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CHAPTER 8

Adaptive kernel density estimation of income distribution and poverty index, *by Z. Baradine, Y. Ciss and A. Diakhaby*

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Abstract. In this paper we propose an estimator of Foster, Greer and Thorbecke class of measures indexed by $\alpha > 0$ and depending on a poverty line $z > 0$ nonparameric methods. The estimator is constructed with adaptive kernel. Uniform almost sure consistency and uniform mean square consistency are established.

Keywords. poverty line; adaptive kernel; uniform almost sure consistency; uniform mean square consistency; rate of convergence.
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Résumé. Nous proposons un estimateur de l'indice de pauvreté de Foster, Greer et Thorbecke, défini par : $P(z,\alpha) = \int_0^z \left(\frac{z-x}{z}\right)^\alpha f(x)\,dx$ où $z > 0$ est le seuil de pauvreté, $f(x)$ la densité de la distribution des revenus et $\alpha \in]0,1[$, est le paramètre d'aversion de la pauvreté. L'estimateur est construit à l'aide du noyau adaptatif de Parzen-Rosenblatt. La convergence uniforme presque sûre et la convergence uniforme en erreur quadratique moyenne sont établies. Nous fournisons ensuite une étude de performance de cet estimateur, sur des données simulées, comparativement à l'estimateur provenant du noyau non adaptatif et à l'estimateur empirique. L'étude montre que estimateur à noyau adaptatif est recommandé.

1. Introduction and motivations

Let $F(x)$ be the cumulative distribution function of the income variable X from a population with continuous density $f(x)$ at a given point x. The FGT Foster, Greer and Thorbecke (1984) class of poverty index measures by the real $\alpha \geq 0$ is defined by:

$$P(z,\alpha) = \begin{cases} \int_0^z \left(\frac{z-x}{z}\right)^\alpha f(x)\,dx & \text{if } z > 0, \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

where z is the poverty line.

Let $X_1, \cdots, X_n$ be a random sample size n from income r.v. with F. The following estimator of $P(z,\alpha)$, (see for example Seidl (1988)),

$$\widehat{P}_n(z,\alpha) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{X_i}{z}\right)_+^\alpha \qquad \text{where } x_+ = \max(0,x)$$

is the empirical estimator of FGT poverty index , which is fully useful in a large range of applications in economics (Widely used in practice in econometrics and actuarial). This empirical estimator is unbiased consistent and has a limit normal law $P(z,\alpha)$ mean and variance equal to $n^{-1}(P(z,2\alpha) - (P(z,\alpha))^2)$.

Lo *et al.* (2009) used empirical processes and extreme-values theories to study this estimator.
Seck (2011), Seck and Lo (2009) used some non-weighted poverty measures, viewed as stochastic processes and indexed by real numbers or

monotone functions, to follow up the poverty evolution between two periods.

1.1. Riemann Sum and Kernel type estimator. Dia (2008) proposes new kernels estimators, based on the Riemann sum, for $\alpha = 0$ and $\alpha \geq 1$ and latter Ciss *et al.* (2015) for $\alpha \in]0, 1[$.

Dia (2008) and Ciss *et al.* (2015) consider the classical estimator of the density $f(x)$ (Parzen-Rosenblatt):

$$\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}K\Big(\frac{X_i - x}{h}\Big)$$

where h is a function of n which tends to zero as n tends to infinity and K verifies the following hypotheses:

$$\text{(1.2)} \qquad \mathbf{H_1} \sup_{x\in\mathbb{R}} |K(x)| < +\infty, \quad \mathbf{H_2} \int_{\mathbb{R}} K(x)\,dx = 1, \quad \mathbf{H_3} \lim_{x\to\pm\infty} |xK(x)| = 0.$$

and proposed as estimator of FGT poverty index, the following estimator :

$$\text{(1.3)} \qquad P_n(z,\alpha) = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=0}^{[z/h]}\Big(\frac{z - ih}{z}\Big)^{\alpha} K\Big(\frac{X_j - ih}{h}\Big)\,dx.$$

The kernel method requires a prior choice of the smoothing parameter, also called window width, which is then set at any point where the distribution is estimated. This constraint finds its limits when the concentration of the data is particularly heterogeneous in the sample.

To answer this problem, we consider the adaptive kernel estimator of the density $f(x)$:

$$\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h\lambda_i}K\Big(\frac{X_i - x}{h\lambda_i}\Big).$$

1.2. Definition of the new estimator. An adaptive kernel approach adapts to the sparseness of the data by using a broader kernel over observations located in regions of low density. This is done by varying the bandwidth inversely with the density. As Silverman (1986) puts it on page 101, "An obvious practical problem is deciding in the first place whether or not an observation is in a region of low density".

Further, in this paper, we assume that the hypotheses $\mathbf{H_1}, \mathbf{H_2}, \mathbf{H_3}$ hold, K is Riemann integrable, and that f is bounded with support included in $\mathbb{R}_+$.We denote by x_0 the infinimum of this support.
Let us substitute in (1.1) f by $\hat{f}$. We obtain

$$\tilde{J}_n(z,\alpha) = \int_0^z \Big(\frac{z-x}{z}\Big)^{\alpha} (nh)^{-1} \sum_{i=1}^{n} \frac{1}{\lambda_i} K\Big(\frac{X_i - x}{h\lambda_i}\Big)\, dx. \tag{1.4}$$

Let $\Delta_{h\lambda_j,i} = [\lambda_j hi, \lambda_j h(i+1)[, 0 \leq i < [\frac{z}{h\lambda_j}]$ be a partition of $[0,z]$ and using the Riemann sum definition of the integral, we establish that it correspond to the integral $\tilde{J}_n(z,\alpha)$ of the following sum:

$$\tilde{P}_n^{\lambda}(z,\alpha) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=0}^{[z/h\lambda_j]} \Big(\frac{z - ih\lambda_j}{z}\Big)^{\alpha} K\Big(\frac{X_j - ih\lambda_j}{h\lambda_j}\Big)\, dx + \mathcal{V}_n(z) \tag{1.5}$$

where $\Big[\frac{\cdot}{h\lambda_i}\Big]$ design the integer part of $\frac{\cdot}{h\lambda_i}$ and also for $\alpha = 0$ or $\alpha \geq 1$, we show that $\mathcal{V}_n(z) \to 0$ almost sure as $n \to +\infty$. Then we propose as estimator of FGT poverty index ($\alpha = 0$ or $\alpha \geq 1$), the following estimator :

$$P_n^{\lambda}(z,\alpha) = \frac{1}{n} \sum_{j=1}^{n} \sum_{i=0}^{[z/h\lambda_j]} \Big(\frac{z - ih\lambda_j}{z}\Big)^{\alpha} K\Big(\frac{X_j - ih\lambda_j}{h\lambda_j}\Big)\, dx, \tag{1.6}$$

where λ_i is a parameter that varies according to the local concentration of the data.

An estimate of the density at the point X_i, denoted $\tilde{f}(X_i)$, measures the concentration of the data around this point: a high value indicates a high concentration while a small value indicates a low concentration. The parameter λ_i can therefore be defined as being inversely proportional to this estimate $\tilde{f}(X_i)$:

$$\lambda_i = \Big[\frac{g}{\tilde{f}(X_i)}\Big]^{\beta}$$

where g is the geometric mean of the $\tilde{f}(X_i)$ and β is the sensitivity parameter, a number satisfying $0 \leq \beta \leq 1$.
The parameter λ_i is even smaller than the density is strong (especially in the center of the distribution) and even greater than the density is low (at the ends of the distrbution). the function $\tilde{f}(X_i)$ is called the pilot estimator, which must be calculated in a first step, before the adaptive estimator can be evaluated.

