A COLLECTION OF PAPERS IN MATHEMATICS AND RELATED SCIENCES,
A FESTSCHRIFT IN HONOUR OF THE LATE GALAYE DIA
EDITORS : PROFS HAMET SEYDI, GANE SAMB LO, ABOUBAKARY DIAKHABY
SPAS-EDS (SAINT-LOUIS, CALGARY, 2018), WWW.STATPAS.ORG/SPASEDS/
IN EUCLID (WWW.PROJECTEUCLID.ORG)
DOI : 10.16929/SBS/2018.100

CHAPTER 6

On the Parameter Estimation by the Moment Method and Wald Type Test for Poisson Processes,*by E. O. Accrachi, A. S. Dabye and A. A. Gounoung*

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Abstract. The method of moments in parameter estimation problem concerning two statistical problems for inhomogeneous Poisson processes is tackled. Asymptotic behavior and statistical tests are discussed.

Keywords. Parameter estimation; Poisson processes; method of moments estimators; consistency; asymptotic normality; test of Wald .
AMS 2010 Mathematics Subject Classification. 62F10; 62F12; 62G10; 62G20; 62M05.

Cite the chapter as :
Accrachi E.O., Dabye A.S. and Gounoung A. A.(2018). On the Parameter Estimation by Method of Moments and Wald Type Test for Poisson Processes. In *A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia* (Editors : Seydi H., Lo G.S. and Diakhaby A.). Spas Editions, Euclid Series Book, pp. 57 –73.
Doi : 10.16929/sbs/2018.100-02-02

A Collection of Papers in Mathematics and Related Sciences, a festschrift in honour of the late Galaye Dia. **Accrachi E.O., Dabye A.S. and Gounoung A. A.(2018). On the Parameter Estimation by Moment Method and Wald Type Test for Poisson Processes**. Pages 57 — 73.

Abstract. We consider two statistical problems for inhomogeneous Poisson processes. The first one concerns the parameter estimation and the second one is about the goodness of fit test. We develop the method of moments in parameter estimation problem and describe the asymptotic behavior of the new test based on the method of moments estimator. We show that the constructed estimators are consistent and asymptotically normal and for the test we find the limit distribution of the test statistics under hypothesis and show its consistency .

1. Introduction and motivations

We consider two problems devoted to the one model of observations inhomogeneous Poisson processes. This class of point processes plays an important role in many applied problems that is why the statistical problems for this model of observations attracts attention of many researchers, see,e.g., Snyder and Miller (1991), Dia (1987), Kutoyants (1998).
One problem is parameter estimation by the method of moments and the second is the hypothesis testing with simple null hypothesis and one-sided parametric alternative. We describe the properties of the estimators and test in the asymptotic of large samples.

The method of moments is the first known method of parameter estimation. This method was developed by P. Chebyshev in 1887. Since that time this method was widely used as a simplest way of parameter estimation and till now is quite attractive in many applied problems. To calculate the empirical moments is computationally much more easy than to maximize the likelihood function for many statistical models. The current theory of parameter estimation by the method of moments can be found, e.g., in Borovkov (1998) or Lehmann (1999).

Another example of application of the method of moments for ergodic diffusion processes can be found in Kutoyants (2004).

In all models the corresponding method of moments estimators are consistent and asymptotically normal, but not asymptotically efficient. This means that such estimators are recommended in the problems where the minimization of the time of calculation is more important than the value of the limit variance.Recently this method was introduced for the inhomogeneous Poisson processes Dabye, Gounoung and Kutoyants (2018).

We suppose that we have n independent observations $X^n = (X_1, \ldots, X_n)$ of inhomogeneous Poisson processes with intensity function $\lambda(\vartheta, \cdot)$ depending on unknown It was shown that the method of moments estimator ϑ_n is consistent and asymptotically normal.

In the present work we remind the statement and give more detailed proofs and examples. Our main problem is the construction of the Wald type test in the case of observations of inhomogeneous Poisson processes with intensity function depending of one-dimensional parameter. We test the simple null hypothesis that this parameter $\vartheta = \vartheta_0$ against alternative$\vartheta > \vartheta_0$. The test is based on the method of moments estimator and has form $\Psi_n^* = \mathbb{I}_{\left\{\sqrt{n}(\vartheta_n^* - \vartheta_0) \geq z_\alpha D(\vartheta_0)\right\}}$. Here ζ_α is $1-\alpha$ quantile of the standard Gaussian law. The asymptotically optimal test in the hypothesis testing problem with simple null hypothesis and one-sided parametric for inhomogeneous Poisson processes was proposed in Kutoyants (1977).

