# CHAPTER 4

# Differential Forms on Manifolds

**4.1. Grassmann algebra.** It may be helpful to precede the formal definition by a short informal discussion. We shall keep a given differentiable manifold $M$ and an arbitrary point $p \in M$ fixed throughout this section. Let $f$ be a $C^1$ function $M \to R$ and $df$ its differential. It was seen in Chapter 3 that one of the two possible interpretations of the value of $df$ at $p$ is a linear functional on the tangent space $M_p$. That is, $df$ at $p$ is a member of the space $M_p^*$ dual to $M_p$. In differential geometry there is a need for functions whose arguments consist of more than one element of $M_p$. The case of two elements is especially prevalent, for instance in the notions of curvature and torsion transformations (see Bishop and Crittenden, 1964). In this monograph the most important case will be $d$ arguments, where $d = \dim M$, since that will be used to construct a measure on $M$ (Section 6.6). In general, then, we are going to define, for every $1 \leq k \leq d$, a differential form $\omega$ of degree $k$, or, simply, a $k$-form. Its value at $p$ is denoted $\omega_p$ and will be defined as a real valued function of a certain kind on the $k$-fold product $M_p \times \cdots \times M_p$. (The extension to $\omega_p$ being vector valued is important in differential geometry, but not for the purpose of this monograph.)

In order to simplify the notation put $W^k = M_p^{\times k} = k$-fold product of $M_p$ with itself ($k \geq 1$). A real valued function $u$ on $W^k$ is said to be **$k$-linear** if $u$ is linear in each of its $k$ arguments separately. The function $u$ is called **alternating** if $u$ changes sign whenever two arguments are interchanged. Equivalently, this can be expressed in terms of arbitrary permutations of the arguments, as follows. Let $G_k$ be the group of permutations $\pi$ of the integers $1, \ldots, k$ and let $\operatorname{sgn}(\pi)$ be $+1$ or $-1$ according as $\pi$ is even or odd. Then $u$ is alternating if

$$u(t_{\pi 1}, \ldots, t_{\pi k}) = \operatorname{sgn}(\pi) u(t_1, \ldots, t_k) \tag{4.1.1}$$

for $t_1, \ldots, t_k \in M_p$, $\pi \in G_k$.

Consider all $u$ on $W^k$ that are $k$-linear and alternating. This is a finite dimensional vector space, say $V_k$. As an example take $k = 2$. Relative to a chosen basis in $M_p$ let $t^i$, $i = 1, \ldots, d$, be the components of $t \in M_p$. Then a 2-linear alternating function $u$ on $W^2$ must be of the form $u(t_1, t_2) = \sum_{ij} a_{ij} t_1^i t_2^j$ (sum over all $i, j = 1, \ldots, d$), with $a_{ji} = -a_{ij}$. Thus, $u$ can be written as

$$u(t_1, t_2) = \sum_{i<j} b_{ij} (t_1^i t_2^j - t_2^i t_1^j) \tag{4.1.2}$$

with arbitrary $b_{ij}$ $(1 \leq i < j \leq d)$. Therefore, the 2-linear alternating functions $t_1^i t_2^j - t_2^i t_1^j$, $1 \leq i < j \leq d$, span $V_2$. Since they are linearly independent, they form a basis of $V_2$ and it follows that $\dim V_2 = d(d-1)/2$. For arbitrary $1 \leq k \leq d$ it will be proved that $\dim V_k = \binom{d}{k}$. For us of particular interest is the case $k = d$, when $V_d$ is spanned by the single function $\sum_\pi \operatorname{sgn}(\pi) t_{\pi 1}^1 \ldots t_{\pi d}^d$. Note that for $k = 1$, $V_1$ is the space of linear functionals on $M_p$, i.e., $V_1 = M_p^*$. It is also convenient to define $V_k$ for $k = 0 : V_0 = R$; then $\dim V_0 = 1$.

