A STRONG LIMIT THEOREM FOR PROCESSES WITH ASSOCIATED INCREMENTS

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A finite collection of random variables $X_1, \ldots, X_m$ is said to be associated if for any two coordinatewise nondecreasing functions f, g on R^m

$$\mathrm{Cov}(f(X_1, \ldots, X_m), g(X_1, \ldots, X_m)) \geq 0$$

whenever the covariance is defined; a stochastic process $X(t)$ is said to have associated increments if for each t, $X(t)$ and the increments of the process in (t, ∞) are associated.

THEOREM. *If $X(t)$ is a separable, mean zero stochastic process with associated increments, and $H(t) \uparrow \infty$ is positive and continuous and such that $\int_0^\infty \frac{d\sigma(t)}{H(t)} < \infty$, where $\sigma(t)$ = standard deviation $(X(t))$, then $\frac{X(t)}{H(t)} \to 0$ a.s.*

1. Introduction. Let $\{X_j, 1 \leq j \leq m\}$ be a collection of random variables. The collection is said to be associated if for any coordinatewise non-decreasing functions f, g on R^m, $\mathrm{Cov}(f(X_1, \ldots, X_m), g(X_1, \ldots, X_m)) \geq 0$ whenever the covariance is defined; an infinite collection is associated if every finite subcollection is associated. In Newman and Wright (1982) it was shown that associated random variables satisfy several of the classical martingale inequalities.

In this paper we consider continuous time stochastic processes. A stochastic process $X(t)$ is said to have associated increments if for each t, $X(t)$ and the increments of the process in (t, ∞) are associated. Such processes have many of the sample function properties that a separable submartingale has. Many of these properties are discussed in Wood (1983), in the more general case of (continuous time) demimartingales. These properties also hold for separable and centered processes with independent increments. In our case we assume the process is separable, mean zero, with *associated* increments. One interesting consequence shown here is that a separable, mean zero process with associated increments is automatically centered.

The properties above allow us to prove the following strong limit theorem: Let $\{X(t), t \geq 0\}$ be a separable, mean zero process with associated increments. If

AMS 1980 subject classifications. 60G17, 60G44, 60J30.

Key words and phrases. Associated random variables, processes with independent increments, infinite divisibility.

$\int_0^\infty \frac{d\sigma(t)}{H(t)} < \infty$, where $\sigma(t)$ = standard deviation $(X(t))$, and $H(t) \uparrow \infty$ is positive and continuous, then $\frac{X(t)}{H(t)} \overset{\text{a.s.}}{\to} 0$.

2. Associated Processes. Let $\{X(t), t \geq 0\}$ be a stochastic process. We say that the process has associated increments if for any $n \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_n$ the random variables $X(t_1)$, $X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are associated.

We first discuss almost sure sample function properties of such processes. Outside a set of probability zero, the sample functions are bounded on every bounded interval, have finite left and right-hand limits at each point, and their discontinuities are jumps, outside a fixed and countable set T^*. These properties are known to hold for separable submartingales and separable centered processes with independent increments (Doob, 1953, pages 361, 422). In Wood (1983) it is mentioned that the first two properties above hold in the demimartingale case.

We will need the following two lemmas:

LEMMA 1. *Let $\{X_j, 1 \leq j \leq n\}$ be a sequence of associated, mean zero random variables, and set $S_j = \sum_{j=1}^n X_j$. Then for $\lambda > 0$*

$$P\left\{\max_{1\leq j\leq n} S_j \geq \lambda\right\} \leq \frac{E|S_n|}{\lambda} \leq \frac{\sigma(S_n)}{\lambda}.$$

The proof may be found in Newman and Wright, 1982, Theorem 3, page 363.

For the next lemma, we first recall a version of Doob's upcrossing inequality which holds for demimartingales:

Let $\{X_j, 1 \leq j \leq n\}$ be as in Lemma 1, and define a sequence of stopping times $J_0 = 0, J_1, J_2 \ldots$ as follows (for $k = 1, 2, \ldots$):

$$J_{2k-1} = \begin{cases} n+1, \text{ if } \{j : J_{2k-2} < j \leq n \text{ and } S_j \leq a\} \text{ is empty} \\ \min\{j : J_{2k-2} < j \leq n \text{ and } S_j \leq a\}, \text{ otherwise} \end{cases}$$

$$J_{2k} = \begin{cases} n+1, \text{ if } \{j : J_{2k-1} < j \leq n \text{ and } S_j \geq b\} \text{ is empty} \\ \min\{j : J_{2k-1} < j \leq n \text{ and } S_j \geq b\}, \text{ otherwise} \end{cases}$$

We define the number of complete upcrossings of [a, b] by $S_1, \ldots, S_n$ by $U_{a,b} = \max\{k : J_{2k} < n+1\}$. We have the following:

LEMMA 2. *If $\{X_j, 1 \leq j \leq n\}$ is a sequence of mean zero, associated random variables, then for any $a < b$,*

$$E(U_{a,b}) \leq \frac{E((S_n - a)^+) - E((S_1 - a)^+)}{b - a}$$

The proof may be found in Newman and Wright, 1982, Theorem 7.