The case of nonparametric alternative and asymptotically optimal test was considered in Ingster and Kutoyants (2007).

The general theory was started with the work by Le Cam (1956). The Le Cam's theory was further developed in the fundamental work of Ibragimov and Khasminskii (1981) and then applied to the models of Poisson processes by Kutoyants (1998).

2. Results and proofs

2.1. Method of Moments for Poisson Processes . Let us construct an estimator of the method of moments in the case of observations of inhomogeneous Poisson processes. Suppose we have $n$independent observations $X^{(n)} = (X_1, X_2, ..., X_n)$, of the Poisson processes $X_j = (X_j(t), 0 \leq t \leq \tau)$ with the intensity function $\lambda(\vartheta, t), 0 \leq t \leq \tau$. Here the parameter $\vartheta \in \Theta \subset \mathbb{R}^d$, where Θ is an open,convex, bounded set.

Remind that

$$\mathbb{E}_\vartheta X_j(t) = \int_0^t \lambda(\vartheta, s)\, ds.$$

Our goal is to construct an estimator of the parameter ϑ and to describe the properties of this estimator in the asymptotic of large samples,i.e., as

$n \to \infty$. We remind that the method of moments for Poisson processes first used in our work Dabye, Gounoung and Kutoyants (2018) and here we give more detailed proofs and add some examples.

Introduce the vector-function $g(s) = (g_1(s), ..., g_d(s)), 0 \le s \le \tau$ and the vector of integrals

$$\begin{cases} I_1(\tau) &= \int_0^\tau g_1(s)\, dX_1(s) \\ & \vdots \\ I_d(\tau) &= \int_0^\tau g_d(s)\, dX_1(s). \end{cases}$$

We have the system

$$\begin{cases} \mathbb{E}_\vartheta I_1(\tau) &= \int_0^\tau g_1(s)\,\lambda(\vartheta, s)\, ds = m_1(\vartheta) \\ & \vdots \\ \mathbb{E}_\vartheta I_d(\tau) &= \int_0^\tau g_d(s)\,\lambda(\vartheta, s)\, ds = m_d(\vartheta). \end{cases}$$

Denote $M(\vartheta) = (m_1(\vartheta), ..., m_d(\vartheta))$ and suppose that the equation

$$M(\vartheta) = a, \qquad a = (a_1, ..., a_d)$$

has a unique solution

$$\vartheta = M^{-1}(a) = H(a), \qquad H(a) = (H_1(a), ..., H_d(a))$$

for the values

$$\mathcal{A} = \{a : \ a = M(\vartheta), \vartheta \in \Theta\}.$$

Then the estimator of the method of moments (MME) ϑ_n^* is defined as in classical statistics by the equation

$$\vartheta_n^* = H(a_n)$$

or more generally

$$\vartheta_n^* = \arg\inf_{\vartheta\in\Theta} \|\vartheta - H(a_n)\|$$

Without loss of generality we use the first equation because from the law of large numbers the probability of the event $a_n \notin \mathcal{A}$ is exponentially small.

Here a_n is the empirical version of the vector a, $a_n = (a_{1,n}, ..., a_{d,n})$ with

$$a_{l,n} = \frac{1}{n}\sum_{j=1}^{n}\int_0^{\tau} g_l\left(s\right) dX_j\left(s\right), \quad l = 1, ..., d.$$

Suppose that the vector-function $H\left(a\right)$ is continuous, then

$$\vartheta_n^* \longrightarrow H\left(a\right) = \vartheta,$$

because by the law of large numbers

$$a_{l,n} \longrightarrow \int_0^{\tau} g_l\left(s\right)\lambda\left(\vartheta, s\right) ds, \quad l = 1, ..., d,$$

and hence

$$H\left(a_n\right) \longrightarrow H\left(a\right) = \vartheta$$

To show asymptotic normality

$$\sqrt{n}\left(\vartheta_n^* - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathcal{D}\left(\vartheta\right)\right),$$

where $\mathcal{D}\left(\vartheta\right)$ is a limit covariance matrix, we suppose that the vector-function $H\left(a\right)$ is continuously differentiable.