Next, define

$$V = V_0 \oplus \cdots \oplus V_d \tag{4.1.3}$$

(direct sum), then $V$ is a vector space of dimension $\sum_{i=0}^d \binom{d}{i} = 2^d$. A multiplication in $V$ will now be defined that makes $V$ into an algebra, called the **Grassmann algebra** over $V_1$. If $u, v \in V$ we shall write $u \wedge v$ for their product, and call it the **wedge product**. The multiplication is required to be distributive so that it suffices to define $u \wedge v$ if $u \in V_k$, $v \in V_\ell$, for all $0 \leq k, \ell \leq d$. If $k$ or $\ell$ equals 0, the multiplication is ordinary multiplication by a real number. Suppose therefore that $1 \leq k,\ \ell \leq d$. Then $u \wedge v$ is defined as the function on $W^{k+\ell}$ such that for $t_i \in M_p$, $i = 1, \dots, k+\ell$, we have

$$(4.1.4)\quad \begin{aligned} &u \wedge v(t_1, \dots, t_{k+\ell}) \\ &\quad = (k!\ell!)^{-1} \sum_{\pi \in G_{k+\ell}} \operatorname{sgn}(\pi) u(t_{\pi 1}, \dots, t_{\pi k}) v(t_{\pi(k+1)}, \dots, t_{\pi(k+\ell)}). \end{aligned}$$

When the sum on the right-hand side of (4.1.4) is written out it is seen that several terms have the same value as a result of the alternating character of $u$ and $v$. For instance, if $k = 1$, $\ell = 2$, then the right-hand side of (4.1.4) has $3! = 6$ terms. But, for instance, two of the terms are $u(t_1)v(t_2, t_3)$ and $-u(t_1)v(t_3, t_2)$, and those are equal since $-v(t_3, t_2) = v(t_2, t_3)$. Thus, for $k = 1$, $\ell = 2$, (4.1.4) simplifies to

$$(4.1.5)\quad u \wedge v(t_1, t_2, t_3) = u(t_1)v(t_2, t_3) - u(t_2)v(t_1, t_3) + u(t_3)v(t_1, t_2),$$

in which the arguments $(t_i, t_j)$ of $v$ have been written so that $i < j$. In general, for any $k$, $\ell$, (4.1.4) can be written as

$$(4.1.6)\quad \begin{aligned} &u \wedge v(t_1, \dots, t_{k+\ell}) \\ &\qquad = \sum_{\pi} \operatorname{sgn}(\pi) u(t_{\pi 1}, \dots, t_{\pi k}) v(t_{\pi(k+1)}, \dots, t_{\pi(k+\ell)}) \end{aligned}$$

summed over all $\pi$ for which $\pi 1 < \cdots < \pi k$ and $\pi(k+1) < \cdots < \pi(k+\ell)$. For instance, if $k = \ell = 2$, then

$$(4.1.7)\quad \begin{aligned} u \wedge v(t_1, \dots, t_4) = {} & u(t_1, t_2)v(t_3, t_4) - u(t_1, t_3)v(t_2, t_4) \\ & + u(t_1, t_4)v(t_2, t_3) + u(t_2, t_3)v(t_1, t_4) \\ & - u(t_2, t_4)v(t_1, t_3) + u(t_3, t_4)v(t_1, t_2). \end{aligned}$$

The function $u \wedge v$ defined by (4.1.4) is easily seen to be $(k+\ell)$-linear and can be verified to be alternating (see, e.g., the examples (4.1.5) and (4.1.7)). Thus, $u \wedge v \in V_{k+\ell}$ provided $k + \ell \leq d$. If $k + \ell > d$, then one of the arguments $t_i$ of $u \wedge v$ must be a linear combination of the remaining arguments and then linearity and alternation of $u \wedge v$ forces its value to be 0. It follows that $u, v \in V$ implies $u \wedge v \in V$. The multiplication is easily checked to be associative so that we can freely write $u \wedge v \wedge w$, etc. In particular, if $u_1, \ldots, u_k \in V_1$, then repeated application of (4.1.4) or (4.1.6) yields

$$(u_1 \wedge \cdots \wedge u_k)(t_1, \ldots, t_k) = \sum_{\pi \in G_k} \operatorname{sgn}(\pi) u_1(t_{\pi 1}) \cdots u_k(t_{\pi k}), \qquad u_1, \ldots, u_k \in V_1. \tag{4.1.8}$$

For $k = 2$ this reads $u_1 \wedge u_2(t_1, t_2) = u_1(t_1)u_2(t_2) - u_1(t_2)u_2(t_1)$, from which follows

$$u \wedge v = -v \wedge u, \quad u, v \in V_1. \tag{4.1.9}$$

Take $u = v$ in (4.1.9), then one obtains

$$u \wedge u = 0, \quad u \in V_1. \tag{4.1.10}$$

Formula (4.1.8) will be used in particular when the $u_i$ are elements of a basis $e_1, \ldots, e_d$ of $V_1$. Then for any $1 \le i_1, \ldots i_k \le d$ the function $e_{i_1} \wedge \cdots \wedge e_{i_k}$ is an element of $V_k$. It follows from (4.1.8) that this function changes sign if any two subscripts on the $e$'s are interchanged (in particular, the function is 0 if two subscripts are equal) so that we only have to consider $i_1 < \cdots < i_k$. It will be shown now that

