)}(g) \right), \tag{1.26}$$

for any function g, defined on $\mathcal{V}_X^2$ such that $\mathbb{E}(g(X^{(1)}, X^{(2)})^2) < +\infty$ is not used directly. Instead, by using Representation (1.22), we have

$$\mathbb{G}_{n,(2)}(g) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(g\left(F_{(2),1}^{-1}(U_j^{(1)}), F_{(2),1}^{-1}(U_j^{(2)})\right) - \mathbb{P}_{((U^1, U^2))}\left(g((F_{(2),1}^{-1}(.), F_{(2),2}^{-1}(.))\right)\right).$$

Hence the correspondence between the function g in Formula (1.26) and f in Formula in (1.23) is the following.

$$f(s,t) = g\left(F^{-1}_{(2),1}(s), F^{-1}_{(2),2}(t)\right),\ (s,t) \in [0,1]^2,. \tag{1.27}$$

All the needed notation are now complete and will allow the expression of the asymptotic theory we undertake here.

2. The asymptotic behavior of the Gini Index

Let X denote the income random variable of one given population with a positive mean $\mu = \mathbb{E}(X)$ and let $\mathcal{V}_X$ denote its support.

$$GI_n = \frac{1}{\mu_n}\left(\frac{1}{n}\sum_{j=1}^{n}\left(\frac{2j-1}{n}-1\right)X_{j,n}\right). \tag{2.1}$$

Set

$$A_n = \frac{1}{n}\sum_{j=1}^{n}\frac{j}{n}X_{j,n}. \tag{2.2}$$

We can write this expression of as

$$A_n = \frac{1}{n}\sum_{j=1}^{n}\mathbb{F}_{n,(1)}(X_j)X_j. \tag{2.3}$$

Formula (2.1) becomes

$$GI_n = \frac{2A_n}{\mu_n} - 1 - \frac{1}{n}. \tag{2.4}$$

Before tackling this study, let us first introduce some notations:

$$\forall x \in \mathcal{V}_X, h(x) = xF_{(1)}(x),\ I_d(x) = x;$$
$$\forall s \in (0,1), \ell(s) = F^{-1}_{(1)}(s), f_s(x) = \mathbf{1}_{(0,F^{-1}_{(1)}(s))}(x),$$

And finally, set for real-valued measurable functions f and g defined on $\mathbb{R}$

$$\gamma_1(f,g) = \int_0^1\int_0^1 f(s)g(t)\left(\min(s,t) - s\,t\right)\,ds\,dt \tag{2.5}$$

Now, we have the following theorems for the asymptotic behavior, the first concerns the statistic A_n and the second concerns that of GI_n. Let us state first the following lemma of the representation.

LEMMA 16. *Define*

$$\beta_{n,(1)}(\ell) = \int_0^1 \ell(s)\mathbb{G}_{n,(1)}(f_s)\, ds.$$

The, the statistic A_n can be represented as follows

$$A_n = \mathbb{P}_{n,1}(h) + \frac{1}{\sqrt{n}}\beta_{n,(1)}(\ell) + o_p(n^{-1/2}). \tag{2.6}$$

PROOF. By decomposing the equation (2.3), we get

$$A_n = \frac{1}{n}\sum_{j=1}^{n)} F_{(1)}(X_j)X_j + \frac{1}{n}\sum_{j=1}^{n} \left(\mathbb{F}_{n,(1)}(X_j) - F_{(1)}(X_j)\right) X_j.$$

Let us denote the residual term by

$$Re_n = \frac{1}{n}\sum_{j=1}^{n} \left(\mathbb{F}_{n,(1)}(X_j) - F_{(1)}(X_j)\right) X_j$$

then

$$Re_n = \sum_{j=1}^{n} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left\{\mathbb{F}_{n,(1)}(\mathbb{F}_{n,(1)}^{-1}(s)) - F_{(1)}(\mathbb{F}_{n,(1)}^{-1}(s))\right\} \mathbb{F}_{n,(1)}^{-1}(s)\, ds,$$

and so

$$Re_n = \int_0^1 \left\{F_{n,(1)}(F_{n,(1)}^{-1}(s)) - F_{(1)}(F_{n,(1)}^{-1}(s))\right\} F_{n,(1)}^{-1}(s)\, ds. \tag{2.7}$$

By using Formulas (1.14), (1.15) and (1.16), we get

$$\begin{aligned}\sqrt{n}Re_n &= -\int_0^1 \sqrt{n}\left\{\mathbb{U}_{n,(1)}\left(\mathbb{V}_{n,(1)}(s)\right) - \mathbb{V}_{n,(1)}(s)\right\} F_{(1)}^{-1}\left(\mathbb{V}_{n,(1)}(s)\right)\, ds \\ &= -\int_0^1 \sqrt{n}\left(s - \mathbb{V}_{n,(1)}(s)\right) F_{(1)}^{-1}\left(\mathbb{V}_{n,(1)}(s)\right)\, ds\end{aligned}$$

$$-\int_0^1 \sqrt{n}\left(\mathbb{U}_{n,(1)}\left(\mathbb{V}_{n,(1)}(s)\right)-s\right)\, F_{(1)}^{-1}\left(\mathbb{V}_{n,(1)}(s)\right)\, ds.$$