Additional hypotheses are made about the kernel K, that is:
(1) $[\mathbf{H_4}]$ K is of bounded variation function $V_{-\infty}^{u}K$ on $\mathbb{R}$ and let $V(\mathbb{R})$ be its total variation.
(2) $[\mathbf{H_5}]$ $\int_{\mathbb{R}} |uK(u)|du < +\infty$.
(3) $[\mathbf{H_6}]$ There exists a nonincreasing function λ such that $\lambda(\dfrac{u}{h}) = O(h)$ on bounded intervals and for any couple of two reals numbers x and y

$$|K(x) - K(y)| \leq \lambda|x - y| \text{ and } \lambda(u) \longrightarrow 0 \text{ when } u \to 0, \text{and } u \geq 0.$$

Now, we can formulate our main results.

2. Convergence of the estimator

Our main results are relative to the following additional about the function $f(x)$:

$\mathbf{C_1}$: $f(x)$ is uniformly continuous.

$\mathbf{C_2}$: $f(x)$ admits almost everywhere derivative $f^{'}(x) \in L_1(\mathbb{R})$.

2.1. The uniform almost sure consistency and the behavior of the bias.

THEOREM 19. Assume that the hypotheses $\mathbf{H_4}$ and $\mathbf{C_1}$ hold. Then for all $b > 0$, estimator $P_n^{\lambda}(z, \alpha)$ converges uniformly almost surely on $[0, b]$ to $P(z, \alpha)$ as $n \to +\infty$ $i.e.$

$$P\Big(\lim_{n\to+\infty} \sup_{z\in[0,b]} |P_n^{\lambda}(z, \alpha) - P(z, \alpha)| = 0\Big) = 1$$

provided $nh^2(\log \log n)^{-1} \to +\infty$ as $n \to +\infty$.

THEOREM 20. Assume that the hypotheses $\mathbf{H_4}, \mathbf{H_5}$ and $\mathbf{C_2}$ hold. Then for all $b > 0$, the estimator $P_n^{\lambda}(z, \alpha)$ converges uniformly almost surely on $[0, b]$ to $P(z, \alpha)$ as $n \to +\infty$ $i.e.$

$$P\Big(\lim_{n\to+\infty}\sup_{z\in[0,b]}|P_n^\lambda(z,\alpha)-P(z,\alpha)|=0\Big)=1$$

provided that $nh^2(\log\log n)^{-1}\to+\infty$ as $n\to+\infty$.

For the demonstration of the theorems, we use the theorem 2 of Kiefer *et al.* (XXX) and the following lemmas showing that $P_n^\lambda(z,\alpha)$ is uniformly asymptotic unbiased on all bounded interval.

Lemma 12. If the hypothesis $\mathbf{C_1}$ holds , then $\forall b>0$, we have

$$\lim_{n\to+\infty}\sup_{z\in[0,b]}|\mathbb{E}(P_n^\lambda(z,\alpha))-P(z,\alpha)|\to 0,\quad n\to+\infty.$$

Lemma 13. If the hypotheses $\mathbf{H_5}$ and $\mathbf{C_2}$ hold, then:

$$\begin{aligned}\sup_{z\in\mathbb{R}}|\mathbb{E}(P_n^\lambda(z,\alpha))-P(z,\alpha)|&\le h\lambda_j\Bigg(\Big(\int_{\mathbb{R}}|f'(x)|\,dx\Big)\Big(\int_{\mathbb{R}}(|u|+1)|K(u)|\,du\Big)\\&+2(\alpha M+Ah)\int_{-\infty}^{+\infty}|K(u)|\,du\Bigg)\end{aligned}\tag{2.1}$$

where

$$M=\sup_{z\in\mathbb{R}}F(z)\quad\text{and}\quad A=\sup_{x\in\mathbb{R}}f(x).$$

Remark 4. If K satisfies the hypothesis $\mathbf{H_5}$, then by using $\mathbf{H_1}$, the kernel $\widehat{K}=\dfrac{K^2}{\int_{\mathbb{R}}K^2(y)\,dy}$ also satisfy it.

From the two previous lemmas, we get the following corollaries:

Corollary 6. Under the assumptions of lemma 12, we have uniformly on $[0,b]$ (resp $\mathbb{R}$)

$$\lim_{n\to+\infty}\mathbb{E}\left(\sum_{i=1}^{[\frac{z}{h\lambda_j}]}\Big(1-\frac{ih\lambda_j}{z}\Big)^{2\alpha}K^2\Big(\frac{X_j-ih\lambda_j}{h\lambda_j}\Big)\right)=\Big(\int_{\mathbb{R}}K(y)\,dy\Big)P(z,2\alpha)$$

Corollary 7. If the assumptions of theorem 20 hold and if $h=O(n^{-1}\log\log n)^{1/4}$, then for all $b>0$, we have almost surely:

$$\sup_{z\in[0,b]}|\mathbb{E}(P_n^\lambda(z,\alpha))-P(z,\alpha)|=O(n^{-1}\log\log n)^{1/4}.$$

2.2. The uniform mean square consistency.

THEOREM 21. If $\mathbf{H_6}$ and $\mathbf{C_1}$ hold. Then:

(1) $\lim_{n\to+\infty} n\mathbb{V}ar(P_n^{\lambda}(z,\alpha)) = \Big(\int_{\mathbb{R}} K^2(y)\,dy\Big)P(z,2\alpha) - \Big(P(z,\alpha)\Big)^2$.

(2) for all $b>0$,

$$\lim_{n\to+\infty}\sup_{z\in[0,b]}\mathbb{E}\Big(P_n^{\lambda}(z,\alpha)-P(z,\alpha)\Big)^2=0.$$

THEOREM 22. Assume that $\mathbf{H_6}$ and $\mathbf{C_2}$ hold. Then

(1) $\lim_{n\to+\infty} n\mathbb{V}ar(P_n^{\lambda}(z,\alpha)) = \Big(\int_{\mathbb{R}^2} K^2(y)\,dy\Big)P(z,2\alpha) - \Big(P(z,\alpha)\Big)^2$.

(2) moreover if $\mathbf{H_5}$ holds, we have for all $b>0$,

$$\lim_{n\to+\infty}\sup_{z\in[0,b]}\mathbb{E}\Big(P_n^{\lambda}(z,\alpha)-P(z,\alpha)\Big)^2=0.$$

For the proof of this theorem, we assume that the hypothesis $\mathbf{C_1}$ or $\mathbf{C_2}$ holds and before that we prove the theorem 23 below using the

LEMMA 14. Let $0\le\theta_i\le 1, i=1,2$. Then for all x,y and $x\neq y$ we have

$$\lim_{n\to+\infty}\sup_{z\in[0,1]}\Big((h\lambda_j)^{-2}\int_{-\infty}^{+\infty}|K\big(\frac{u-x+\theta_1}{h\lambda_j}\big)K\big(\frac{u-y+\theta_2}{h\lambda_j}\big)|f(u)\,du\Big)=0.$$

THEOREM 23. Assume that the hypothesis $\mathbf{H_6}$ holds. Then for all $b>0$,

$$\lim_{n\to+\infty}\sup_{z\in[0,b]}\sum_{0\le i\neq j\le[\frac{z}{h\lambda_k}]}\Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha}\Big(1-\frac{jh\lambda_k}{z}\Big)^{\alpha}\int_{\mathbb{R}}K\Big(\frac{u-ih\lambda_k}{h\lambda_k}\Big) K\Big(\frac{u-jh\lambda_k}{h\lambda_k}\Big)f(u)\,du=0.$$

REMARK 5. The estimator $P_n^{\lambda}(z,\alpha)$ has asymptotic efficiency with respect to $\widehat{P}_n(z,\alpha)$, $e(z,\alpha)=\Big((\int_{\mathbf{R}}K^2(y)\,dy)P(z,2\alpha)-(P(z,\alpha))^2\Big)/P(z,2\alpha)-(P(z,\alpha))^2$. The integral $\int_{\mathbf{R}}K^2(y)\,dy$ is strictly less than 1 for the conventional kernels (see Parzen (1962) p.1068). Then we have in this case $e(z,\alpha)<1$. In theorem 22, the speed of convergence in mean square is of the order of $O(\frac{1}{n})$ if $h\lambda_k$ is of the order of $O(\frac{1}{\sqrt{n}})$.