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \le i_1 < \cdots < i_k \le d\} \tag{4.1.11}$$

is a basis of $V_k$. In order to shorten the notation define $S =$ all sequences $s = (i_1, \ldots, i_k)$ with $1 \le i_1 < \cdots < i_k \le d$. Put

$$\varepsilon_s = e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad (i_1, \ldots, i_k) = s \in S. \tag{4.1.12}$$

Let $(t_1, \ldots, t_d)$ be the basis of $M_p$ dual to $(e_1, \ldots, e_d)$ and define

$$\tau_s = (t_{i_1}, \ldots, t_{i_k}), \quad (i_1, \ldots, i_k) = s \in S. \tag{4.1.13}$$

If in (4.1.8) on the left-hand side $u_1 \wedge \cdots \wedge u_k$ is replaced by $\varepsilon_s$ and $(t_1, \dots, t_k)$ by $\tau_s$, defined by (4.1.12) and (4.1.13), then on the right-hand side of (4.1.8) only the term with $\pi$ the identity permutation survives and yields 1:

$$\varepsilon_s(\tau_s) = 1, \quad s \in S. \tag{4.1.14}$$

On the other hand, again by (4.1.8),

$$\varepsilon_s(\tau_{s'}) = 0 \quad \text{if } s' \neq s, \tag{4.1.15}$$

in which $s'$ is an arbitrary sequence $(i_1, \dots, i_k)$. Now if $u$ is an arbitrary element of $V_k$, then the properties of linearity and alternation imply that $u$ is determined by its values on the $\tau_s$ of (4.1.13). I.e., if two elements of $V_k$ coincide on each $\tau_s$, then they must be the same function. It follows then from (4.1.14) and (4.1.15) that

$$u = \sum_{s \in S} u(\tau_s)\varepsilon_s \tag{4.1.16}$$

since the right-hand side is also an element of $V_k$ and both sides agree on each $\tau_s$, $s \in S$. Moreover, again by (4.1.14) and (4.1.15), the $\varepsilon_s$ are linearly independent. Hence, (4.1.16) shows that (4.1.11) is a basis of $V_k$. It also follows that $\dim V_k$ = number of elements of $S = \binom{d}{k}$.

We have seen above that every element of $V_k$ $(1 \leq k \leq d)$ is a linear combination of wedge products of the from (4.1.12). The factors in such a product are elements of $V_1$ and the coefficients are in $R$, i.e., are elements of $V_0$. Therefore, $V$ defined in (4.1.3) is generated by $V_0$ and $V_1$ (i.e., is a sum of products of elements of $V_0$ and $V_1$). One can also define a Grassmann algebra $V$ over $V_1$ abstractly, where now $V_1$ is a given $d$-dimensional vector space over a field $F$, by requiring the following properties of $V$: (i) $V$ is an associative algebra over $F$ with an identity element; (ii) $V$ contains $V_1$; (iii) $u \wedge u = 0$ for every $u \in V_1$; (iv) $V$ is generated by $F$ and $V_1$; (v) $\dim V = 2^d$. It can be shown that these conditions determine the algebra uniquely (Cohn, 1957, Theorem 4.1.1; Bishop and Crittenden, 1964, Section 4.3, Remark (2)). Our Grassmann algebra, defined in terms of multilinear alternating functions, is a special case with $F = R$. The identity element is the number 1, and condition (iii) is satisfied in view of (4.1.10).

**4.2. Differential forms.** In Section 4.1 a Grassmann algebra $V$ has been defined at an arbitrary $p \in M$. In order to show its dependence on $p$ write $V(p)$. It is a sum of elements of the spaces $V_k(p)$, by (4.1.3), where $V_0(p) = R$, and for $1 \leq k \leq d$, $V_k(p)$ is a $\binom{d}{k}$-dimensional vector space of $k$-linear alternating functions on $M_p^{\times k}$. A member of $V_k(p)$ is called **homogeneous** of degree $k$. If at $p$ a chart is chosen with local coordinates $x_1, \ldots, x_d$, then $dx_1, \ldots, dx_d$ is a basis of $M_p^* = V_1(p)$. Therefore, after replacing in (4.1.11) the $e_i$ by the $dx_i$, we have that elements $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ of $V_k(p)$, $1 \leq i_1 < \cdots < i_k \leq d$, form a basis of $V_k(p)$ so that an arbitrary element $\omega_p$ of $V_k(p)$ can be written as

$$\omega_p = \sum \alpha_{i_1,\ldots,i_k}(p) dx_{i_1} \wedge \cdots \wedge dx_{i_k}, \tag{4.2.1}$$

where the sum is over all $1 \leq i_1 < \cdots < i_k \leq d$, and the $\alpha_{i_1,\ldots,i_k}(p)$ are any real numbers.