We have

$$\begin{aligned} \sqrt{n}\left(\vartheta_n^* - \vartheta\right) &= \sqrt{n}\left(H\left(a_n\right) - H(a)\right) \\ &= \sqrt{n}\left(H\Big(\frac{1}{n}\sum_{j=1}^{n}\int_0^{\tau} g_l\left(s\right) dX_j\left(s\right)\Big) - H\left(a\right)\right). \end{aligned}$$

For the vector of stochastic integrals, we can write

$$\begin{aligned} &\frac{1}{n}\sum_{j=1}^{n}\int_0^{\tau} g_l\left(s\right) dX_j\left(s\right) - \int_0^{\tau} g_l\left(s\right)\lambda\left(\vartheta, s\right) ds \\ =& \frac{1}{n}\sum_{j=1}^{n}\int_0^{\tau} g_l\left(s\right)\left[dX_j\left(s\right) - \lambda\left(\vartheta, s\right) ds\right] \\ =& \frac{1}{\sqrt{n}}\eta_{l,n}. \end{aligned}$$

The vector $\eta_n = (\eta_{1,n}, ..., \eta_{d,n})$ is asymptotically normal

$$\eta_n \Longrightarrow \eta \sim \mathcal{N}\left(0, K\left(\vartheta\right)\right),$$

where the matrix

$$K\left(\vartheta\right) = \left(K_{l,m}\left(\vartheta\right)\right)_{l,m=1,...d}, \quad K_{l,m}\left(\vartheta\right) = \int_0^\tau g_l\left(s\right) g_m\left(s\right) \lambda\left(\vartheta, s\right) ds.$$

Therefore

$$\begin{aligned}
\sqrt{n}\left(\vartheta_n^* - \vartheta\right) &= \sqrt{n}\left(H\big(a + \frac{1}{\sqrt{n}}\eta_n\big) - H\left(a\right)\right) \\
&= \sqrt{n}\left(H(a) + \frac{1}{\sqrt{n}}\frac{\partial H}{\partial a}\eta_n - H(a) + o(\frac{1}{\sqrt{n}})\right) \\
&= \frac{\partial H}{\partial a}\eta_n + o(1).
\end{aligned}$$

For example, for the the n-th component, we have

$$\begin{aligned}
&\sqrt{n}\left(\vartheta_{l,n}^* - \vartheta_l\right) = \sum_{m=1}^{d} \frac{\partial H_l}{\partial a_m}\eta_{m,n} + o\left(1\right) \\
&= \frac{1}{\sqrt{n}} \sum_{m=1}^{d} \frac{\partial H_l}{\partial a_m} \sum_{j=1}^{n} \int_0^\tau g_m\left(s\right)\left[dX_j\left(s\right) - \lambda\left(\vartheta, s\right) ds\right] + o\left(1\right) \\
&\Longrightarrow \sum_{m=1}^{d} \frac{\partial H_l}{\partial a_m}\eta_m.
\end{aligned}$$

Hence

$$\sqrt{n}\left(\vartheta_n^* - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \frac{\partial H^*}{\partial a} K\left(\vartheta\right) \frac{\partial H}{\partial a}\right).$$

Therefore, we proved the following theorem

THEOREM 15. *Suppose that the vector-function $g\left(\cdot\right)$ is such that the identifiability condition for any $\vartheta \in \Theta$ and any $\nu > 0$*

$$\inf_{\left|\vartheta_0 - \vartheta\right| > \nu} \left|M\left(\vartheta\right) - M\left(\vartheta_0\right)\right| > 0$$

is fulfilled and the function $H(a)$ is continuously differentiable.

Then the MME ϑ_n^* is asymptotically

$$\sqrt{n}\left(\vartheta_n^* - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \quad \frac{\partial H^*}{\partial a} K(\vartheta) \frac{\partial H}{\partial a}\right). \tag{2.1}$$

Example 1.

Suppose that the observed inhomogeneous Poisson process has the intensity function

$$\lambda(\vartheta, t) = \sum_{l=1}^{d} \vartheta_l h_l(s) + \lambda_0, 0 \leq s \leq \tau.$$

Introduce the vector-function $g(\cdot)$ and the corresponding integrals

$$I_l(\tau) = \int_0^\tau g_l(s)\, dX_j(s), \quad l = 1, ..., d,$$

then

$$\begin{aligned} \mathbb{E}_\vartheta I_k(\tau) = m_k(\vartheta) &= \int_0^\tau g_k(s)\, \lambda(\vartheta, s)\, ds \\ &= \sum_{l=1}^{d} \vartheta_l \int_0^\tau g_k(s)\, h_l(s)\, ds + \lambda_0 \int_0^\tau g_k(s)\, ds \\ &= \sum_{l=1}^{d} \vartheta_l A_{k,l} + \lambda_0 G_k = a_k \end{aligned}$$

in obvious notations.