From Shorack and Wellner (1986) (page 585), we have

$$\sup_{0\leq s\leq 1}\left|\mathbb{U}_{n,(1)}\left(\mathbb{V}_{n,(1)}(s)\right)-s\right|\leq \frac{1}{n}.$$

Using the notations $\alpha_{n,(1)}$ and $\gamma_{n,(1)}$, we get

$$\sqrt{n}Re_n = -\int_0^1 \sqrt{n}\left(s-\mathbb{V}_{n,(1)}(s)\right) F_{(1)}^{-1}\left(\mathbb{V}_{n,(1)}(s)\right)\, ds + o_p(1)$$

$$= -\int_0^1 \gamma_{n,(1)}(s) F_{(1)}^{-1}(s)\, ds + o_p(1) \tag{2.8}$$

By using the following Bahadur's representation (See Formula 1.20) and by applying Formula 1.19, we get

$$\sqrt{n}Re_n = \int_0^1 \mathbb{G}_{n,(1)}(f_s)\,\ell(s)\, ds + o_p(1),$$

then by identification we get

$$\beta_{n,(1)}(\ell) = \int_0^1 \mathbb{G}_{n,(1)}(f_s)\,\ell(s)\, ds,$$

which closes the proof. □

Here is the full representation of the asymptotic distribution of Gini's statistic.

THEOREM 24. *Let $\mathbb{P}_X(h^2)$ is finite and the function ℓ is bounded, then when n tends to ∞, $\sqrt{n}\left(A_n-\mathbb{P}_X(h)\right)\rightarrow_d \mathbb{A}(h)=\mathbb{G}_{(1)}(h)+\beta_{(1)}(\ell)$, with $\mathbb{A}(h)\rightsquigarrow \mathcal{N}\left(0,\sigma_A^2\right)$,*

$$\sigma_A^2 = \Gamma(h,h)+\Gamma(\beta_{(1)}(\ell),\beta_{(1)}(\ell))+2\Gamma(h,\beta_{(1)}(\ell)), \tag{2.9}$$

with

$$\Gamma(h,h) = \int \left(h(x)-\mathbb{P}_X\left(h\right)\right)^2\, dF_{(1)}(x), \tag{2.10}$$

$$\Gamma(\beta_{(1)}(\ell),\beta_{(1)}(\ell)) = \gamma_1(\ell,\ell), \tag{2.11}$$

where the function $\gamma_1(.,.)$ is defined in Formula (2.5), and

$$\Gamma(h, \beta_{(1)}(\ell)) = \int_0^1 \ell(s) \left(\int_{\left(x \leq F_{(1)}^{-1}(s)\right)} h(x)\, dF_{(1)}(x) \right) ds - \left(\mathbb{P}_X(h)\right)^2. \tag{2.12}$$

PROOF. By using the previous lemma, its easy to see that

$$\sqrt{n}\left(A_n - \mathbb{P}_X(h)\right) = \mathbb{G}_{n,(1)}(h) + \beta_{n,(1)}(\ell) + o_p(1)$$

which tends to a centered Gaussian process $\mathbb{A}(h) = \mathbb{G}_{(1)}(h) + \beta_{(1)}(\ell)$.

Now, let us find the variance of this centered process. We have

$$\sigma_A^2 = \mathbb{E}\left(\left(\mathbb{G}_{(1)}(h) + \beta_{(1)}(\ell)\right)^2\right)$$

$$\begin{aligned} \sigma_A^2 &= \mathbb{E}\left(\mathbb{G}_{(1)}(h)^2\right) + \mathbb{E}\left(\beta_{(1)}(\ell)^2\right) + 2\mathbb{E}\left(\mathbb{G}_{(1)}(h)\beta_{(1)}(\ell)\right) \\ &\equiv \Gamma(h,h) + \Gamma(\beta_{(1)}(\ell), \beta_{(1)}(\ell)) + 2\Gamma(h, \beta_{(1)}(\ell)). \end{aligned}$$

By Equation (2.10) of the definition of the covariance function, we find

$$\Gamma(h,h) = \int \left(h(x) - \mathbb{P}_X(h)\right)^2 dG(x).$$

Let us compute now the remaining terms as follows.

$$\Gamma(\beta_{(1)}(\ell), \beta_{(1)}(\ell)) = \mathbb{E}\left(\int_0^1 \int_0^1 \ell(s)\ell(t)\mathbb{G}_{(1)}(f_s)\mathbb{G}_{(1)}(f_t)\, ds\, dt\right)$$

which gives, by applying Fubini's Theorem,

$$\Gamma(\beta_{(1)}(\ell), \beta_{(1)}(\ell)) = \int_0^1 \int_0^1 \ell(s)\ell(t)\mathbb{E}\left(\mathbb{G}_{(1)}(f_s)\mathbb{G}_{(1)}(f_t)\right) ds\, dt.$$