3. Simulations

In this section, we make simulations giving the mean square error and variance of 50 samples of size n of the three estimators that we compared. Our adaptive kernel estimator and the classical one are evaluated by a Gaussian kernel checking assumptions $H_i, i = 1, ..., 6$, taking $h = 1/\sqrt{nlogn}$.

For a Pareto distribution type on $[0, 1]$ with parameters $x_0 = 0.02$ and $b = 0.2$, we calculated the mean square error msqe1 of $(P^{\lambda}_{n,1}(z, \alpha), ..., P^{\lambda}_{n,50}(z, \alpha))$, msqe2 of $(P_{n,1}(z, \alpha), ..., P_{n,50}(z, \alpha))$ and msqe3 of $(\widehat{P}_{n,1}(z, \alpha), ..., \widehat{P}_{n,50}(z, \alpha))$ and the respective variances σ_1, σ_2 and σ_3 for different values of z by the following formulas:

$$\overline{P^{\lambda}_n(z, \alpha)} = \frac{1}{50} \sum_{i=1}^{50} P^{\lambda}_{n,i}(z, \alpha),$$

$$msqe1 = \frac{1}{50} \sum_{i=1}^{50} (P^{\lambda}_{n,i}(z, \alpha) - P(z, \alpha))^2,$$

$$\sigma_1 = \frac{1}{50} \sum_{i=1}^{50} (P^{\lambda}_{n,i}(z, \alpha) - \overline{P^{\lambda}_n(z, \alpha)})^2.$$

Similarly, $\overline{P_n(z, \alpha)}$ and $\overline{\widehat{P}_n(z, \alpha)}$, ($msqe2, \sigma_2$) and ($msqe3, \sigma_3$) are respectively calculated for the estimator $P_n(z, \alpha)$ and $\widehat{P}_n(z, \alpha)$.
The studies cases $P(z, 0), P(z, 1), P(z, 2)$ are commonly and respectively called the poverty rate or headcount ratio, the depth of poverty or poverty gap index and the severity of poverty (Foster, Greer and Thorbecke (1984)) .

A comparison of simulation results shows that for small samples, each point z, our adaptive kernel estimator provides a much lower error for the three values of α considered. Thus, we can conclude that our estimator is recommended for small samples.

4. Details of the Proofs

4.1. Construction of the estimator. For $z > 0$ and

$$\Delta_{h\lambda_j,i} = [\lambda_j hi, \lambda_j h(i+1)[\qquad i = 0, \cdots, [\frac{z}{h\lambda_j}]$$

We have the following Riemann sum over the interval $[0, z]$:

z	0.1	0.2	0.4	0.5	0.8
$\alpha = 0 \quad n = 1000$					
mqse1	0.0004314689	0.02100396	0.102017	0.09753305	0.1513109
mqse2	0.1073076	0.1595671	0.2489659	0.2177836	0.2539262
mqse3	0.1082529	0.157403	0.2487593	0.2211496	0.2538144
σ_1	0.0004267454	0.0004228734	0.0003446658	0.0003776458	0.0002897363
σ_2	0.0001819032	0.0002026087	0.0002041675	0.0002699001	0.0002439786
σ_3	0.0002024935	0.0002441559	0.0002028167	0.0001985616	0.0002449996
$\alpha = 1 \quad n = 1000$					
mqse1	1.766574	1.697019	1.657922	1.645952	1.629851
mqse2	2.623803	2.413041	2.20105	2.13593	2.005052
mqse3	2.558946	2.413723	2.201034	2.13585	2.0049
σ_1	0.000241857	0.0001549475	0.0002890191	0.0002380051	0.000218397
σ_2	0.001543131	9.051537e-05	0.0001249713	0.0001305388	0.0001539388
σ_3	0.007770306	9.062371e-05	0.0001251423	0.0001307005	0.0001541145
$\alpha = 2 \quad n = 1000$					
mqse1	1.800745	1.584588	1.47656	1.454531	1.419009
mqse2	2.574427	2.266523	2.015189	1.945325	1.810318
mqse3	2.582422	2.265496	2.015418	1.945211	1.809821
σ_1	0.0002007089	0.0001324068	0.0001766105	0.0002801858	0.0002075677
σ_2	$1.078946e-05$	0.0001383426	9.296009e-05	0.000103433	0.0001234406
σ_3	9.753499e-06	4.783931e-05	0.0001064848	0.0001216999	0.0001535933

TABLE 1. Comparative table of results simulations for small samples

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{i=0}^{[\frac{z}{h\lambda_j}]-1}\Big(1-\frac{ih\lambda_j}{z}\Big)^{\alpha}K\Big(\frac{X_j-ih\lambda_j}{h\lambda_j}\Big)+\Big(z-h\lambda_j[\frac{z}{h\lambda_j}]\Big)$$
$$\sum_{j=1}^{n}\left(1-\frac{h\lambda_j[\frac{z}{h\lambda_j}]}{z}\right)^{\alpha}\frac{1}{n}\frac{1}{h\lambda_j}K\left(\frac{X_j-[\frac{z}{h\lambda_j}]h\lambda_j}{h\lambda_j}\right)$$

corresponding to the integral

$$\int_0^z\Big(\frac{z-x}{z}\Big)^{\alpha}\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h\lambda_j}K\Big(\frac{X_j-x}{h\lambda_j}\Big)\,dx.$$

This sum can be written as

$$\frac{1}{n}\sum_{j=1}^{n}\sum_{i=0}^{[\frac{z}{h\lambda_j}]}\Big(1-\frac{ih\lambda_j}{z}\Big)^{\alpha}K\Big(\frac{X_j-ih\lambda_j}{h\lambda_j}\Big)+\mathcal{V}_n(z)$$

with

$$\mathcal{V}_n(z)=\frac{1}{n}\sum_{j=1}^{n}\frac{\Big(z-h\lambda_j[\frac{z}{h\lambda_j}]\Big)-h\lambda_j}{h\lambda_j}\Big(1-\frac{h\lambda_j[\frac{z}{h\lambda_j}]}{z}\Big)^{\alpha}K\Big(\frac{X_j-[\frac{z}{h\lambda_j}]h\lambda_j}{h\lambda_j}\Big).$$

Now we have to show that

$$\mathcal{V}_n\longrightarrow 0,\qquad \text{almost sure} \quad \text{as} \qquad n\to+\infty$$

we have

$$|\mathcal{V}_n(z)|\leq\begin{cases}\sup_{x\in\mathbb{R}}|K(x)|\sum_{k=1}^{n}\Big(1-\frac{h\lambda_k[\frac{z}{h\lambda_k}]}{z}\Big)^{\alpha} & \text{if} \quad (\alpha>0)\\ \frac{2}{n}\sum_{k=1}^{n}K(\frac{X_k-[\frac{z}{h\lambda_k}]h\lambda_k}{h\lambda_k}) & \text{if} \quad (\alpha=0)\end{cases}\tag{4.1}$$

We obtain for $\alpha>0$,

$$\mathcal{V}_n(z)\longrightarrow 0 \quad p.s. \qquad \text{as} \quad n\to+\infty.$$

For $\alpha=0$, we have also the same result since

$$\begin{aligned}\Big|\mathbb{E}\Big[\frac{1}{n}\sum_{k=1}^{n}K\Big(\frac{X_k-[\frac{z}{h\lambda_k}]h\lambda_k}{h\lambda_k}\Big)\Big]\Big| &= \Big|h\lambda_k\int K(u)f(x)du\Big|\\ & h\lambda_k\sup_{u\in\mathbb{R}}\int K(u)du=O(h\lambda_k).\end{aligned}\tag{4.2}$$