Therefore in matrix form we have the equation:

$$A\vartheta + \lambda_0 G = a,$$

where $A = (A_{k,l})_{d\times d}$ is a matrix, $G_k = (G_1, ..., G_d)$ and $a_n = (a_1, ..., a_d)$ are the vectors.

Suppose that the vector-function $g(\cdot)$ is such that we have the matrix A is nondegenerate. Then the solution of this equation is

$$\vartheta = A^{-1}(a - \lambda_0 G).$$

The MME ϑ_n^* is given by the equality:

$$\vartheta_n^* = A^{-1}(a_n - \lambda_0 G),$$

where $a_n = (a_{1,n}, ..., a_{d,n})$ is given by

$$a_{l,n} = \frac{1}{n}\sum_{j=1}^{n}\int_0^\tau g_l(s)\, dX_j(s).$$

By the law of large numbers

$$a_{l,n} \longrightarrow \int_0^\tau g_l(s)\,\lambda(\vartheta, s)\, ds.$$

Hence

$$\vartheta_n^* \longrightarrow A^{-1}\left(\int_0^\tau g(s)\,\lambda(\vartheta, s)\, ds - \lambda_0 G\right) = \vartheta,$$

and MME ϑ_n^* is consistent.

To study the normalized difference we write

$$\sqrt{n}\,(\vartheta_n^* - \vartheta) = A^{-1}\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\int_0^\tau g(s)\,[dX_j(s) - \lambda(\vartheta, s)\, ds].$$

The vector

$$\eta_n = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\int_0^\tau g(s)\,[dX_j(s) - \lambda(\vartheta, s)\, ds]$$

is asymptotically normal:

$$\eta_n \Longrightarrow \eta \rightsquigarrow \mathcal{N}(0, K(\vartheta)),$$

where the matrix $K(\vartheta) = (K_{l,m}(\vartheta))_{d\times d}$ is

$$K_{l,m}\left(\vartheta\right) = \vartheta_m \int_0^\tau g_l\left(s\right) h_m\left(s\right) ds + \lambda_0 \int_0^\tau g_l\left(s\right) ds.$$

Hence

$$\sqrt{n}\left(\vartheta_n^* - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathcal{D}\left(\vartheta\right)\right)$$

where the matrix

$$\mathcal{D}\left(\vartheta\right) = A^{-1} K\left(\vartheta\right) A^{-1}.$$

Example 2.

Suppose that we observe n independent Poisson processes $X^n = \left(X_1, \ldots, X_n\right)$, where $X_j = \left(X_j\left(t\right)\right), 0 \leq t < \infty$ and the intensity function is

$$\lambda\left(\vartheta, t\right) = \frac{t^{\beta-1}\alpha^\beta}{\Gamma\left(\beta\right)} \exp\left(-\alpha t\right), t \geq 0,$$

i.e, we have a Poisson process with Gamma intensity function. The unknown parameter is $\vartheta = \left(\alpha, \beta\right)$.

We know (see Example 1), that

$$\begin{aligned} m_1\left(\vartheta\right) &= \int_0^\infty t\,\lambda\left(\vartheta, t\right) dt = \frac{\beta}{\alpha} \\ m_2\left(\vartheta\right) &= \int_0^\infty t^2 \lambda\left(\vartheta, t\right) dt = \frac{\beta\left(\beta+1\right)}{\alpha^2}. \end{aligned}$$

Hence, if we take $g(t) = \left(g_1(t), g_2\left(t\right)\right) = \left(t, t^2\right)$, then the system

$$\begin{cases} m_1\left(\vartheta\right) = a_1 \\ m_2\left(\vartheta\right) = a_2 \end{cases}$$

has the unique solution

$$\begin{cases} \alpha = \frac{a_1}{a_2 - a_1^2} \\ \beta = \frac{a_1^2}{a_2 - a_1^2}. \end{cases}$$