Since we have

$$\mathbb{E}\left(\mathbb{G}_{(1)}(f_s)\mathbb{G}_{(1)}(f_t)\right) = \min(s,t) - s\,t,$$

we get

$$\Gamma(\beta_{(1)}(\ell), \beta_{(1)}(\ell)) = \int_0^1 \int_0^1 \ell(s)\ell(t)\left(\min(s,t) - s\,t\right) ds\, dt.$$

By Equation (2.5), we have

$$\Gamma(\beta_{(1)}(\ell), \beta_{(1)}(\ell)) = \gamma_1(\ell, \ell).$$

For $\Gamma(h, \beta_{(1)}(\ell)) = \mathbb{E}\left(\mathbb{G}_{(1)}(h)\beta_{(1)}(\ell)\right)$, we obtain

$$\Gamma(h, \beta_{(1)}(\ell)) = \int_0^1 \ell(s) m(h, f_s)\, ds$$

with

$$\begin{aligned} m(h, f_s) &= \mathbb{E}\left(\mathbb{G}_{(1)}(h)\mathbb{G}_{(1)}(f_s)\right) \\ &= \int_{\left(x \leq F_{(1)}^{-1}(s)\right)} h(x)\, dF_{(1)}(x) - s\,\mathbb{P}_X(h). \end{aligned}$$

Finally, we conclude that

$$\Gamma(h, \beta_{(1)}(\ell)) = \int_0^1 \ell(s) \left(\int_{\left(x \leq F_{(1)}^{-1}(s)\right)} h(x)\, dF_{(1)}(x) - s\,\mathbb{P}_X(h)\right) ds. \tag{2.13}$$

This completes the proof of Theorem 24. □

For the last part of this section, let's define the continuous form of the Gini index as follows

$$GI = \frac{2\,A}{\mu} - 1 \text{ with } A = \mathbb{P}_X(h).$$

Then we are able to expose the following Theorem.

THEOREM 25. *Assume that $\mu \neq 0$, ℓ is bounded, the quantities $\mathbb{P}_X(h^2)$ and $\mathbb{P}_X(I_d^2)$ are finite, then the following assertion holds :*

$$\sqrt{n}\,(GI_n - GI) = \frac{2}{\mu}\left(\mathbb{G}_{n,(1)}(h - \frac{A}{\mu}I_d) + \beta_{n,(1)}(\ell)\right) + o_p(1).$$

This quantity tends to a centered Gaussian process with variance σ^2_{GI} which is giving by

$$\sigma^2_{GI} = \frac{4}{\mu^2}\left(\sigma^2_A + \frac{A^2}{\mu^2}\Gamma(I_d, I_d) - \frac{2A}{\mu}\left(\Gamma(h, I_d) + \Gamma(I_d, \beta_{(1)}(\ell))\right)\right)$$

where σ^2_A is giving in Theorem 24; $\Gamma(I_d, I_d) = \mathbb{E}\left((X-\mu)^2\right)$ is the variance of the random variable X;

$$\Gamma(h, I_d) = \int \left(h(x) - \mathbb{P}_X(h)\right)(x-\mu)\, dF_{(1)}(x);$$

$$\Gamma(I_d, \beta_{(1)}(\ell)) = \int_0^1 \ell(s) \left(\int_{\left(x \leq F_{(1)}^{-1}(s)\right)} x\, dF_{(1)}(x) \right) ds - A\,\mu.$$

PROOF.

$$\begin{aligned}\sqrt{n}\,(GI_n - GI) &= 2\left(\frac{\sqrt{n}\,(A_n - A)}{\mu_n} - \frac{A}{\mu\mu_n}\sqrt{n}\,(\mu_n - \mu) \right) - \frac{1}{\sqrt{n}} \\ &= \frac{2}{\mu}\left(\mathbb{G}_{n,(1)}(h) + \beta_n(\ell) - \frac{A}{\mu}\mathbb{G}_{n,(1)}(I_d) \right) + o_p(1)\end{aligned}$$

which tends to a centered Gaussian process $\frac{2}{\mu}\left(\mathbb{A} - \frac{A}{\mu}\mathbb{G}_{(1)}(I_d)\right)$. Compute now the expression of the variance. We get

$$\begin{aligned}\sigma_{GI}^2 &= \mathbb{E}\left(\left(\frac{2}{\mu}\left(\mathbb{A} - \frac{A}{\mu}\mathbb{G}_{(1)}(I_d) \right) \right)^2 \right) \\ &= \frac{4}{\mu^2}\left(\sigma_A^2 + \frac{A^2}{\mu^2}\Gamma(I_d, I_d) - \frac{2A}{\mu}\left(\Gamma(h, I_d) + \Gamma(I_d, \beta_{(1)}(\ell)) \right) \right).\end{aligned}$$

Applying the equation (2.10) to the functions h and I_d, we obtain the expression of $\Gamma(h,h)$, $\Gamma(I_d, I_d)$ and $\Gamma(h, I_d)$. And by replacing the function h by I_d in equation 2.13 we obtain

$$\Gamma(I_d, \beta_{(1)}(\ell)) = \int_0^1 \ell(s) \left(\int_{\left(x \leq F_{(1)}^{-1}(s)\right)} I_d(x)\, dF_{(1)}(x) - s\,\mathbb{P}_X(I_d) \right) ds,$$

but, remember that I_d is the identity function and

$$\mathbb{P}_X(I_d) \int_0^1 s\,\ell(s)\, ds = \mu\, A$$

then

$$\Gamma(I_d, \beta_\ell) = \int_0^1 \ell(s) \left(\int_{\left(x \leq F_{(1)}^{-1}(s)\right)} x\, dF_{(1)}(x) \right) ds - A\,\mu.$$

This completes the proof of this part. □

Let us move to the variation of the Gini's statistics.