A **differential form of degree** $k$ $(1 \leq k \leq d)$ is a function $\omega$ that assigns to each $p \in M$ an element $\omega_p$ of $V_k(p)$. Then if $X_1, \ldots, X_k$ are any vector fields on $M$, $\omega(X_1, \ldots, X_k)$ is a real valued function on $M$ whose value at $p \in M$ equals $\omega_p(X_1(p), \ldots, X_k(p))$. In order to make the notion of differential form useful more smoothness is needed.

4.2.1. DEFINITION. *Let $M_0$ be an open subset of the $C^\infty$ manifold $M$ and $\omega$ a differential form of degree $k$. Then $\omega$ is said to be a $C^\infty$* differential form of degree *$k$, or simply a* $k$-**form**, *on $M_0$ if every point of $M_0$ has a neighborhood $U$ such that for any $C^\infty$ vector fields $X_1, \ldots, X_k$ on $U$ the function $\omega(X_1, \ldots, X_k)$ is $C^\infty$ on $U$. An* analytic *$k$-form is similarly defined, with "$C^\infty$" replaced everywhere by "analytic."*

In applications $M_0$ is usually all of $M$. A 1-form is also called a **Pfaffian** form; a special case of this is the differential $df$ of a $C^\infty$ real valued function $f$. It is also convenient to define a 0-form as any $C^\infty$ real valued function on $M$.

A definition equivalent to Definition 4.2.1 can be given in terms of charts. Let $M_0$ be covered by a family of charts and consider in (4.2.1) the $\alpha$'s as functions of the local coordinates. Then $\omega$ is $C^\infty$ (resp. analytic) if and only if the $\alpha$'s are $C^\infty$ (resp. analytic). This definition does not depend on the choice of admissible charts. It may not always be necessary to require the $\alpha$'s to be $C^\infty$. For instance, in some applications it may be sufficient for the $\alpha$'s to be continuous, in which case $\omega$ is called a **continuous differential form.**

A $k$-form is a special case of a **covariant tensor** of order $k$. The latter entity is more general in that the alternating property is not imposed; only the $k$-linearity is retained. In this monograph tensors will not be used.

**4.3. Change of variables.** If at $p \in M$ a second admissible chart is chosen with local coordinates $y_1, \dots, y_d$, then $\omega_p$ can also be written in terms of the $dy_j$. In order to derive this expression from (4.2.1) first express the $dx_i$ in terms of the $dy_j$, using (3.3.8) with the roles of the $x$'s and $y$'s interchanged:

$$dx_i = \sum_{j=1}^{d} \left.\frac{\partial x_i}{\partial y_j}\right|_p dy_j, \quad i = 1, \dots, d. \tag{4.3.1}$$

Then substitute the expressions (4.3.1) into the right-hand side of (4.2.1). From (4.1.8) it is seen that a wedge product is linear in each factor, hence the substitution of (4.3.1) into (4.2.1) produces a linear combination of terms of the form $dy_{j_1} \wedge \cdots \wedge dy_{j_k}$, where we need consider only $1 \le j_1 < \cdots < j_k \le d$ by virtue of (4.1.9) and (4.1.10). For us the most important case is $k = d$ and then $V_d$ is spanned by the single form $dx_1 \wedge \cdots \wedge dx_d$. Substitution of (4.3.1) and linearity produces

$$dx_1 \wedge \cdots \wedge dx_d = \sum \left.\frac{\partial x_1}{\partial y_{j_1}} \cdots \frac{\partial x_d}{\partial y_{j_d}}\right|_p dy_{j_1} \wedge \cdots \wedge dy_{j_d} \tag{4.3.2}$$

in which the summation is over all $j_i = 1, \dots, d$, $i = 1, \dots, d$. However, $dy_{j_1} \wedge \cdots \wedge dy_{j_d} = 0$ unless $j_1, \dots, j_d$ is a permutation, say $\pi \in G_d$, of $1, \dots, d$ and then $dy_{j_1} \wedge \cdots \wedge dy_{j_d} = \mathrm{sgn}(\pi) dy_1 \wedge \cdots \wedge dy_d$.