Therefore the MME is

$$\alpha_n^* = \frac{1}{n}\sum_{j=1}^{n}\int_0^\infty t dX_j(t)\Big(\frac{1}{n}\sum_{j=1}^{n}\int_0^\infty t^2 dX_j(t) - \Big(\frac{1}{n}\sum_{j=1}^{n}\int_0^\infty t dX_j(t)\Big)^2\Big)^{-1},$$

$$\beta_n^* = \Big(\frac{1}{n}\sum_{j=1}^{n}\int_0^\infty t dX_j(t)\Big)^2\Big(\frac{1}{n}\sum_{j=1}^{n}\int_0^\infty t^2 dX_j(t) - \Big(\frac{1}{n}\sum_{j=1}^{n}\int_0^\infty t dX_j(t)\Big)^2\Big)^{-1}.$$

As in the Example 1, this estimator is consistent, because

$$\frac{1}{n}\sum_{j=1}^{n}\int_0^\infty t dX_j(t) \longrightarrow \frac{\beta}{\alpha}, \quad \frac{1}{n}\sum_{j=1}^{n}\int_0^\infty t^2 dX_j(t) \longrightarrow \frac{\beta(\beta+1)}{\alpha^2}$$

and we have

$$\alpha_n^* \longrightarrow \frac{\beta}{\alpha}\left(\frac{\beta(\beta+1)}{\alpha^2} - \frac{\beta^2}{\alpha^2}\right)^{-1} = \alpha,$$

$$\beta_n^* \longrightarrow \frac{\lambda^2}{\alpha^2}\left(\frac{\beta(\beta+1)}{\alpha^2} - \frac{\beta^2}{\alpha^2}\right)^{-1} = \beta.$$

It is possible to verify, that

$$\sqrt{n}\left(\vartheta_n^* - \vartheta\right) = \sqrt{n}\begin{pmatrix}\alpha_n^* - \alpha \\ \beta_n^* - \beta\end{pmatrix} \Longrightarrow \mathcal{N}\left(0, \mathcal{D}\left(\vartheta\right)\right),$$

where the matrix of covariance $\mathcal{D}(\vartheta)$ is

$$\mathcal{D}(\vartheta) = \begin{pmatrix} \frac{\beta}{\alpha^2} & \frac{2\beta(\beta+1)}{\alpha^3} \\ \frac{2\beta(\beta+1)}{\alpha^3} & \frac{2\beta(\beta+1)(2\beta+3)}{\alpha^4}\end{pmatrix}.$$

Suppose for example $\vartheta = \beta$, $g_1(t) = t$ and $g_2(t) = t^2$. The parameter α is known.

Then if we put

$$X_1 = \int_0^\infty t dX(t), \qquad X_2 = \int_0^\infty t^2 dX(t),$$

then we have

$$\mathbb{E}_\vartheta\left(X_1\right)=\frac{\beta}{\alpha}=m_1\left(\vartheta\right),\quad \mathbb{E}_\vartheta\left(X_1^2\right)=\frac{\beta\left(\beta+1\right)}{\alpha^2}=\mathbb{E}_\vartheta\left(X_2\right),$$

and

$$\mathbb{E}_\vartheta\left(X_1X_2\right)=\frac{\beta\left(\beta+1\right)\left(\beta+2\right)}{\alpha^3},\quad \mathbb{E}_\vartheta\left(X_2^2\right)=\frac{\left(\beta+3\right)}{\alpha}\mathbb{E}_\vartheta\left(X_1X_2\right).$$

Hence

$$Var\left(X_1\right)=\frac{\beta}{\alpha^2},\ Var\left(X_2\right)=\frac{2\beta\left(\beta+1\right)\left(2\beta+3\right)}{\alpha^4}$$

and

$$Cov\left(X_1,X_2\right)=\frac{2\beta\left(\beta+1\right)}{\alpha^3}.$$

Example 3.

Suppose that we observe n independent Poisson processes $X^n=(X_1,\ldots,X_n)$, where $X_j=\left(X_j\left(t\right),t\in\mathbb{R}\right)$ and the intensity function is Gaussian

$$\lambda\left(\vartheta,t\right)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{\left(t-\alpha\right)^2}{2\sigma^2}\right\},\ t\in\mathbb{R}.$$

Then, of course, the MLE has no explicit expression. Suppose that the unknown parameter $\vartheta=\sigma^2$ and α is known.