3. Variation of the Gini Index between two dates

We fully use the setting described in Subsection 1.1 regarding the two phase approach. We need to adapt the notation and the results found in Theorems 24 and 25 to follow the consequences of the moving from Formula (1.26) to Formula (1.23) through Formula (1.27). Accordingly, define $\forall\ (u,v)\ \in\ [0,1]^2$ and $\forall\, j\ =\ 1,\ 2$:

$$\ell_j(u) = F^{-1}_{(2),j}(s),\ h_j(u) = u\,\ell_j(u);\ L_j(u) = \frac{2}{\mu(j)}\ell_j(u);$$
$$\tilde{f}_j(u,v) = \ell_j \circ \pi_j(u,v),\ f_{j,h}(u,v) = h_j \circ \pi_j(u,v),$$
$$f_{j,s}(u,v) = \pi_i\left(\mathbf{1}_{(0\leq s)}(u), \mathbf{1}_{(0\leq s)}(v)\right)$$

where π_j is the j^{th} projection;

$$F^*_{j,h}(u,v) = \frac{2}{\mu(j)} f_{j,h}(u,v);$$
$$F^*(u,v) = F^*_{2,h}(u,v) - F^*_{1,h}(u,v);$$
$$\tilde{F}^*(u,v) = 2\left(\frac{A(2)}{\mu(2)^2}\tilde{f}_2(u,v) - \frac{A(1)}{\mu(1)^2}\tilde{f}_1(u,v)\right).$$

Let

$$\beta^*_{n,(2)}(L) = \int_{[} 0,1] \left(L_2(s)\mathbb{G}_{n,(2)}(f_{2,s}) - L_1(s)\mathbb{G}_{n,(2)}(f_{1,s})\right)\, ds$$

be the bidimensional residual process.

We can now expose our main theorem which concerns the variation of the Gini Index.

THEOREM 26. *Assume that, for all $j=1,2$, $\mu(j)$ is finite and not null; L_j is bounded; the functions $f_{j,s}$, $f_{j,h}$, $\tilde{f}_j$, F^* and $\tilde{F}^*$ are square integrable, then we have the following convergence in distribution as n tends to infinity:*

$$\sqrt{n}\left(\Delta GI_n(1,2) - \Delta GI(1,2)\right) \to_d \mathcal{G}_{\Delta GI} \rightsquigarrow \mathcal{N}\left(0, \sigma^2_{\Delta GI}\right)$$

with

$$\sigma^2_{\Delta GI} = \Gamma^*\left(F^*, F^*\right) + \Gamma^*\left(\tilde{F}^*, \tilde{F}^*\right) + \Gamma^*\left(\beta^*_L, \beta^*_L\right)$$
$$-2\left(\Gamma^*\left(F^*, \tilde{F}^*\right) + \Gamma^*\left(\tilde{F}^*, \beta^*_L\right) - \Gamma^*\left(F^*, \beta^*_L\right)\right)$$

where

$$\Gamma^*\left(\beta_L^*, \beta_L^*\right) = \gamma_1(L_1, L_1) + \gamma_1(L_2, L_2) - 2\gamma_2(L_1, L_2);$$

$$\begin{aligned}
\Gamma^*\left(F^*, \beta_L^*\right) &= \int_{[0,1]} \left\{ L_2(s) \int_{[0,1]\times(0,s)} F^*(u,v)dC(u,v) \right\} ds \\
&- \int_{[0,1]} \left\{ Ł_1(s) \int_{(0,s)\times[0,1]} F^*(u,v)dC(u,v) \right\} ds \\
&-2\left(\frac{A(2)}{\mu(2)} - \frac{A(1)}{\mu(1)}\right) \mathbb{P}_{(U^{(1)},U^{(2)})}\left(F^*\right)
\end{aligned}$$

$\Gamma^*\left(\tilde{F}^*, \beta_L^*\right)$ *is obtained by replacing* F^* *by* $\tilde{F}^*$ *in the previous expression. And we get the covariances of* $\Gamma^*\left(F^*, F^*\right)$, $\Gamma^*\left(\tilde{F}^*, \tilde{F}^*\right)$ *and* $\Gamma^*\left(F^*, \tilde{F}^*\right)$ *by the equation (1.24).*