Lemma 15. *Let* $0\leq\theta<1$. *If* f *is uniformly continuous and bounded we have uniformly relatively to* $(\theta,y)\subset[0,1]\times\mathbb{R}$

$$\lim_{n\to+\infty}\int_{\mathbb{R}}(h\lambda_i)^{-1}K(\frac{u}{h\lambda_i})f(u+y-\theta h\lambda_i)\,du=f(y)\int_{\mathbb{R}}K(u)\,du.$$

Let $\varepsilon>0$ then, there exist $\eta>0$ such that $\theta h\lambda_i<\eta$ we have:

$$
\begin{aligned}
&|\int_{\mathbb{R}}(h\lambda_i)^{-1}K(\frac{u}{h\lambda_i})f(u+y-\theta h\lambda_i)\,du - \int_{\mathbb{R}}(h\lambda_i)^{-1}K(\frac{u}{h\lambda_i})f(u+y)\,du| \\
&\leq \int_{\mathbb{R}}(h\lambda_i)^{-1}|K(\frac{u}{h\lambda_i})||f(u+y-\theta h\lambda_i)-f(u+y)|\,du \leq \varepsilon \int_{\mathbb{R}}(h\lambda_i)^{-1}|K(\frac{u}{h\lambda_i})|\,du
\end{aligned}
$$

For all y; choose then n large enough such that $h\lambda_i < \eta$ to get the inequality for all (θ, y).

Since

$$
\lim_{n\to+\infty}\int_{\mathbb{R}}(h\lambda_i)^{-1}K(\frac{u}{h\lambda_i})f(u+y)\,du = f(y)\int_{\mathbb{R}}K(u)\,du.
$$

uniformly by the theorem (1) of Parzen (1962), the proof of the lemma is complete.

4.2. Convergence of the estimator.

Proof of lemma 12. Let's firstly observe that $F(z)$ is bounded by definition.

Let $\bar{\Delta}_{h\lambda_k,i} = \Delta_{h\lambda_k,i} \cap [0,z]$; and χ_B the indicator function of the set B. Let $z \in [0,b]$.

We have

$$
\mathbb{E}(P_n^{\lambda}(z,\alpha)) = \sum_{i=1}^{[\frac{z}{h\lambda_k}]}\left(1-\frac{ih\lambda_k}{z}\right)^{\alpha}\int_{\mathbb{R}}K(u)f(uh\lambda_k - ih\lambda_k)\,du
$$

which can be written in the following form

$$
\begin{aligned}
&\int_0^z \sum_{i=1}^{[\frac{z}{h\lambda_k}]}\chi_{\bar{\Delta}_{h\lambda_k,i}}(x)\left(1-\frac{ih\lambda_k}{z}\right)^{\alpha}\int_{-\infty}^{+\infty}K(u)f(uh\lambda_k - ih\lambda_k)\,dudx \\
&+ (h\lambda_k([\frac{z}{h\lambda_k}]+1)-z)\left(1-\frac{h\lambda_k[\frac{z}{h\lambda_k}]}{z}\right)^{\alpha}\int_{-\infty}^{+\infty}K(u)f(uh\lambda_k+\frac{h\lambda_k[\frac{z}{h\lambda_k}]}{z})\,du
\end{aligned}
\tag{4.3}
$$

We have

(4.4)

$$\sup_{(z)\in\mathbb{R}} |(h\lambda_k([\frac{z}{h\lambda_k}]+1)-z)\Big(1-\frac{h\lambda_k[\frac{z}{h\lambda_k}]}{z}\Big)^\alpha \int_{-\infty}^{+\infty} K(u)f(uh\lambda_k+\frac{h\lambda_k[\frac{z}{h\lambda_k}]}{z})\,du| \leq h\lambda_k \sup_{x\in\mathbb{R}} f(x)\int_{-\infty}^{+\infty}|K(u)|\,du$$

Since we have $|h\lambda_i([\frac{z}{h\lambda_i}]+1)-z| \leq h\lambda_i$.

Because of assumption $\mathbf{H_2}$, we can write

$$P(z,\alpha)=\int_0^z \Big(1-\frac{x}{z}\Big)^\alpha K(u)\,du f(x)\,dx \quad (4.5)$$

Let $x\in\bar{\Delta}_{h\lambda_k,i}$. By considering the terms in (4.3) et de (4.5) we get

$$\begin{aligned}
&|\Big(1-\frac{ih\lambda_k}{z}\Big)^\alpha \int_{-\infty}^{+\infty} K(u)f(uh\lambda_k-ih\lambda_k)\,du-\Big(1-\frac{x}{z}\Big)^\alpha K(u)\,du f(x)| \\
&= |\int_{-\infty}^{+\infty}\Big[\Big(1-\frac{ih\lambda_k}{z}\Big)^\alpha f(uh\lambda_k-ih\lambda_k)-\Big(1-\frac{x}{z}\Big)^\alpha f(x)\Big]K(u)\,du| \\
&\leq \int_{-\infty}^{+\infty}|\Big(1-\frac{ih\lambda_k}{z}\Big)^\alpha-\Big(1-\frac{x}{z}\Big)^\alpha|f(x)|K(u)|\,du \\
&+\int_{-\infty}^{+\infty}|\Big(1-\frac{ih\lambda_k}{z}\Big)^\alpha||f(uh\lambda_k-ih\lambda_k)-f(x)||K(u)|\,du
\end{aligned} \quad (4.6)$$

Let $x\in\Delta_{h\lambda_k,i}$, we have by the first order Taylor formula applied to the fonction

$$g(x)=\Big(1-\frac{x}{z}\Big)^\alpha,$$

for $c\in]h\lambda_i i,x[$,

$$\begin{aligned}
\Big|\Big(1-\frac{ih\lambda_k}{z}\Big)^\alpha-\Big(1-\frac{x}{z}\Big)^\alpha\Big| &= \Big|\Big(1-\frac{c}{z}\Big)^\alpha(ih\lambda_k-x)\Big| \\
&\leq 2\frac{\alpha h\lambda_k}{z}.
\end{aligned}$$

Therefore, denoting by $I_1^i(x)$ the first integral of the right hand-side of (4.6) and

$$I_1(x) = \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \chi_{\bar{\Delta}_{h\lambda_k,i}}(x) I_1^i(x)$$

We have

(4.7)
$$\int_0^z I_1(x)\,dx \le 2\frac{\alpha h\lambda_i}{z}\int_0^z (\int_{-\infty}^{+\infty} f(x)|K(u)|\,du)\,dx = 2\frac{\alpha h\lambda_i}{z}(\int_{-\infty}^{+\infty} |K(u)|\,du)F(z)$$

Denoting by $I_2^i(x)$ the second integral of the right hand-side of (4.6) and

$$I_2(x) = \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \chi_{\bar{\Delta}_{h\lambda_k,i}}(x) I_2^i(x)$$

We get

$$I_2^i(x)2\frac{\alpha h\lambda_k}{z} \le \Bigg(\int_{-\infty}^{+\infty} |f(uh\lambda_k - ih\lambda_k) - f(ih\lambda_k)||K(u)|\,du + \int_{-\infty}^{+\infty} |f(ih\lambda_k) - f(x)||K(u)|\,du\Bigg) \quad (4.8)$$

Let $\varepsilon > 0$, since $f(x)$, uniformly continuous, there exists $\eta_0 = \eta_0(z) > 0$ such that $|ih\lambda_k - x| \le \eta_0$ hence if $h\lambda_k \le \eta_0$, we have $|f(uh\lambda_k - ih\lambda_k) - f(x)| < \dfrac{\varepsilon}{b}$. Therefore,

$$\int_0^z I_2(x)\,dx \le 2\frac{\alpha h\lambda_k}{z}(\sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\lambda_k \int_{-\infty}^{+\infty} |f(uh\lambda_k - ih\lambda_k) - f(ih\lambda_k)||K(u)|\,du + \varepsilon\int_{-\infty}^{+\infty} |K(u)|\,dudx).$$