Then we can take $g(t)=\left(t-\alpha\right)^2$ and to have the equality

$$(\vartheta)=\int_{-\infty}^{+\infty}\left(t-\alpha\right)^2\ \lambda\left(\vartheta,t\right)dt=\vartheta.$$

Then the estimator of the method of moments is

$$\vartheta_n^*=\frac{1}{n}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\left(t-\alpha\right)^2dX_j(t).$$

This estimator is consistent and asymptotically normal, i.e.

$$\vartheta_n^* \longrightarrow \int_{-\infty}^{+\infty} (t-\alpha)^2 \lambda(\vartheta, t)\, dt = \vartheta$$

and

$$\begin{aligned}\sqrt{n}\,(\vartheta_n^* - \vartheta_0) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_{-\infty}^{+\infty} (t-\alpha)^2 \left[dX_j(t) - \lambda(\vartheta, t)\, dt\right] \\ &\Longrightarrow \mathcal{N}\left(0, \mathcal{D}(\vartheta)\right)\end{aligned}$$

where

$$\mathcal{D}(\vartheta) = \int_{-\infty}^{+\infty} (t-\alpha)^4 \lambda(\vartheta, t)\, dt = 3\vartheta^2.$$

2.2. Wald Type Test. Suppose that we have n independent observations $(X_1, \ldots, X_n)$ of inhomogeneous Poisson process, where $X_j = \{X_j(t),\ 0 \le t \le T\}$, $j = 1, \ldots, n$. The mean and intensity functions denoted as $\Lambda(\vartheta, \cdot)$ and $\lambda(\vartheta, \cdot)$ respectively, i.e.

$$\mathbb{E}_\vartheta X_j(t) = \Lambda(\vartheta, t) = \int_0^t \lambda(\vartheta, s) ds, \quad \vartheta \in \Theta = (\alpha,\ \beta), \quad \Lambda(\vartheta, T) < \infty.$$

Here ϑ is unknown one-dimensional parameter and $\mathbb{E}_\vartheta$ is the mathematical expectation. The problems of hypothesis testing for inhomogeneous Poisson processes were considered by many authors. Let us mention here some of them Bar-David (1969), Davies (1977), Gounoung (2016).

Let us consider the following hypothesis testing problem.

$$\begin{aligned}\mathcal{H}_0 &: \vartheta = \vartheta_0 \\ \mathcal{H}_1 &: \vartheta > \vartheta_1\end{aligned}$$

i.e., we have a simple hypothesis against one sided composite alternative. Our goal is to construct a test Ψ_n^* which belongs to the class $\mathcal{K}_\alpha$ of test of asymptotic level $1-\alpha$:

$$\mathcal{K}_\alpha = \left\{\bar{\Psi}_n, \quad \lim_{n \longrightarrow \infty} \mathbb{E}_{\vartheta_0} \bar{\Psi}_n = \alpha\right\}, \quad ;\text{i.e. };\quad \mathbb{E}_{\vartheta_0} \bar{\Psi}_n = \alpha + o(1)$$

and is consistent under alternative: for any $\vartheta > \vartheta_0$ we have

$$\lim_{n\longrightarrow\infty} \mathbb{E}_{\vartheta}\bar{\Psi}_n = 1.$$

We do not seek the optimal test which can be based on the MLE $\hat{\vartheta}_n$ of ϑ and we prefer to use non optimal but much more easy calculated MME ϑ_n^*. The well-known test of Wald has the following form

$$\Psi_n = \mathbb{I}_{\left\{\sqrt{n}\left(\hat{\vartheta}_n-\vartheta_0\right)>z_\alpha\, I(\vartheta_0)^{-1/2}\right\}} \tag{2.2}$$

Here z_α is the quantile order $1-\alpha$ of the Gaussian law $\mathbb{P}_{\vartheta_0}\left(\zeta > z_\alpha\right) = \alpha$, where $\zeta \sim \mathcal{N}\left(0,1\right)$ and $I\left(\vartheta_0\right)$ is the Fisher information.

This test is locally asymptotically uniformly most powerful, i.e., asymptotically optimal. The construction of the MLE $\hat{\vartheta}_n$ for many non linear models can be computationally difficult problem. This is the reason why we replace the MLE by the MME.

In accordance with the relation (2.2), we propose to use in the construction of the test the MME ϑ_n^*. Therefore

$$\Psi_n^* = \mathbb{I}_{\left\{\sqrt{n}(\vartheta_n^*-\vartheta_0)>z_\alpha\, D(\vartheta_0)^{1/2}\right\}}. \tag{2.3}$$

Here we suppose that the conditions of the Theorem 15 are fulfilled and under $\mathcal{H}_0$ we have the asymptotic normality

$$\sqrt{n}\left(\vartheta_n^* - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, D\left(\vartheta_0\right)\right),$$

with $D\left(\vartheta_0\right) = \left(H'(a)\right)^2 K(\vartheta)$.