Proof. We get

$$\begin{aligned}
&\sqrt{n}\left(\Delta GI_n(1,2) - \Delta GI(1,2)\right) \\
&= \frac{2}{\mu(2)}\left(\mathbb{T}_{n,(2)}\left(f_{2,h}\right) + \beta_{n,(2)}^*\left(\ell_2\right) - \frac{A(2)}{\mu(2)}\mathbb{T}_{n,(2)}\left(\tilde{f}_2\right)\right) \\
&- \frac{2}{\mu(1)}\left(\mathbb{T}_{n,(2)}\left(f_{1,h}\right) + \beta_{n,(2)}^*\left(\ell_1\right) - \frac{A(1)}{\mu(1)}\mathbb{T}_{n,(2)}\left(\tilde{f}_1\right)\right) + o_p(1) \\
&= 2\mathbb{T}_{n,(2)}\left(\frac{f_{2,h}}{\mu(2)} - \frac{f_{1,h}}{\mu(1)}\right) + 2\beta_{n,(2)}^*\left(\frac{\ell_2}{\mu(2)} - \frac{\ell_1}{\mu(1)}\right) \\
&-2\mathbb{T}_{n,(2)}\left(\frac{A(2)}{\mu^2(2)}\tilde{f}_2 - \frac{A(1)}{\mu^2(1)}\tilde{f}_1\right) + o_p(1).
\end{aligned}$$

We find the next expression

$$\begin{aligned}
\sqrt{n}\left(\Delta GI_n(1,2) - \Delta GI(1,2)\right) &= \mathbb{T}_{n,(2)}\left(F^*\right) + \beta_{n,(2)}^*\left(L\right) \\
&- \mathbb{T}_{n,(2)}\left(\tilde{F}^*\right) + o_p(1) \\
\to_d \mathcal{T}_{\Delta GI} = \mathbb{T}_{(2)}\left(F^*\right) + \beta^*\left(L\right) - \mathbb{T}_{(2)}\left(\tilde{F}^*\right) &\rightsquigarrow \mathcal{N}\left(0, \sigma_{\Delta GI}^2\right).
\end{aligned}$$

Now, we search the expression of the variance.

$$\sigma_{\Delta GI}^2 = \mathbb{E}\left(\left(\mathbb{T}_{(2)}(F^*) + \beta^*(L)\right)^2 + \mathbb{T}_{(2)}\left(\tilde{F}^*\right)^2\right)$$

$$\begin{aligned}
&-2\mathbb{E}\left(\left(\mathbb{T}_{(2)}(F^*)+\beta^*(L)\right)\mathbb{T}_{(2)}\left(\tilde{F}^*\right)\right)\\
&=\mathbb{E}\left(\mathbb{T}_{(2)}(F^*)^2\right)+\mathbb{E}\left(\beta^*(L)^2\right)+2\mathbb{E}\left(\mathbb{T}_{(2)}(F^*)\beta^*(L)\right)\\
&+\mathbb{E}\left(\mathbb{T}_{(2)}(\tilde{F}^*)^2\right)-2\mathbb{E}\left(\mathbb{T}_{(2)}(F^*)\mathbb{T}_{(2)}(\tilde{F}^*)\right)-2\mathbb{E}\left(\mathbb{T}_{(2)}(\tilde{F}^*)\beta^*(L)\right).
\end{aligned}$$

Then

$$\begin{aligned}
\sigma^2_{\Delta GI}&=\Gamma^*\left(F^*,F^*\right)+\Gamma^*\left(\tilde{F}^*,\tilde{F}^*\right)+\Gamma^*\left(\beta^*_L,\beta^*_L\right)\\
&-2\left(\Gamma^*\left(F^*,\tilde{F}^*\right)+\Gamma^*\left(\tilde{F}^*,\beta^*_L\right)-\Gamma^*\left(F^*,\beta^*_L\right)\right).
\end{aligned}$$

And we get the expressions of the covariances of $\Gamma^*\left(F^*,F^*\right)$, $\Gamma^*\left(\tilde{F}^*,\tilde{F}^*\right)$ and $\Gamma^*\left(F^*,\tilde{F}^*\right)$ by the equation (1.24). For the tree rest, let's find them.

Firstly, compute $\Gamma^*\left(\beta^*_L,\beta^*_L\right)$.

$$\begin{aligned}
\Gamma^*\left(\beta^*_L,\beta^*_L\right)&=\mathbb{E}\left(\beta^*(L)^2\right)\\
&=\mathbb{E}\left(\int_D\left(L_2(s)\mathbb{T}_{(2)}(f_{2,s})-L_1(s)\mathbb{T}_{(2)}(f_{1,s})\right)\,ds\right)^2\\
&=\mathbb{E}\left(\int_{[0,1]^2}\left(L_2(s)\mathbb{T}_{(2)}(f_{2,s})-L_1(s)\mathbb{G}(f_{1,s})\right)\left(L_2(t)\mathbb{T}_{(2)}(f_{2,t})-L_1(t)\mathbb{T}_{(2)}(f_{1,t})\right)\,ds\,dt\right)\\
&=\int_{[0,1]^2}L_2(s)L_2(t)\Gamma^*\left(f_{2,s},f_{2,t}\right)\,ds\,dt+\int_{[0,1]^2}L_1(s)L_1(t)\Gamma^*\left(f_{1,s},f_{1,t}\right)\,ds\,dt\\
&-\int_{[0,1]^2}L_1(t)L_2(s)\Gamma^*\left(f_{1,t},f_{2,s}\right)\,ds\,dt-\int_{[0,1]^2}L_1(s)L_2(t)\Gamma^*\left(f_{1,s},f_{2,t}\right)\,ds\,dt.
\end{aligned}$$