Thus, the right-hand side of (4.3.2) reduces to $dy_1 \wedge \cdots \wedge dy_d$ multiplied by the coefficient

$$\sum_{\pi \in G_d} \operatorname{sgn}(\pi) \frac{\partial x_1}{\partial y_{\pi 1}} \cdots \frac{\partial x_d}{\partial y_{\pi d}} \bigg|_p = \frac{\partial(x)}{\partial(y)} \bigg|_p \tag{4.3.3}$$

in which

$$\frac{\partial(x)}{\partial(y)} = \det\left(\left(\frac{\partial x_i}{\partial y_j}\right)\right). \tag{4.3.4}$$

(There is a slight conflict of notation with (3.1.1) in that in (4.3.4) we do not take the absolute value of the right-hand side. It will usually be clear from the context which of the two definitions of Jacobian is intended.) From (4.3.2) and (4.3.3) we get

$$dx_1 \wedge \cdots \wedge dx_d = \frac{\partial(x)}{\partial(y)} dy_1 \wedge \cdots \wedge dy_d, \tag{4.3.5}$$

in which we now consider both sides as a $d$-form by letting $p$ vary over a neighborhood in which the local coordinates $x_1, \ldots, x_d$ as well as $y_1, \ldots, y_d$ are defined. This shows that on such a neighborhood the $d$-form $dx_1 \wedge \cdots \wedge dx_d$ can be used for integration as a volume element since (4.3.5) is the usual formula for the transformation of a volume element under a change of variables.

The result of the computation that led from (4.3.1) to (4.3.5) is restated below as a lemma in a form useful for later applications. Note that the roles of $x$ and $y$ are interchanged.

4.3.1 LEMMA. *Let $dx_1, \ldots, dx_d$ and $dy_1, \ldots, dy_d$ be* 1-forms *related by $dy = Adx$, in which $dx$ and $dy$ are $d \times 1$ column vectors with elements $dx_i$, $dy_i$, $i = 1, \ldots, d$, respectively, and $A$ is $d \times d$. Then*

$$dy_1 \wedge \cdots \wedge dy_d = (\det A) dx_1 \wedge \cdots \wedge dx_d. \tag{4.3.6}$$

This result is used to determine how the Lebesgue measure of a subset of $R^d$ transforms under a linear transformation of $R^d$. On $R^d$

Lebesgue measure $\lambda$ is $\lambda(dx) = dx_1 \wedge \cdots \wedge dx_d$, except that we discard any negative sign (this will be made more precise in Section 6.6). Define $(dx)$ to be the absolute value of the wedge product $dx_1 \wedge \cdots \wedge dx_d = \wedge_i dx_i$, say. I.e., $(dx) = |\wedge_i dx_i|$, and similarly $(dy) = |\wedge_i dy_i|$. Then (4.3.6) reads

$$(dy) = |\det A|(dx). \tag{4.3.7}$$

**4.4. Orientation.** A $C^\infty$ manifold $M$ of dimension $d$ is called **orientable** if it admits a continuous $d$-form that does not vanish anywhere on $M$. (See Chevalley, 1946, Chap. V, §VI. For a different definition that is equivalent for paracompact spaces (defined in Section 13.3) see Bishop and Crittenden, 1964, Section 4.5, Lemma 3.) In terms of coordinates, if $M$ is covered by a family of charts, then on each chart with local coordinates $x_1, \ldots, x_d$, the $d$-form $\omega$ is represented by an expression of the form

$$\omega_{p(x)} = \alpha(x) dx_1 \wedge \cdots \wedge dx_d, \tag{4.4.1}$$

where $p(x) \in M$ is the point corresponding to the coordinates $x = (x_1, \ldots, x_d)$. Then $M$ is orientable if and only if on every chart $\alpha$ is continuous and never 0. This clearly does not depend on the choice of charts since under a change of variables from $x$ to $y$ the new function $\alpha$ is the old one multiplied by (4.3.4) which is continuous and $\neq 0$.