By the uniform continuity of $f(x)$ we have

$$\exists \eta_1 = \eta_1(z) > 0, |uh\lambda_k| < \eta_1 \Rightarrow |f(uh\lambda_k - ih\lambda_k) - f(ih\lambda_k)||K(u)| < \frac{\varepsilon}{b}.$$

Hence

$$\begin{aligned}
&\sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\lambda_k \int_{-\infty}^{+\infty} |f(uh\lambda_k - ih\lambda_k) - f(ih\lambda_k)||K(u)|\,du \\
&\leq \sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\lambda_k \int_{|uh\lambda_k|<\eta_1} \frac{\varepsilon}{b}|K(u)|\,du \\
&\quad + \sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\lambda_k \int_{|uh\lambda_k|\geq\eta_1} |f(uh\lambda_k - ih\lambda_k) - f(ih\lambda_k)||K(u)|\,du \\
&\leq \varepsilon \int_{-\infty}^{+\infty} |K(u)|\,du + \sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\lambda_k \int_{|uh\lambda_k|\geq\eta_1} |f(uh\lambda_k - ih\lambda_k)||K(u)|\,du \\
&\quad + \sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\lambda_k \int_{|uh\lambda_k|\geq\eta_1} |f(ih\lambda_k)||K(u)|\,du
\end{aligned}$$

Since $f(x)$ is continuous, it is Riemann-integrable, hence

$$\sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\lambda_k|f(ih\lambda_k)| \to \int_0^b f(x)\,dx \qquad n \to +\infty$$

because

$$(h\lambda_k([\frac{b}{h\lambda_k}]+1) - b)f(h\lambda_k[\frac{b}{h\lambda_k}]) \to 0, \quad n \to +\infty.$$

Therefore the sum

$$\sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\lambda_k|f(ih\lambda_k)| \to \int_0^b f(x)\,dx$$

is bounded. Let A be the latter.

By the change of variable $v = uh\lambda_i$, we have

$$\sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\lambda_k \int_{|uh\lambda_k|\geq\eta_1} |f(uh\lambda_k - ih\lambda_k)||K(u)|\,du$$
$$+\sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\int_{|uh\lambda_k|\geq\eta_1} |f(ih\lambda_k)||K(u)|\,du$$
$$\leq \sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\lambda_k \int_{|v|\geq\eta_1} |f(v - ih\lambda_k)|\frac{1}{h\lambda_k}|K(\frac{v}{h\lambda_k})|\,dv + A\int_{|uh\lambda_k|\geq\eta_1} |K(u)|\,du$$

Because $(\mathbf{H_3})$ there exists $C > 0$ fixed, such that $\left|\frac{v}{h\lambda_k}\right| \geq C$ we have

$$\left|\frac{v}{h\lambda_k}\right|\left|K(\frac{v}{h\lambda_k})\right| \leq \frac{\eta_1\varepsilon}{b}.$$

Let $\eta = \inf(\eta_1, Ch) = Ch\lambda_k, h\lambda_k$ being small enough, then

$$\frac{1}{\eta_1}\int_{|v|\geq\eta_1} |f(v - ih\lambda_k)|\frac{v}{h\lambda_k}|K(\frac{v}{h\lambda_k})|\,dv \leq \frac{\varepsilon}{b}\int_{|v|\geq\eta_1} |f(v - ih\lambda_k)|\,dv$$
$$\leq \frac{\varepsilon}{b}\int_{\mathbb{R}} |f(x)|\,dx = \frac{\varepsilon}{b}$$

Hence

$$\sum_{i=1}^{[\frac{z}{h\lambda_k}]} h\lambda_k \int_{|v|\geq\eta_1} |f(v - ih\lambda_k)|\frac{1}{h\lambda_k}|K(\frac{v}{h\lambda_k})|\,dv \leq \frac{z\varepsilon}{b} \leq \varepsilon.$$

Since $A\int_{|uh\lambda_k|\geq\eta_1} |K(u)|\,du \to 0, n \to +\infty$, we have together with (4.7) and (4.4)

$$\lim_{n\to+\infty} \sup_{z\in[0,b]} |\mathbb{E}(P_n^\lambda(z,\alpha)) - P(z,\alpha)| \leq 2\varepsilon \int_{-\infty}^{+\infty} K(u)\,du + \varepsilon.$$

The proof of the lemma is complete.

Proof of lemma 13. For $x \in \bar{\Delta}_{h\lambda_k,i} \qquad i = 1, \cdots, [\frac{z}{h\lambda_k}]$ we have

$$\int_{-\infty}^{+\infty} \left|\left(1 - \frac{ih\lambda_k}{z}\right)^{\alpha}\right| |f(uh\lambda_k + ih\lambda_k) - f(x)||K(u)|\,du$$
$$\leq \left(\int_{-\infty}^{+\infty} \int_{x}^{x+h\lambda_k(|u|+1)} |f^{'}(t)||K(u)|\,du \right)$$

Hence $I_2(x)$ as defined in the previous lemma 12, we have

$$\int_0^z I_2(x)\,dx \leq \int_0^z \Big(\int_{-\infty}^{+\infty} \int_{x}^{x+h\lambda_k(|u|+1)} |f^{'}(t)||K(u)|\,du\Big)\,dx. \tag{4.9}$$

By the change of variable $t = x + h\lambda_k(|u|+1)v$ we have by Fubini's theorem

$$\int_0^z I_2(x)\,dx$$
$$\leq h\lambda_k \int_{\mathbb{R}} (|u|+1)|K(u)| \Big(\int_{\mathbb{R}} |f^{'}(x + h\lambda_i(|u|+1)v)||K(u)|\,du\Big)\,dx\,du \int_0^1 dv$$
$$= \int_{\mathbb{R}} |f^{'}(x)|\,dx.$$

This inequality together with (4.4) and (4.7) lead to completion of the proof.

Proof of theorem 19 and theorem 20. Let $\widehat{F}_n$ be the empirical distribution of the sample $(X_1, X_2, \cdot, \cdot, \cdot, X_n)$ defined by

$$F_n(l) = n^{-1} \sum_{i=1}^{n} (\chi_{X_i < l})$$

where χ_A stands for the indicator function A. We can write

$$P_n^{\lambda}(z,\alpha) = \int_{\mathbb{R}} \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \Big(1 - \frac{ih\lambda_k}{z}\Big)^{\alpha} K\Big(\frac{l - ih\lambda_k}{h\lambda_k}\Big)\,d\hat{F}_n(l)$$

and

$$\mathbb{E}(P_n^{\lambda}(z,\alpha)) = \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \int_{\mathbb{R}} \Big(1 - \frac{ih\lambda_k}{z}\Big)^{\alpha} K\Big(\frac{l - ih\lambda_k}{h\lambda_k}\Big)\,dF(l).$$

We have

$$|P_n^{\lambda}(z,\alpha) - \mathbb{E}(P_n^{\lambda}(z,\alpha))| = \left| \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \int_{\mathbb{R}} \Big(1 - \frac{ih\lambda_k}{z}\Big)^{\alpha} K\Big(\frac{l - ih\lambda_k}{h\lambda_k}\Big) \left(d\hat{F}_n(l) - dF(l)\right) \right|.$$

