
Hamet Seydi. Email : hseydi@gmail.com.
University Cheikh Anta Diop, Dakar (SENEGAL).

Teylama Hervé Miabey. Email : thmiabey@ymail.com, teylama.miabey@udc.edu
Mathematics Department, University of the District of Columbia Community College, USA.

Maimouna Salou. Email : maimouna idi@yahoo.fr
Département de Mathématiques et d’informatique, Université A. Moumouni de Niamey, NIGER.

Abstract. The present paper gives the solution of a Hartshorne conjecture: Every non singular real algebraic variety is affine. Other related results are provided.

Keywords. Hartshorne’s conjecture; Projective algebraic varieties; Schemes

AMS 2010 Mathematics Subject Classification. 14-XX; 14P25; 14N05

Cite the paper as:
Doi : 10.16929/sbs/2018.100-01-02


1. Introduction and motivations.

Let $K$ be a normal closed field, $A_n = K[T_1, \cdots, T_n]$ the ring of polynomials in $n$ variables over $K$, $X_n = K^n$ provided with the Zarisky topology i.e the topology on $X_n$ the set of $K$-rational roots $Y_n = Spec(A_n)$ by the Zarisky topology on $Y_n$ the seaf on $X_n$ by the canonical sheaf on $Y_n$.

A space $(V, O_V)$ is said to be an algebraic variety over $X_n$ if $(V, O_V)$ is isomorphic to a space $(W, R_W)$ where $W$ is the set of $K$-rational points of a finitely generated and separated $(Y, O_Y)$ over $K$ and $R_W$ is the restriction to $W$ of the topology on $Y$. The seaf $O_V$ is also denoted $R_V$ and called the sheaf of regular functions on $V$.

If $Y = P^n_K$ the projective space of dimension $n$ over $\mathbb{R}$, $W$ is denoted $P^n(K)$ and called the projective algebraic space of dimension $n$ over $K$. An algebraic sub-variety of a projective space $P^n(K)$ is called a projective algebraic variety.

An algebraic variety over $K$ $(V, O_V)$ is said to be affine if $(Y, O_Y)$ is defined to be an algebraic sub-variety of $X_n$ for some integer $n$.

2. Results and proofs.

THEOREM 1. The projective algebraic space $P^n(K)$ of dimension $n$ over $K$ is an affine algebraic variety.

For the proof of this proposition we need the following lemma.

LEMMA 1. Let $I$ be an ideal of $A_n$. Then there exist an element $f$ of $I$ such that $V(I) = V(f)$. Furthermore, if $I = \sigma f_i A_n$, $f$ can be chosen to be equation $P(f_1, \cdots, f_n)$ where $P$ is a homogenous polynomial in $q$ variables with coefficients in $K$.

Proof of Lemma 1. Since $K$ is not algebraically closed, there exist a finite extension $K'$ of $K$ such that $n = [K' : K] > m$.

Let $B = x_1, \cdots, x_m$ be a base of $K'$ over $K$. Then there exist a homogenous polynomial $P_1$ in $n$ variables with coefficients in $K$ such that if $x \in K'$, $x = \sigma \lambda_i x_i$ with $\lambda_i \in K'$ for $1 \leq i \leq n$, and if $f_x$ is the multiplication by $x$ in $K'$, then $def f_x = P_1(\lambda_1, \cdots, \lambda_m)$. 
$P_1(\alpha_1, \cdots, \alpha_m) = 0$ if and only if $\alpha_1 = \cdots, \alpha_m = 0$.

We will prove by induction on $k > 1$ that there exist a homogenous polynomial $P_k$ in $k$ variables with coefficients in $K$ of degree $m^k$ such that $P_k(\alpha_1, \cdots, \alpha_{m^k}) = 0$ if and only if $\alpha_1 = \cdots, \alpha_{m^k} = 0$.

Assume that this assertion is true up to $k > 1$. Let $X_{ij}, 1 < i < m, 1 < j < m^k$ be $m^{k+1}$ variables et $Q_k = P_k(X_{i1}, \cdots, X_{im^k})$. It is clear that $P_{k+1} = P_1(Q_1, \cdots, Q_m)$ is a homogenous polynomial in the $m^{k+1}$ variables $X_{ij}, 1 < i < m, 1 < j < m^k$ of degree $m^{k+1}$ such that $P_{k+1}(\alpha_{11}, \cdots, \alpha_{1m^k}, \cdots, \alpha_{m1}, \cdots, \alpha_{mm^k}) = 0$ if and only if $\alpha_{ij} = 0$ for $1 \leq i \leq m, 1 \leq j \leq m^k$. Hence the assertion is true up to $k + 1$. Since the assertion is true for $k = 1$, we conclude that the assertion is true for every integer $k > 1$.

More assume that if $I = \sum_{i=1}^{n} f_i A_n$, there exist an integer $k > 1$ such that $m^k > q$. It is clear that $f = P_k(f_1, \cdots, f_q, 0, \cdots, 0)$ is an element of $I$ because the constant term of $P_k$ is equal to 0 and $\alpha = (\alpha_1, \cdots, \alpha_m) \in K^m$ is a zero of $f$ if and only if $\alpha$ is a zero of $f_1, \cdots, f_q$ ie $V(f) = V(I)$.

**Proof of Theorem 1.** Let $R = K[X_1, \cdots, X_{m+1}]$ the ring of polynomials in $n + 1$ variables over $K$ and $M$ the maximal ideal of $R$ generated by $X_1, \cdots, X_{m+1}$. Then there exist a homogenous polynomial $f \in M$ such that $V(M) = V(f)$.

It is well know that $Y = V(f)$ is an affine open subset of $P^n(K)$. It is also clear that $P^n(K)$ is contained in $Y$ because of $\alpha \in P^n(K) \cap D(X_i)$. Then $\alpha = (\alpha_1, \cdots, \alpha_{i-1}, 1, \alpha_{i+1}, \cdots, \alpha_n)$ with $\alpha_j \in K$ for $j = 1, \cdots, i - 1, i + 1, \cdots, n$. So,

\[ f(\alpha_1, \cdots, \alpha_{i-1}, 1, \alpha_{i+1}, \cdots, \alpha_n) \neq 0. \]

Hence $P^n(K)$ which is the set of $K$–maximal points of $P^n(K)$ is an affine algebraic variety.

**Corollary 1.** Every projective algebraic variety $(V, O_V)$ over $K$ is affine.

**Corollary 2.** Every compact non singular real algebraic variety is affine.

**Proof.** By a theorem of Nash (1995), every compact non singular algebraic variety is projective, so the conclusion follows from corollary 1

**Theorem 2** (Harstshorne’s conjecture). Every non singular real algebraic variety is affine.
Proof of Theorem 2. Let (V, O_V) be a non singular real algebraic variety. Assume that V is the set of K–maximal points of separated finitely generated scheme over K, (Y, O_Y) and O_V = O_Y/V. By Nagata (1962)’s compactification theorem there exist a complete scheme over K, (Z, O_Z) such that Y is an open subset of Z and O_Y = O_Z)/Y and by Hiromota’s resolution of singularities theorem we can assume that (Z, O_Z) is a regular scheme. Let W be a set of K–rational points of (Z, O_Z) and O_W = O_Z)/W. Then (W, O_W) is a compact non singular real algebraic variety. So (W, O_W) is an affine real algebraic variety (of Corollary 2), and by Lemma 1. There exist f ∈ T(W, O_W) such that W\V = V(f). Hence V = D(f), so (V, O_V) is an affine real algebraic variety.
Bibliography