If $M$ is orientable and $\omega$ is a continuous nonvanishing $d$-form on $M$, then $C^\infty$ vector fields $X_1, \ldots, X_d$ on $M$ exist such that $\omega(X_1, \ldots, X_d) > 0$ everywhere on $M$. It amounts to the same by saying that for every $p \in M$ there is an ordered basis of $M_p$ such that $\omega_p$ evaluated at this basis is positive. One says that $M$ is **positively oriented** by this choice of basis. In contrast, $M$ is **negatively oriented** by vector fields $Y_1, \ldots, Y_d$ if $\omega(Y_1, \ldots, Y_d) < 0$ everywhere on $M$ (take, e.g., the $Y$'s an odd permutation of the $X$'s).

The geometric meaning of orientability can best be understood by some examples. In the Euclidean plane $R^2$ with coordinates $x$, $y$, the 2-form $\omega = dx \wedge dy$ is continuous and defined on all of $R^2$. For the vector fields $\partial/\partial x$, $\partial/\partial y$, we have $\omega(\partial/\partial x, \partial/\partial y) = dx(\partial/\partial x)dy(\partial/\partial y) -$

$dx(\partial/\partial y)dy(\partial/\partial x) = 1 - 0 = 1$. Thus, $(\partial/\partial x, \partial/\partial y)$ orients $R^2$ positively. On the other hand, $(\partial/\partial y, \partial/\partial x)$ orients $R^2$ negatively. Similar considerations show that $R^n$ is orientable for every $n \geq 1$. For $n = 3$ and $\omega = dx \wedge dy \wedge dz$, the three even permutations of $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ orient $R^3$ positively, the odd permutations negatively.

We shall show now that the unit circle $C$ is also orientable. Let $C$ analytically be defined by the unit interval $0 \leq x \leq 1$ with $x = 0$ and $x = 1$ identified as the same point, say $p_0$. We cover $C$ by two charts; one is $(0,1)$ with $x$ as its coordinate; the other is the union of $0 \leq x < \frac{1}{2}$ and $\frac{1}{2} < x \leq 1$ with coordinate $u = x$ if $0 \leq x < \frac{1}{2}$ and $u = x - 1$ if $\frac{1}{2} < x \leq 1$, so that $-\frac{1}{2} < u < \frac{1}{2}$. Let $\omega$ be defined as $dx$ on the first chart and as $du$ on the second. Wherever the charts overlap it is seen that $dx = du$; thus, $\omega$ is well defined. Also, $\omega$ is continuous and nowhere 0. The vector field $X$ that equals $\partial/\partial x$ on the first, $\partial/\partial u$ on the second chart, orients $C$ positively, whereas $-X$ orients it negatively. This corresponds to the two distinct ways of going around the circle. Similarly, the 2-sphere $x^2 + y^2 + x^2 = 1$ is orientable.

The circle example can be extended by crossing $C$ with the real line $R$, obtaining a cylinder $C \times R$. Since both $C$ and $R$ are orientable, so is $C \times R$. Analytically a cylinder can be represented by $\{(x,y) : 0 \leq x \leq 1,\ -\infty < y < \infty,\ (0,y) = (1,y) \text{ for every } y\}$. That is, the cylinder is a vertical strip in the plane with the left and right edges identified. Now change this example by identifying those edges in opposite direction:

$$M = \{(x,y) : 0 \leq x \leq 1, -\infty < y < \infty,\quad (0,y) = (1,-y) \quad \text{for every } y\}. \tag{4.4.2}$$

Then $M$ is a **Moebius strip** and is *not* orientable, as will be shown now. Cover $M$ by two charts. The first covers all points with $0 < x < 1$, and we may take $(x,y)$ as the coordinates in this chart. The second covers the points with $0 \leq x < \frac{1}{2}$, and $\frac{1}{2} < x \leq 1$. On this chart the coordinates are chosen $(u,v)$, with $u = x$, $v = y$ when $0 \leq x < \frac{1}{2}$

and $u = x - 1$, $v = -y$ when $\frac{1}{2} < x \leq 1$. Suppose there were a continuous and nowhere vanishing 2-form $\omega$ on $M$, represented on the first chart by $\alpha(x,y)dx \wedge dy$ and on the second by $\beta(u,v)du \wedge dv$, with $\alpha, \beta$ continuous in their arguments. These two expressions are to be equated at all points where the charts overlap. For the points with $0 < x < \frac{1}{2}$, therefore $0 < u < \frac{1}{2}$, this gives $\beta(u,v) = \alpha(u,v)$, and for the points with $-\frac{1}{2} < u < 0$ we get $\beta(u,v) = -\alpha(u+1,-v)$ (using $du \wedge dv = -dx \wedge dy$ on this set). Since $\alpha$ is continuous on the strip $0 < x < 1$ and nowhere 0, it must be of one sign. WLOG suppose $\alpha > 0$ on $0 < x < 1$. Then the equations $\beta(u,v) = \alpha(u,v)$ for $u > 0$ and $\beta(u,v) = -\alpha(u+1,-v)$ for $u < 0$ show that $\beta(u,v) > 0$ for $u > 0$ and $< 0$ for $u < 0$. If $\beta$ is to be continuous at $u = 0$ we must have $\beta(0,v) = 0$ for every $v$. But then $\omega = 0$ in the points $(0,y)$, contradicting the assumption that $\omega$ does not vanish anywhere.