The integration by parts yields

$$\begin{aligned} |P_n^{\lambda}(z,\alpha) - \mathbb{E}(P_n^{\lambda}(z,\alpha))| &\leq \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \int_{\mathbb{R}} \left| dK\Big(\frac{l - ih\lambda_k}{h\lambda_k}\Big)\right| \sup_{l\in\mathbb{R}} |\hat{F}_n(l) - F(l)| \\ &\leq \sum_{i=1}^{[\frac{z}{h\lambda_k}]} \int_{\mathbb{R}} dK_{-\infty}^{\frac{l-ih\lambda_k}{h\lambda_k}} \sup_{l\in\mathbb{R}} |\hat{F}_n(l) - F(l)| \\ &\leq \Big[\frac{z}{h\lambda_k}\Big] V(\mathbb{R}) \sup_{l\in\mathbb{R}} |\hat{F}_n(l) - F(l)| \end{aligned}$$

Remarking that

$$\begin{aligned} |P_n^{\lambda}(z,\alpha) - (P(z,\alpha))| &= |P_n^{\lambda}(z,\alpha) - \mathbb{E}(P_n^{\lambda}(z,\alpha)) + \mathbb{E}(P_n^{\lambda}(z,\alpha)) - P(z,\alpha)| \\ &\leq |P_n^{\lambda}(z,\alpha) - \mathbb{E}(P_n^{\lambda}(z,\alpha))| + |\mathbb{E}(P_n^{\lambda}(z,\alpha)) - P(z,\alpha)| \end{aligned}$$

By lemma 12 we have

$$|\mathbb{E}(P_n^{\lambda}(z,\alpha)) - P(z,\alpha)| \to 0 \qquad n \to +\infty$$

and the previous results we have

$$|P_n^{\lambda}(z,\alpha) - \mathbb{E}(P_n^{\lambda}(z,\alpha))| \to 0 \qquad n \to +\infty$$

4.3. The uniform mean square consistency.

Proof lemma 14. We suppose that C_1 holds. Let $\delta > 0$.

Define

$$I_n(x,y) = (h\lambda_k)^{-2}\int_{-\infty}^{+\infty} |K(\frac{u-x+\theta_1 h\lambda_k}{h\lambda_k})K(\frac{u-y+\theta_2 h\lambda_k}{h\lambda_k})|f(u)\,du$$
$$= \int_{-\infty}^{+\infty} ((h\lambda_k)^{-1}K(\frac{v}{h\lambda_k}))|((h\lambda_k)^{-1}K(\frac{v+x-\theta_1-y+\theta_2 h\lambda_k}{h\lambda_k})|f(x+v-\theta_1 h\lambda_k)\,du$$
$$= \int_{|v-\theta_1 h\lambda_k|\leq\delta} + \int_{|v-\theta_1 h\lambda_k|>\delta}$$

Since $f(x)$ is continuous, it is bounded on $I=[x-\delta, x+\delta]$. We assume n large enough such that $x+v\pm\theta_1 h\lambda_k \in I$. Therefore

(4.10)
$$\int_{|v-\theta_1 h\lambda_k|\leq\delta} \leq \sup_{|v-\theta_1 h\lambda_k|\leq\delta} f(x+v-\theta_1 h\lambda_k)\int_{-\frac{\delta}{h\lambda_k}+\theta_1\leq u\leq\frac{\delta}{h\lambda_k}+\theta} |K(u)|$$
$$\times |K(\frac{x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)|(h\lambda_k)^{-1}\,du$$

$$= \sup_{|v-\theta_1 h\lambda_k|\leq\delta} f(x+v-\theta_1 h\lambda_k)\int_{-\infty}^{+\infty} \chi_{-\frac{\delta}{h\lambda_k}+\theta_1\leq u\leq\frac{\delta}{h\lambda_k}+\theta_1}(u)|K(u)|$$
$$\times |K(\frac{x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)(h\lambda_k)^{-1}|\,du$$

For every u

$$\lim_{n\to+\infty} |K(\frac{x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)(h\lambda_k)^{-1}| = 0.$$

Write

$$\Big|K(\frac{x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)(h\lambda_k)^{-1}\Big| =$$
$$\Big|(\frac{x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k}{h\lambda_k}+u)K(\frac{x-\theta_1 - y+\theta_2 h\lambda_k}{h\lambda_k}+u)\Big| *$$
$$\Big|\frac{1}{x-\theta_1 h\lambda_k - y+\theta_2 h\lambda_k + h\lambda_k u}\Big|.$$

We have

$$\Big|\frac{1}{x-\theta_1 h\lambda_k - y + \theta_2 h\lambda_k + h\lambda_k u}\Big| = \frac{1}{|x-y||1-\frac{\theta_1-\theta_2-u}{x-y}h\lambda_k|}.$$

Since $|u| \leq \frac{\delta}{h\lambda_k} + \theta_1$ we may choose δ small enough such that for $n \geq n_0$ we have

$$\Big|\frac{\theta_1-\theta_2-u}{x-y}h\lambda_k\Big| \leq \frac{3h\lambda_k+\delta}{|x-y|} = \eta < 1.$$

Therefore

$$\Big|\frac{1}{x-\theta_1 h\lambda_k - y + \theta_2 h\lambda_k + h\lambda_k u}\Big| \leq \frac{1}{|x-y|(1-\eta)} \tag{4.11}$$

since $\mathbf{H_3}$ implies there exists B such that

$$\Big|(\frac{x-\theta_1 h\lambda_k - y + \theta_2 h\lambda_k}{h\lambda_k} + u)K(\frac{x-\theta_1 h\lambda_k - y + \theta_2 h\lambda_k}{h\lambda_k} + u)| \leq B$$

Then, we have

$$\Big|K(\frac{x-\theta_1 h\lambda_k - y + \theta_2 h\lambda_k}{h\lambda_k} + u)(h\lambda_k)^{-1}\Big| \leq \frac{B}{|x-y|(1-\eta)}$$

$|K(u)|$ being integrable, by dominated convergence

$$\int_{|v-\theta_1 h\lambda_k|\leq\delta} \to 0 \quad \text{as} \quad n \to +\infty.$$

Let $\int_{|v-\theta_1 h\lambda_k|>\delta}$ write it in the form

$$\int_{|v-\theta_1 h\lambda_k|>\delta} = \int_{|v-\theta_1 h\lambda_k|>\delta} \Big|v(h\lambda_k)^{-1}K(\frac{v}{h\lambda_k})((h\lambda_k)^{-1} \times K(\frac{v+x-\theta_1-y+\theta_2 h\lambda_k}{h\lambda_k}))\frac{f(x+v-\theta_1 h\lambda_k)}{v}\Big|\,dv.$$

We get

$$\int_{|v-\theta_1 h\lambda_k|>\delta} \leq \frac{2}{\delta-\theta_1 h\lambda_k} \sup_{|v-\theta_1 h\lambda_k|>\delta} \Big|\frac{v}{h\lambda_k}K(\frac{v}{h\lambda_k})\Big| \int_{|v-\theta_1 h\lambda_k|>\delta} ((h\lambda_k)^{-1} \times K(\frac{v+x-\theta_1-y+\theta_2 h\lambda_k}{h\lambda_k}))|f(x+v-\theta_1 h\lambda_k)|\,dv. \tag{4.12}$$

Let the change of variable defined by

$$v + x - \theta_1 h\lambda_k - y + \theta_2 h\lambda_k = u.$$

Then

$$\int_{|v-\theta_1 h\lambda_k|>\delta} \leq \frac{2}{\delta - \theta_1 h\lambda_k} \sup_{|v-\theta_1 h\lambda_k|>\delta} \left| \frac{v}{h\lambda_k} K(\frac{v}{h\lambda_k}) \right| \int_{\mathbb{R}} |(h\lambda_k)^{-1} K(\frac{u}{h\lambda_k})| \times f(u + y - \theta_2 h\lambda_k)\, du. \tag{4.13}$$

Lemma 12 (replacing K by $|K|$) and $(\mathbf{H_3})$ we have

$$|\int_{|v-\theta_1 h\lambda_k|>\delta}| \to 0 \quad \text{as} \quad n \to +\infty$$

and the convergence is uniform.

Thus, the proof of lemma 14.