THEOREM 16. *Suppose that the conditions of the theorem 15 are fulfilled. Then the test Ψ_n^* belongs to the class $\mathcal{K}_\alpha$.*

Proof.

We have

$$\mathbb{E}_{\vartheta_0}\Psi_n^* = \mathbb{P}_{\vartheta_0}\left(\sqrt{n}\left(\vartheta_n^* - \vartheta_0\right) > z_\alpha\, D\left(\vartheta_0\right)^{1/2}\right).$$

We know that the estimator ϑ_n^* is asymptotically normal (see 2.1). Therefore

$$\lim_{n\to\infty} \mathbb{E}_{\vartheta_0}\Psi_n^* = \mathbb{P}_{\vartheta_0}\left(\zeta > z_\alpha\right) = \alpha$$

Proposition 1. *The test Ψ_n^* defined by (2.3) is consistent against any alternative: $\vartheta = \vartheta_1 > \vartheta_0$*

Proof. Indeed, we can write

$$\begin{aligned}
\mathbb{E}_{\vartheta_1}\Psi_n^* &= \mathbb{P}_{\vartheta_1}\left(\sqrt{n}\left(\vartheta_n^* - \vartheta_0\right) > z_\alpha\, D\left(\vartheta_0\right)^{1/2}\right)\\
&= \mathbb{P}_{\vartheta_1}\left(\sqrt{n}\left(\vartheta_n^* - \vartheta_1\right) + \sqrt{n}\left(\vartheta_1 - \vartheta_0\right) > z_\alpha\, D\left(\vartheta_0\right)^{1/2}\right)\\
&= \mathbb{P}_{\vartheta_1}\left(\sqrt{n}\left(\vartheta_n^* - \vartheta_0\right) > z_\alpha\, D\left(\vartheta_0\right)^{1/2} - \sqrt{n}\left(\vartheta_1 - \vartheta_0\right)\right).
\end{aligned}$$

Therefore we have

$$\sqrt{n}\left(\vartheta_1 - \vartheta_0\right) \xrightarrow[n\to+\infty]{} +\infty.$$

Hence

$$\mathbb{E}_{\vartheta_1}\Psi_n^* = \mathbb{P}_{\vartheta_1}\left(\sqrt{n}\left(\vartheta_n^* - \vartheta\right) > z_\alpha\, D\left(\vartheta_0\right)^{1/2}\right) \xrightarrow[n\to+\infty]{} 1$$

and the test is consistent.

For the local alternatives $\vartheta_1 = \vartheta_0 + \frac{u}{\sqrt{n}}$, we obtain the following expression for the power function

$$\begin{aligned}
\beta_n\left(u, \Psi_n^*\right) &= \mathbb{E}_{\vartheta_0 + \frac{u}{\sqrt{n}}}\Psi_n^* = \mathbb{P}_{\vartheta_0 + \frac{u}{\sqrt{n}}}\left(\sqrt{n}\left(\vartheta_n^* - \vartheta_0\right) > z_\alpha\, D\left(\vartheta_0\right)^{1/2}\right)\\
&= \mathbb{P}_{\vartheta_0 + \frac{u}{\sqrt{n}}}\left(\sqrt{n}\left(\vartheta_n^* - \vartheta_0 - \frac{u}{\sqrt{n}}\right) + u > z_\alpha\, D\left(\vartheta_0\right)^{1/2}\right)\\
&\xrightarrow[n\to+\infty]{} \mathbb{P}_{\vartheta_0}\left(\zeta\, d\left(\vartheta_0\right) + u > z_\alpha\, D\left(\vartheta_0\right)^{1/2}\right)\\
&= \mathbb{P}_{\vartheta_0}\left(\zeta > z_\alpha - \frac{u}{D\left(\vartheta_0\right)^{1/2}}\right) \equiv \beta^*\left(u\right).
\end{aligned}$$

Remind that in the case of asymptotically optimal test of Wald we have

$$
\begin{aligned}
&\mathbb{E}_{\vartheta_0+\frac{u}{\sqrt{n}}}\mathbb{I}_{\left\{\sqrt{n}\left(\hat{\vartheta}_n-\vartheta_0\right)>z_\alpha\, I(\vartheta_0)^{-1/2}\right\}} \\
\underset{n\to+\infty}{\longrightarrow}\quad &\mathbb{P}_{\vartheta_0}\left(\zeta > z_\alpha - u\, I\left(\vartheta_0\right)^{-1/2}\right) \equiv \hat{\beta}\left(u\right).
\end{aligned}
$$

By the Cramer-Rao inequality we have the relation

$$
I\left(\vartheta_0\right) \geq D\left(\vartheta_0\right)^{-1}
$$

Hence for all $u > 0$ we have

$$
\hat{\beta}\left(u\right) \geq \beta^*\left(u\right).
$$

Recall that advantage of our approach is in the simplicity of the computation.