**4.5. Adjoint of a differential.** Let $M$ and $N$ be $C^\infty$ manifolds, with $\dim M = d$, $\dim N = e$, and let $f : M \to N$ be a $C^\infty$ mapping. We have seen in Chapter 3 that *df* is a linear transformation $M_p \to N_q$, for $p \in M$ and $q = f(p)$. This linear transformation also transforms differential forms, but in the opposite direction. Let $\theta$ be a $k$-form on $N$, then it determines a $k$-form $\omega$ on $M$ by the formula

$$\omega(X_1, \dots, X_k) = \theta(df X_1, \dots, df X_k), \tag{4.5.1}$$

for arbitrary $C^\infty$ vector fields $X_1, \dots, X_k$ on $M$. We shall denote this linear map $\theta \to \omega$ by $\delta f$ and call $\delta f$ the **adjoint** of *df*. (It is a special case of the general notion of the adjoint of a linear transformation on one linear space into another. See Dunford and Schwartz, 1958, VI 2.1.) Thus, (4.5.1) can also be written $\omega = \delta f(\theta)$. In terms of charts, suppose $p \in M$ has a neighborhood $U_p$ with local coordinates $x = (x_1, \dots, x_d)$ and $q = f(p) \in N$ has a neighborhood $V_q \supset f(U_p)$ with local coordinates $y = (y_1, \dots, y_e)$, then a $k$-form on $V_q$ has the form

$$\theta = \sum \beta_{i_1,\dots,i_k}(y) dy_{i_1} \wedge \cdots \wedge dy_{i_k} \tag{4.5.2}$$

where the sum is over all $1 \leq i_1 < \cdots < i_k \leq e$. In order to write $\omega = \delta f(\theta)$ in terms of the $x$'s, express in (4.5.2) $y$ as a function of $x$ with help of $f$; the $dy_i$ transform as in (3.3.8). With the notation $D_{ij} = \partial y_i / \partial x_j$ and taking as an example $k = 2$, the 2-form $\theta = dy_1 \wedge dy_2$ becomes $\delta f(\theta) = (\sum D_{1j_1} dx_{j_1}) \wedge (\sum D_{2j_2} dx_{j_2}) = \sum (D_{1j_1} D_{2j_2} - D_{1j_2} D_{2j_1}) dx_{j_1} \wedge dx_{j_2}$, where the last sum is over all $1 \leq j_1 < j_2 \leq d$.

Now suppose that $\dim M = \dim N = d$ and suppose that $f$ is a diffeomorphism $M \to N$. At $p \in M$ let there be a chart with local coordinates $x = (x_1, \ldots, x_d)$, and similarly at the corresponding point $q = f(p) \in N$ a chart with $y = (y_1, \ldots, y_d)$. Then locally $y$ is a $C^\infty$ function of $x$. Consider on the $y$-chart the $d$-form $\theta = dy_1 \wedge \cdots \wedge dy_d$. Then its image $\delta f(\theta)$ on the $x$-chart is

$$\delta f(dy_1 \wedge \cdots \wedge dy_d) = \frac{\partial(y)}{\partial(x)} dx_1 \wedge \cdots \wedge dx_d \tag{4.5.3}$$

by the same computation that led to (4.3.5) (the Jacobian on the right-hand side of (4.5.3) is defined in (4.3.4)). In particular, suppose $M = N$ and $G$ is a group with $C^\infty$ action on $M$. Then (4.5.3) can be applied to each diffeomorphism of $M$ with itself determined by $g \in G$. This will be used in Section 5.3 when $M$ is $G$ itself.