But, we have

$$\int_{[0,1]^2}L_1(t)L_2(s)\Gamma^*\left(f_{1,t},f_{2,s}\right)\,ds\,dt=\int_{[0,1]^2}L_1(s)L_2(t)\Gamma^*\left(f_{1,s},f_{2,t}\right)\,ds\,dt$$

then

$$\Gamma^*\left(\beta^*_L,\beta^*_L\right)=\int_{[0,1]^2}L_1(s)L_1(t)\Gamma^*\left(f_{1,s},f_{1,t}\right)\,ds\,dt$$

$$
\begin{aligned}
&+\int_{[0,1]^2} L_2(s)L_2(t)\Gamma^*\left(f_{2,s},f_{2,t}\right)\,ds\,dt\\
&-2\int_{[0,1]^2} L_1(t)L_2(s)\Gamma^*\left(f_{1,t},f_{2,s}\right)\,ds\,dt\\
&\equiv \gamma_1(L_1,L_1)+\gamma_1(L_2,L_2)-2\gamma_2(L_1,L_2).
\end{aligned}
$$

But

$$
\begin{aligned}
\Gamma^*\left(f_{1,s},f_{1,t}\right)&=\mathbb{E}\left(\mathbf{1}_{(0,s)}\left(U\right)\mathbf{1}_{(0,t)}\left(U\right)\right)-st=\min(s,t)-s\,t,\\
\Gamma^*\left(f_{2,s},f_{2,t}\right)&=\mathbb{E}\left(\mathbf{1}_{(0,s)}\left(U^{(2)}\right)\mathbf{1}_{(0,t)}\left(U^{(2)}\right)\right)-st=\min(s,t)-s\,t,
\end{aligned}
$$

and

$$
\Gamma^*\left(f_{1,s},f_{2,t}\right)=\mathbb{E}\left(\mathbf{1}_{(0,s)}\left(U\right)\mathbf{1}_{(0,t)}\left(U^{(2)}\right)\right)-st=C(s,t)-st.
$$

Then by identification we get

$$
\begin{aligned}
\gamma_1(L_1,L_1)&=\int_{[0,1]^2} L_1(s)L_1(t)\left(\min(s,t)-s\,t\right)\,ds\,dt;\\
\gamma_1(L_2,L_2)&=\int_{[0,1]^2} L_2(s)L_2(t)\left(\min(s,t)-s\,t\right)\,ds\,dt;\\
\gamma_2(L_1,L_2)&=\int_{[0,1]^2} L_1(s)L_2(t)\left(C(s,t)-s\,t\right)\,ds\,dt.
\end{aligned}
$$

Secondly, let's find the expression of $\Gamma^*\left(F^*,\beta_L^*\right)$.

$$
\begin{aligned}
\Gamma^*\left(F^*,\beta_L^*\right)&=\mathbb{E}\left(\mathbb{T}_{(2)}(F^*)\beta_L^*\right)\\
&=\int_D L_2(s)\Gamma^*\left(F^*,f_{2,s}\right)\,ds-\int_D L_1(s)\Gamma^*\left(F^*,f_{1,s}\right)\,ds
\end{aligned}
$$

or

$$
\Gamma^*\left(F^*,f_{2,s}\right)=\mathbb{E}\left(F^*(U^{(1)},U^{(2)})f_{2,s}(U^{(1)},U^{(2)})\right)-\mathbb{P}_{(U^{(1)},U^{(2)})}\left(F^*\right)\mathbb{P}_{(U^{(1)},U^{(2)})}(f_{2,s})
$$

$$
\begin{aligned}
\mathbb{P}_{(U^{(1)},U^{(2)})}(f_{2,s})&=\int_{[0,1]^2} f_{2,s}(u,v)\,dC(u,v)\\
&=\int_{[0,1]^2}\mathbf{1}_{(0,s)}(v)\,dC(u,v)=C(1,s)=s.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left(F^*(U^{(1)},U^{(2)})f_{2,s}(U^{(1)},U^{(2)}\right) &= \mathbb{E}\left(F^*(U^{(1)},U^{(2)})\mathbf{1}_{(0,s)}(U^{(2)})\right)\\
&= \int_{[0,1]\times(0,s)} F^*(u,v)\,dC(u,v),
\end{aligned}
$$

and so, we arrive at

$$
\Gamma^*\left(F^*, f_{2,s}\right) = \int_{[0,1]\times(0,s)} F^*(u,v)\,dC(u,v) - s\,\mathbb{P}_{(U^{(1)},U^{(2)})}\left(F^*\right).
$$

Similarly, we get

$$
\Gamma^*\left(F^*, f_{1,s}\right) = \int_{(0,s)\times[0,1]} F^*(u,v)\,dC(u,v) - s\,\mathbb{P}_{(U^{(1)},U^{(2)})}\left(F^*\right).
$$