Remark 6. If condition $\mathbf{C_2}$ is verified, then the integral of the right hand-side of (4.13) becomes

$$\begin{aligned}
\int_{\mathbb{R}} |(h\lambda_k)^{-1} K(\frac{u}{h\lambda_k})||f(u + y - \theta_2 h\lambda_k) - f(\frac{u}{h\lambda_k}) + f(\frac{u}{h\lambda_k})|du \\
&\leq \int_{\mathbb{R}} |(h\lambda_k)^{-1} K(\frac{u}{h\lambda_k})| \int_{\frac{u}{h\lambda_k}}^{u+y-\theta_2 h\lambda_k} |f'(t)|\, dt du \\
&\quad + \int_{\mathbb{R}} f(\frac{u}{h\lambda_k})|(h\lambda_k)^{-1} K(\frac{u}{h\lambda_k})|\, du \\
&\leq \int_{\mathbb{R}} |(h\lambda_k)^{-1} K(\frac{u}{h\lambda_k})|\, du \int_{\mathbb{R}} |f'(t)|\, dt \\
&+ \int_{\mathbb{R}} |K(u)| f(u)\, du.
\end{aligned}$$

The integrals of the right hand-side of this last inequality. Hence the theorem is valid under the hypothesis $\mathbf{C_2}$.

Proof of theorem 23. We suppose condition $\mathbf{C}_1$ verified. Let $\Delta = [0, z] \times [0, z]$. We can write

$$\sum_{0 \leq i \neq j \leq [\frac{z}{h\lambda_k}]} \Big(1 - \frac{ih\lambda_k}{z}\Big)^{\alpha} \Big(1 - \frac{jh\lambda_k}{z}\Big)^{\alpha} \int_{\mathbb{R}} |K(\frac{u - ih\lambda_k}{h\lambda_k})K(\frac{u - jh\lambda_k}{h\lambda_k})| f(u)\, du$$
$$= \int_{\{(x,y)\in\Delta : |x-y|>0\}} \Phi_n(x, y)\, dxdy$$

where

$$\Phi_n(x, y) = \frac{1}{(h\lambda_k)^2} \sum_{0 \leq i \neq j \leq [\frac{z}{h\lambda_k}]} \chi_{\Delta_{h\lambda_k,i} \times \Delta_{h\lambda_k,j}}(x, y) \Big(1 - \frac{ih\lambda_k}{z}\Big)^{\alpha} \Big(1 - \frac{jh\lambda_k}{z}\Big)^{\alpha}$$
$$\times \int_{\mathbb{R}} |K(\frac{u - ih\lambda_k}{h\lambda_k})K(\frac{u - jh\lambda_k}{h\lambda_k})| f(u)\, du.$$

If $(x, y) \in \Delta_{h\lambda_k,i} \times \Delta_{h\lambda_k,j} \quad i \neq j$ with the representation

$$x = h\lambda_k i + \theta_1 h\lambda_k, \quad y = h\lambda_k j + \theta_2 h\lambda_k \quad 0 \leq \theta_l < 1, \quad l = 1, 2$$

We have

(4.14)
$$\frac{1}{(h\lambda_k)^2} \Big(1 - \frac{ih\lambda_k}{z}\Big)^{\alpha} \Big(1 - \frac{jh\lambda_k}{z}\Big)^{\alpha} \int_{\mathbb{R}} |K(\frac{u - x + \theta_1 h\lambda_k}{h\lambda_k})K(\frac{u - y + \theta_2 h\lambda_k}{h\lambda_k})| f(u)\, du$$
$$\leq \frac{1}{(h\lambda_k)^2} \int_{\mathbb{R}} |K(\frac{u - x + \theta_1 h\lambda_k}{h\lambda_k})K(\frac{u - y + \theta_2 h\lambda_k}{h\lambda_k})| f(u)\, du$$

The right hand-side of (4.14) tends to zero as $n \to +\infty$ by lemma 14.

Let $\delta = \frac{z}{2}$. Write

$$\frac{1}{(h\lambda_k)^2} \int_{\mathbb{R}} |K(\frac{u - x + \theta_1 h\lambda_k}{h\lambda_k})K(\frac{u - y + \theta_2 h\lambda_k}{h\lambda_k})| f(u)\, du = \int_{|v| \leq \delta} + \int_{|v| > \delta}.$$

Then, we have

$$\int_{\{(x,y)\in\Delta:|x-y|>0\}} \Phi_n(x,y)\,dxdy$$
$$\leq \int_{\{(x,y)\in\Delta:|x-y|>0\}} \sum_{0\leq i\neq j\leq [\frac{z}{h\lambda_k}]} \chi_{\Delta_{h\lambda_k,i}\times\Delta_{h\lambda_k,j}}(x,y)\Big(\int_{|v|\leq\delta} + \int_{|v|>\delta}\Big).$$

The proof of the remainder is conducted as follow:

First consider

$$\int_{\{(x,y)\in\Delta:|x-y|>0\}} \int_{|v|\leq\delta}.$$

Let $A = \sup_{x\in[0,z]} f(x)$. The notations being as in the proof lemma 13 with $\delta = \frac{z}{2}$, we have in accordance with inequality (4.10)

$$\int_{|v|\leq\delta} \leq A\int_{-\infty}^{+\infty} \chi_{-\frac{\delta}{h\lambda_k}\leq u\leq\frac{\delta}{h\lambda_k}} |K(u)||K(\frac{x-\theta_1 h\lambda_k - y + \theta_2 h\lambda_k}{h\lambda_k} + u)(h\lambda_k)^{-1}|\,du.$$

For every u

$$\lim_{n\to+\infty} \Big|K(\frac{x-\theta_1 h\lambda_k - y + \theta_2 h\lambda_k}{h\lambda_k} + u)(h\lambda_k)^{-1}\Big| = 0.$$

We have

$$\Big|K(\frac{x-\theta_1 h\lambda_k - y + \theta_2 h\lambda_k}{h\lambda_k} + u)(h\lambda_k)^{-1}\Big| = \Big|K(\frac{x-\theta_1 h\lambda_k - y + \theta_2 h\lambda_k}{h\lambda_k} + u)$$
$$- K(\frac{2z + x - \theta_1 - y + \theta_2 h\lambda_k}{h\lambda_k} + u) + K(\frac{2z + x - \theta_1 - y + \theta_2 h\lambda_k}{h\lambda_k} + u)\Big|(h\lambda_k)^{-1}$$
$$\leq \left(\lambda(\frac{2z}{h\lambda_k}) + \Big|K(\frac{2z + x - \theta_1 h\lambda_k - y + \theta_2 h\lambda_k}{h\lambda_k} + u)\Big|\right)(h\lambda_k)^{-1}.$$

Moreover,

$$\Big|K(\frac{2z+x-\theta_1-y+\theta_2 h\lambda_k}{h\lambda_k}+u)\Big|(h\lambda_k)^{-1}$$
$$= \Big|\frac{2z+x-y+h\lambda_k u}{h\lambda_k}\Big|\Big|K(\frac{2z+x-\theta_1-y+\theta_2 h\lambda_k}{h\lambda_k}+u\Big|\frac{1}{|2z+x-y+h\lambda_k u|}$$

Let $B=\sup_{y\in\mathbb{R}}|y||K(y)|$ and $C=\sup_{y\in\mathbb{R}}|K(y)|$, then we have

$$\Big|\frac{2z+x-y+h\lambda_k u}{h\lambda_k}\Big|\Big|K(\frac{2z+x-\theta_1-y+\theta_2 h\lambda_k}{h\lambda_k}+u)\Big|\leq B+2h\lambda_k C.$$

Therefore, we have

$$\Big|K(\frac{2z+x-\theta_1-y+\theta_2 h\lambda_k}{h\lambda_k}+u)\Big|(h\lambda_k)^{-1}\leq\frac{B+2h\lambda_k C}{|2z+x-y+h\lambda_k u|}$$

Hence

$$\Big|K(\frac{x-\theta_1 h\lambda_k-y+\theta_2 h\lambda_k}{h\lambda_k}+u)(h\lambda_k)^{-1}\Big|\leq\lambda(\frac{2z}{h\lambda_k})+\frac{B+2h\lambda_k C}{|2z+x-y+h\lambda_k u|}$$

We conclude that for $h\lambda_k$ small enough

$$\begin{aligned}\int_{|v|\leq\delta} &\leq \frac{A}{|2z+x-y+h\lambda_k u|}\int_{\mathbb{R}}|K(u)|(B+2h\lambda_k C)\,du\\ &< \frac{AD}{|2z+x-y+h\lambda_k u|}\\ &\leq \frac{AD}{|2z+x-y+h\lambda_k u|}\\ &\leq \frac{AD}{(2z+x-y+h\lambda_k u)}\end{aligned}$$

D being the finite bound of $\int_{\mathbb{R}}|K(u)|(B+2C)du$.