Consider again arbitrary $C^\infty$ manifolds $M$ and $N$ and $f : M \to N$ a $C^\infty$ mapping. Suppose $\theta_1$ is a $k$-form and $\theta_2$ an $\ell$-form on $N$; let $\theta_1 \wedge \theta_2$ be their wedge product. Then by using (4.1.4) or (4.1.6) it is easily verified that

$$\delta f(\theta_1 \wedge \theta_2) = \delta f(\theta_1) \wedge \delta(\theta_2). \tag{4.5.4}$$

This can of course be extended to any number of factors, and is especially useful if each factor is a 1-form. We shall apply it in Section 5.3.

**4.6. Exterior differentiation.** (This concept will not be used in the sequel but is described here briefly since it fits in naturally with differential forms.) Let $\omega$ be a $k$-form on an open subset of $M$, $1 \leq k \leq d$. From $\omega$ we build a $(k+1)$-form, written $d\omega$ and called the **exterior derivative** of $\omega$. At any $p \in M$ where $\omega$ is defined and for arbitrary $t_1, \dots, t_{k+1} \in M_p$ we have to define the value of $(d\omega)_p(t_1, \dots, t_{k+1})$. For this purpose let $X_1, \dots, X_{k+1}$ be any $C^\infty$ vector fields such that $X_i(p) = t_i$, $i = 1, \dots, k+1$. Also, in order to shorten the notation, if $(a_1, \dots, a_n)$ is a sequence of any $n$ objects, let $a_{(n)\setminus i}$ stand for the sequence $(a_1, \dots, a_n)$ with $a_i$ deleted. Similarly, $a_{(n)\setminus i,j}$ is the sequence with both $a_i$ and $a_j$ deleted. Then define

$$(4.6.1)\qquad \begin{aligned}(d\omega)_p(t_1, \dots, t_{k+1}) = \Bigg\{ &\sum_{1 \le i \le k+1} (-1)^{i-1} X_i\, \omega(X_{(k+1)\setminus i}) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \omega([X_i, X_j], X_{(k+1)\setminus i,j}) \Bigg\}(p)\end{aligned}$$

(i.e., the function in curly brackets on the right-hand side of (4.6.1) is to be evaluated at $p$). Note that the $i$th term of the first sum on the right-hand side of (4.6.1) is the derivative with respect to the tangent vector $X_i(p)$ of the function $\omega(X_{(k+1)\setminus i})$. It can be shown that the right-hand side of (4.6.1) is independent of the choice of $X_1, \dots, X_{k+1}$ provided $X_i(p) = t_i$, and that $(d\omega)_p \in V_{k+1}(p)$ (see Bishop and Crittenden, 1964, Section 4.6). As an example let $k = 1$, so that $d\omega$ is a 2-form. Then (4.6.1) reads

$$(4.6.2)\qquad (d\omega)_p(t_1, t_2) = \{X_1\omega(X_2) - X_2\omega(X_1) - \omega([X_1, X_2])\}(p),$$

with any $X_1, X_2$ such that $X_i(p) = t_i$.

With help of a chart it is easy to write down $d\omega$ in terms of $\omega$ in a neighborhood of a point. Suppose in terms of local coordinates $x_1, \dots, x_d$ at $p$ we have

$$(4.6.3)\qquad \omega = \sum \alpha_{i_1, \dots, i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

where the summation is over all $1 \le i_1 < \dots < i_k \le d$, and the $\alpha_{i_1, \dots, i_k}$ are $C^\infty$ functions. Then

$$(4.6.4)\qquad d\omega = \sum d\alpha_{i_1, \dots, i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

This follows from (4.6.1) by taking for $(X_1, \ldots, X_{k+1})$ the sequence $(\partial/\partial x_{j_1}, \ldots, \partial/\partial x_{j_{k+1}})$ for every choice of $1 \leq j_1 < \cdots < j_{k+1} \leq d$, and by observing that these vector fields have zero brackets. The exterior derivative can also be defined with help of (4.6.4) (as is done in Cohn, 1957, Section 4.3) but it has to be shown then that this definition does not depend on the choice of charts.

If $\omega$ is a 0-form, then $\omega$ is a $C^\infty$ function, say $f$. In that case we define $d\omega$ simply as $df$. For any $k$, $0 \leq k \leq d$, the exterior derivative is linear as a function on the space of $k$-forms into the space of $(k+1)$-forms. Additionally, the following properties can be shown: (i) if $\omega$ is a $k$-form and $\theta$ an $\ell$-form on $M$, then $d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^k \omega \wedge (d\theta)$; (ii) $d^2 = 0$, i.e., $d(d\omega) = 0$ for any differential form $\omega$.