Therefore, we have

$$
\begin{aligned}
\Gamma^*\left(F^*, \beta_L^*\right) = &\int_{[0,1]} \left\{L_2(s)\int_{[0,1]\times(0,s)} F^*(u,v)dC(u,v)\right\}\,ds\\
&-\int_{[0,1]} \left\{\text{Ł}_1(s)\int_{(0,s)\times[0,1]} F^*(u,v)dC(u,v)\right\}ds\\
&-\mathbb{P}_{(U^{(1)},U^{(2)})}\left(F^*\right)\int_{[0,1]} s(L_2(s)-L_1(s))ds,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{[0,1]} s(L_2(s)-L_1(s))ds &= 2\int_D \left(\frac{s\,\ell_2(s)}{\mu(2)} - \frac{s\,\ell_1(s)}{\mu(1)}\right)\\
&= 2\left(\frac{A(2)}{\mu(2)} - \frac{A(1)}{\mu(1)}\right).
\end{aligned}
$$

Finally we get the expression of $\Gamma^*\left(\tilde{F}^*, \beta_L^*\right)$ by the same way. This achieves the proof of this theorem. □

4. Mutual influence with the GPI

4.1. Reminder on the GPI. We consider a class of poverty measures called the Generalized Poverty Index (GPI) introduced by *and al.* Lo *et al.* (2006) as an attempt to gather a large class of poverty measures reviewed in Zheng (1997). This class includes the most popular indices such as those of Sen (1976), Kakwani (1980), Shorrocks (1995), Clark *et al.* (1981)), (Foster *et al.* (1984)), etc. See Lo (Lo (2013)) for a review of the GPI.

For the variation of the GPI, we need the functions g_i and ν_i provided by the theorem of the representation of the GPI Lo and Sall (2010). Put accordingly with these functions:

$$g_i(x) = c\left(F_{(2),i}(x)\right) q_i(x) \;\text{ and } \nu_i(s) = c'(s) q_i\left(F_{(2),i}^{-1}(s)\right).$$

We define for all $(u,v) \in [0,1]^2$

$$f_{i,s}(u,v) = \pi_i\left(\mathbf{1}_{(o,s)}(u), \mathbf{1}_{(o,s)}(v)\right),$$
$$F_{i,J}^*(u,v) = g_i \circ \tilde{f}_i(u,v) = g_i \circ F_{(2),i}^{-1} \circ \pi_i(u,v),$$

and

$$F_J^*(u,v) = F_{2,J}^*(u,v) - F_{1,J}^*(u,v).$$

And denote the residual term for the *GPI* by

$$\beta_{(2)}^*(\nu) = \int_{[0,1]} \left(\mathbb{T}_{(2)}\left(f_{2,s}\right)\nu_2(s) - \mathbb{T}_{(2)}\left(f_{1,s}\right)\nu_1(s)\right) ds.$$

THEOREM 27. *Let $\mu(i)$ finite for $i=1,2$. Suppose that $\mathbb{P}_{\left(U^{(1)},U^{(2)}\right)}\left(\left(f_{1,s}\right)^2\right)$, $\mathbb{P}_{\left(U^{(1)},U^{(2)}\right)}\left(\left(f_{2,s}\right)^2\right)$ and $\mathbb{P}_{\left(U^{(1)},U^{(2)}\right)}\left(F_J^{*2}\right)$ are finite.*

Then $\sqrt{n}\left(\Delta J_n(1,2) - \Delta J(1,2)\right)$ converges to $\mathcal{G}_{\Delta GPI} = \mathbb{T}_{(2)}\left(F_J^*\right) + \beta^*(\nu)$ which is a centered Gaussian process of variance-covariance function:

$$\sigma_{\Delta GPI}^2 = \Gamma^*\left(F_J^*, F_J^*\right) + \Gamma^*\left(\beta_\nu^*, \beta_\nu^*\right) + 2\,\Gamma^*\left(F^*, \beta_\nu^*\right)$$

where

$$\Gamma^*\left(F_J^*, F_J^*\right) = \int_{[0,1]^2} \left(F_J^*(u,v) - \mathbb{P}_{\left(U^{(1)},U^{(2)}\right)}\left(F_J^*\right)\right)^2 dC(u,v);$$

$$\Gamma^*\left(\beta_\nu^*, \beta_\nu^*\right) = \gamma_1(\nu_1, \nu_1) - 2\,\gamma_2(\nu_1, \nu_2) + \gamma_1(\nu_2, \nu_2)$$

with the covariance functions $\gamma_1(.,.)$ *and* $\gamma_2(.,.)$ *are respectively defined in Equation (2.5) and Equation (1.25);*
and

$$\begin{aligned}\Gamma^*\left(F^*, \beta_\nu^*\right) &= \int_{[0,1]} \left\{\nu_2(s) \int_{[0,1]\times(0,s)} F_J^*(u,v)\, dC(u,v)\right\} ds \\ &- \int_{[0,1]} \left\{\nu_1(s) \int_{(0,s)\times[0,1]} F_J^*(u,v)\, dC(u,v)\right\} ds \\ &- \mathbb{P}_{\left(U^{(1)},U^{(2)}\right)}\left(F_J^*\right) \int_{[0,1]} s\left(\nu_2(s) - \nu_1(s)\right)\, ds.\end{aligned}$$

PROOF. See Lo and Mergane (2013). □

We are now able to stable our main results.