Example 4.

Suppose that the intensity function is Gamma from the Example 2 and the unknown parameter is $\vartheta = \beta$. The null hypothesis is $\mathcal{H}_0 : \beta = \vartheta_0$ and the alternative $\mathcal{H}_1 : \beta = \vartheta > \vartheta_0$.

Then the MLE $\hat{\beta}_n$ has no explicit expression and its numerical to calculation is a difficult problem. This is not the case for the MME of the parameter ϑ. Remind that

$$
\beta = \frac{a_1^2}{a_2 - a_1^2}
$$

and the MME is

$$
\vartheta_n^* = \Big(\frac{1}{n}\sum_{j=1}^{n}\int_0^\infty t dX_j(t)\Big)^2\Big(\frac{1}{n}\sum_{j=1}^{n}\int_0^\infty t^2 dX_j(t) - \Big(\frac{1}{n}\sum_{j=1}^{n}\int_0^\infty t dX_j(t)\Big)^2\Big)^{-1}.
$$

This estimator is asymptotically normal

$$
\sqrt{n}\left(\vartheta_n^* - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathcal{D}\left(\vartheta\right)\right),
$$

with

$$D\left(\vartheta_0\right) = \frac{2\beta\left(\beta+1\right)\left(2\beta+3\right)}{\alpha^4}$$

Therefore the Wald-type test is

$$\Psi_n^* = \mathbb{I}_{\left\{\sqrt{n}(\vartheta_n^* - \vartheta_0) > z_\alpha\, D(\vartheta_0)^{1/2}\right\}}.$$

This test belongs to $\mathcal{K}_\alpha$ and is consistent against any alternative $\vartheta_1 > \vartheta_0$.

Bibliography

Bar-David, I. (1969) Communication under Poisson regime. *IEEE Transactions on Information Theory*, 15(1), pp. 31–37.

Borovkov, A. A. (1998) *Mathematical Statistics*. Gordon & Breach, Amsterdam.

Dabye A.S., Gounoung A. A. & Kutoyants Y. A (2018) Method of moments estimators and multi-step MLE-processes for Poisson processes. *To appear in Journal of Contemporary Mathematical Analysis*.

Davies, R. (1977) Testing the hypothesis that a point process is Poisson. *Advances in Applied Probability*, 9(4), pp. 724-746.

Dia G. (1987) Estimation of a regression function on a Poisson process. *Statistics and Probability Letters*, 6(1), pp. 47-54.

Gounoung A. A. (2016) *Estimation Paramétrique et Tests d'hypothèses pour les Processus de Poisson*. Université Gaston Berger, Saint-Louis.

Ibragimov, I.A. and Khasminskii R.Z. (1981) *Statistical Estimation. Asymptotic Theory.* Springer, New York.

Ingster, Yu. I., Kutoyants Yu. A. (2007) Nonparametric hypothesis testing for intensity of Poisson process, *Mathem. Methods Statist.*, 16,217-245.

Kutoyants Yu. A. (1977) Asymptotically optimal algorithm of detecting Poisson signals, *Radiotehnika i Elektronika*, 22(3), pp. 621-625.

Kutoyants Yu. A. (1998) *Statistical Inference for Spatial Poisson Processes*. Springer-Verlag, New York.

Kutoyants, Yu. A. (2004) *Statistical Inference for Ergodic Diffusion Processes.* Springer, London.

Kutoyants Yu. A. (2018) *Introduction to Statistics of Poisson Processes*. In progress

Le Cam, L. (1956) On the asymptotic theory of estimation and testing hypotheses. *Proc. 3rd Berkeley Symposium I*, pp. 355-368.

Lehmann, E.L. (1999) *Elements of Large-Sample Theory.* Springer, New York.

Snyder, D.R. & Miller, M.I. (1991) *Random Point Processes in Time and Space.* Springer, New York.