Finally, we have

$$\int_{|v|\leq\delta}\leq\frac{AD}{(2z+x-y+h\lambda_k u)}+O(h\lambda_k).$$

Since $-\delta \leq h\lambda_k u \leq \delta$ we have $\frac{z}{2} \leq 2z + x - y + h\lambda_k u \leq \frac{7z}{2}$.

Hence

$$\int_{|v|\leq\delta} \leq \frac{2AD}{z} + O(h\lambda_k).$$

Therefore, by Lebesgue-dominated convergence, we have

$$\lim_{n\to+\infty} \int\int_{\Delta} \Big(\int_{|v|\leq\delta}\Big)\, dxdy = \int\int_{\Delta} \lim_{n\to+\infty} \Big(\int_{|v|\leq\delta}\Big)\, dxdy = 0 \tag{4.15}$$

Consider then $\int_{|v|>\delta}$.

We use the second part, by analogous reasoning, of the proof of lemma 13

$$\int_{|v|>\delta} \leq \frac{2}{\delta} \sup_{|v|>\delta} \Big|\frac{v}{h\lambda_k}\Big|\Big|K(\frac{v}{h\lambda_k})\Big| \int_{\mathbb{R}} \Big|(h\lambda_k)^{-1}K(\frac{u}{h\lambda_k})\Big| f(u+y-\theta_2 h\lambda_k)\, du. \tag{4.16}$$

We have

$$\int_{|v|>\delta} \to 0, \qquad n\to+\infty \quad \text{uniformly}.$$

Hence

$$\lim_{n\to+\infty} \int_{\Delta}\int_{\mathbb{R}} \to 0, \quad n\to+\infty$$

since Δ is bounded. The proof of the lemma is complete.

REMARK 7. *If C_2 is verified, then the theorem is a gain valid.*

Indeed it suffices to apply remark 6 to inequality (4.16).

Proof of Theorem 21. . We suppose condition (C_1) satisfied.

$$\begin{aligned} n\mathbb{V}ar(P_n^{\lambda}(z,\alpha)) &= \mathbb{E}\Bigg(\sum_{i=0}^{[\frac{z}{h\lambda_k}]} \Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha} K\Big(\frac{X_j - ih\lambda_k}{h\lambda_k}\Big)\Bigg)^2 \\ &\quad - \mathbb{E}^2\Bigg(\sum_{i=0}^{[\frac{z}{h\lambda_k}]} \Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha} K\Big(\frac{X_j - ih\lambda_k}{h\lambda_k}\Big)\Bigg) \end{aligned} \tag{4.17}$$

Since

$$
\begin{aligned}
&\mathbb{E}\Bigg(\sum_{i=0}^{[\frac{z}{h\lambda_k}]}\Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha}K\Big(\frac{X_j-ih\lambda_k}{h\lambda_k}\Big)\Bigg)^2\\
&=\mathbb{E}\Bigg[\Bigg(\sum_{i=0}^{[\frac{z}{h\lambda_k}]}\Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha}K\Big(\frac{X_j-ih\lambda_k}{h\lambda_k}\Big)\Bigg)^2\Bigg]\\
&=\mathbb{E}\Bigg[\Bigg\{\sum_{i=0}^{[\frac{z}{h\lambda_k}]}\Big(1-\frac{ih\lambda_k}{z}\Big)^{2\alpha}K^2\Big(\frac{X_k-ih\lambda_k}{h\lambda_k}\Big)\\
&+\sum_{i\neq j}^{[\frac{z}{h\lambda_k}]}\Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha}\Big(1-\frac{jh\lambda_k}{z}\Big)^{\alpha}K\Big(\frac{X_k-ih\lambda_k}{h\lambda_k}\Big)K\Big(\frac{X_k-jh\lambda_k}{h\lambda_k}\Big)\Bigg\}\Bigg]\\
&=\sum_{i=0}^{[\frac{z}{h\lambda_k}]}\Big(1-\frac{ih\lambda_k}{z}\Big)^{2\alpha}\int_{\mathbb{R}}K^2\Big(\frac{u-ih\lambda_k}{h\lambda_k}\Big)f(u)\,du\\
&+\sum_{i\neq j}^{[\frac{z}{h\lambda_k}]}\Big(1-\frac{ih\lambda_k}{z}\Big)^{\alpha}\Big(1-\frac{jh\lambda_k}{z}\Big)^{\alpha}\int_{\mathbb{R}}K\Big(\frac{u-ih\lambda_k}{h\lambda_k}\Big)K\Big(\frac{u-jh\lambda_k}{h\lambda_k}\Big)f(u)\,du.
\end{aligned}
\tag{4.18}
$$

It follows by corollary 6, of lemma 13 and theorem 23 that, as $n\to+\infty$, the first term of the right hand-side of (4.18) tends to $\left(\int_{\mathbb{R}}K^2(y)P(z,2\alpha)\right)$ and the second term tends to zero uniformly on $[0,b]$.

then

$$n\mathbb{V}ar(P_n^{\lambda}(z,\alpha))\to\left(\int_{\mathbb{R}}K^2(y)P(z,2\alpha)-P^2(z,\alpha)\right)$$

According to the Lemma 12, we have

$$\mathbb{E}(P_n^{\lambda}(z,\alpha))\to P(z,\alpha)\quad n\to+\infty$$

therefore

$$\mathbb{E}^2(P_n^{\lambda}(z,\alpha))\to P^2(z,\alpha)\quad n\to+\infty.$$

Define

$$biais(P_n^\lambda(z,\alpha)) = \mathbb{E}(P_n^\lambda(z,\alpha)) - \mathbb{V}ar(P_n^\lambda(z,\alpha)).$$

We have

$$\mathbb{E}(P_n^\lambda(z,\alpha) - P(z,\alpha))^2 = biais^2(P_n^\lambda(z,\alpha)) + \mathbb{V}ar(P_n^\lambda(z,\alpha))$$

and

$$\left|\left(\int_{\mathbb{R}} K^2(y)P(z,2\alpha) - P^2(z,\alpha)\right)\right| \leq \int_{\mathbb{R}} K^2(y) + 1.$$

Hence

$$\mathbb{V}ar(P_n^\lambda(z,\alpha)) = O(\frac{1}{n}).$$

By Lemma 12, we have

$$|\mathbb{E}(P_n^\lambda(z,\alpha)) - P_n^\lambda(z,\alpha)| \to 0, \quad n \to +\infty,$$

therefore

$$biais^2(P_n^\lambda(z,\alpha)) \to 0, \quad n \to +\infty$$

hence

$$\mathbb{E}(P_n^\lambda(z,\alpha) - P(z,\alpha))^2 \to 0, \quad n \to +\infty.$$

If condition $\mathbf{C_2}$ is satisfied, the Theorem is again valid, by Corollary 6, of Lemma 13 and using Remark 7 of Theorem 23 and Theorem 20.

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