4.2. Mutual influence. Let

$$R = \frac{\Delta J(1,2)}{\Delta GI(1,2)},\ a = \frac{1}{\Delta GI(1,2)} \text{ and } b = \frac{\Delta J(1,2)}{\left(\Delta GI(1,2)\right)^2}.$$

THEOREM 28. *Supposing that the above mentioned hypotheses are true, then*

$$\left(\sqrt{n}\left(\Delta J_n(1,2) - \Delta J(1,2)\right), \sqrt{n}\left(\Delta GI_n(1,2) - \Delta GI(1,2)\right)\right)^t \to_d \mathcal{N}_2\left(0, \Sigma\right),$$

with

$$\Sigma = \begin{pmatrix} \sigma^2_{\Delta GPI} & \sigma_{\Delta GPI,\Delta GI} \\ & \\ \sigma_{\Delta GPI,\Delta GI} & \sigma^2_{\Delta GI} \end{pmatrix}$$

where

$$\begin{aligned}\sigma_{\Delta GPI,\Delta GI} &= \Gamma^*\left(F_J^*, F^*\right) + \Gamma^*\left(F_J^*, \beta_L^*\right) - \Gamma^*\left(F_J^*, \tilde{F}^*\right) \\ &+ \Gamma^*\left(F^*, \beta_\nu^*\right) + \Gamma^*\left(\beta_L^*, \beta_\nu^*\right) - \Gamma^*\left(\tilde{F}^*, \beta_\nu^*\right)\end{aligned}$$

with

$$\Gamma^*\left(\beta_L^*, \beta_\nu^*\right) = \gamma_1(L_1,\nu_1) + \gamma_1(L_2,\nu_2) - \gamma_2(L_1,\nu_2) - \gamma_2(L_2,\nu_1);$$

$\sigma^2_{\Delta GPI}$ *and* $\sigma^2_{\Delta GI}$ *are given in the previous theorems.*

Further,

$\sqrt{n}\left\{R_n(1,2) - R(1,2)\right\}$ *tends to a functional Gaussian process*

$$a\,\mathcal{G}_{\Delta GPI} \,-\, b\,\mathcal{G}_{\Delta GI}$$

of variance-covariance function

$$\sigma_R^2 = a^2\,\sigma^2_{\Delta GPI} + b^2\,\sigma^2_{\Delta GI} -\, 2\,a\,b\,\sigma_{\Delta GPI,\Delta GI}.$$

Proof. By Theorem 26 and Theorem 27, it is clear that from (van der Vaart and Wellner (1996), p. 81), the bivariate random variable

$$\left(\sqrt{n}\left(\Delta J_n(1,2) - \Delta J(1,2)\right), \sqrt{n}\left(\Delta GI_n(1,2) - \Delta GI(1,2)\right)\right)$$

is asymptotically Gaussian $(\mathcal{G}_{\Delta GPI}, \mathcal{G}_{\Delta GI})$ with

$$\sigma_{\Delta GPI,\Delta GI} = \mathbb{E}\left(\mathcal{G}_{\Delta GPI}\,\mathcal{G}_{\Delta GI}\right).$$

But let's recall that

$$\mathcal{G}_{\Delta GPI} = \mathbb{G}_{(2)}\left(F_J^*\right) + \beta^*(\nu)$$

and

$$\mathcal{G}_{\Delta GI} = \mathbb{G}_{(2)}\left(F^*\right) + \beta^*(L) - \mathbb{G}_{(2)}\left(\tilde{F}^*\right),$$

then

$$\sigma_{\Delta GPI,\Delta GI} = \mathbb{E}\left(\mathcal{G}_{\Delta GPI}\mathcal{G}_{\Delta GI}\right);$$

by expanding this, we find the following expression

$$\begin{aligned}\sigma_{\Delta GPI,\Delta GI} = \Gamma^*\left(F_J^*, F^*\right) + \Gamma^*\left(F_J^*, \beta_L^*\right) - \Gamma^*\left(F_J^*, \tilde{F}^*\right)\\ +\Gamma^*\left(F^*, \beta_\nu^*\right) + \Gamma^*\left(\beta_L^*, \beta_\nu^*\right) - \Gamma^*\left(\tilde{F}^*, \beta_\nu^*\right),\end{aligned}$$

and we can obtain the complete expression for each covariance by using the same procedure as in the theorems 26 and 27. □

5. Final Comments

We have shown that the approach we used here, once set up, leads to powerful asymptotic laws. Besides, the constructions we used allow to couple the results on the Gini index with other results on indices as long as they are expressed in the current frame. We will not need to begin from scratch.

However, the variances and co-variance may not have simple forms. But this is not a major concern in the modern time of powerful computers. For example the variances and co-variance given here may easily be performed with the free R software.

To avoid making the paper lengthy, we decided to prepare and publish, in a very near future, papers devoted to computational methods and simulations in which we will share the codes and papers with focus on data analysis.

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