We now prove the theorem of this section.

THEOREM 3. *Let* $\{X(t), t \geq 0\}$ *be a separable,* mean zero *process with associated increments. Then outside a set of probability zero, the sample functions have the following properties:*

(a) they are bounded on every finite interval,

(b) they have finite right and left-hand limits at every point, and

(c) their discontinuities are jumps, except possibly for some points $t \epsilon T^*$ *where* T^* *is a fixed countable set.*

PROOF. For $T > 0$ finite, $0 = t_1 < t_2 < \cdots < t_n = T$, and $\lambda > 0$ we have, by Lemma 1 and the definition of a process with associated increments,

$$\lambda P\left\{\max_{1\leq j\leq n} X_{t_j} \geq \lambda\right\} \leq \int_\Omega |X_T| dP < \infty.$$

Taking successively finer partitions of $[0, T]$ and using separability,

$$\lambda P\left\{\sup_{t\epsilon[0,T]} X_t > \lambda\right\} \leq \int_\Omega |X_T| dP < \infty.$$

It follows that the process is bounded from above with probability one. Also, from the definition of separability, with probability one, for $\lambda > 0$,

$$\lambda P\left\{\inf_{t\epsilon[0,T]} X_t < -\lambda\right\} = \lambda P\left\{\inf_{t_j\epsilon[0,T]} X_{t_j} < -\lambda\right\} = \lambda P\left\{\sup_{t_j\epsilon[0,T]} -X_{t_j} > \lambda\right\},$$

where $\{t_j, j = 1, 2, \ldots\}$ is a separating set. But $\{-X_t, t \geq 0\}$ is a mean zero process with associated increments, so it follows from Lemma 1 that

$$\lambda P\left\{\sup_{t\epsilon[0,T]} -X_t > \lambda\right\} \leq E|X_T| < \infty,$$

or

$$\lambda P\left\{\inf_{t\epsilon[0,T]} X_t < -\lambda\right\} \leq E|X_T| < \infty.$$

It follows that the process is bounded from below with probability one, completing the proof of (a).

The proof of (b) follows directly from Doob, 1953, page 361, using our Lemma 2 in place of the usual upcrossing inequality.

For (c), we note that for a sample function satisfying (a) and (b) above, if $\{t_j, j \geq 1\}$ is a separating set for the process, any discontinuity at a point other than a t_j must be a jump. Since the separating set is denumerable, we have proved (c).

3. Centered Processes. Recall that a stochastic process $\{Z_t, t \geq 0\}$ is said to be centered if:

(a) For each $t > 0$, and sequence $s_n \to t$ with $s_n < t, \lim_{n\to\infty} Z_{s_n} = Z_{t-}$ exists with probability one, and for each $t \geq 0$ and sequence $s_n \to t$ with $s_n > t$, $\lim_{n\to\infty} Z_{s_n} = Z_{t+}$ exists with probability one.

(b) If any difference $Z_t - Z_s$ (or $Z_{t+} - Z_s, Z_t - Z_{t-}$, etc.), is identically constant with probability one, the constant is equal to zero.

(c) Except (possibly) for the points of an enumerable set $S \subset [0, \infty)$, the following holds with probability one: $Z_{t-} = Z_t = Z_{t+}$.

We have shown that any separable, mean zero process with associated increments satisfies (a) and (b). That such a process also satisfies (c), and hence is centered, follows from the following theorem:

Let $\{X(t), t \geq 0\}$ be a stochastic process. If for every $t > 0$ at least one of the limits in probability $\lim_{s\uparrow t} X_s = X_{t-}, \lim_{s\downarrow t} X_s = X_{t+}$ exists, then there is an at most enumerable subset $T \subset (0, \infty)$ such that for all $t \epsilon (0, \infty) \setminus T$, both stochastic limits X_{t-} and X_{t+} are defined, and $X_{t-} = X_{t+} = X_t$ with probability 1 (Doob, 1953, Theorem 11.1, page 356).

In particular, a separable, mean zero process with independent increments is centered. It is easy to show that any mean zero, L^2 process with independent increments satisfies (a) and (b) of the definition above (Wright, 1982, Theorem 2, page 110), but may not be centered, if it is not separable.

A trivial example of an L^2, separable and infinitely divisible process with independent increments which is *not* centered is the following:

$$Z(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < \infty. \end{cases}$$

4. A Strong Limit Theorem. In this section we prove a strong limit theorem for stochastic processes with associated increments. For its proof we make use of separability, Lemma 1, Theorem 3, and the easily proven fact that for a process $\{X(t), t \geq 0\}$ with associated increments, $\sigma(X(t) - X(s)) \leq \sigma(t)$ for $s < t$.

THEOREM 4. *Let $\{X(t), t \geq 0\}$ be an L^2, mean zero and separable process with associated increments. If $H(t) \uparrow \infty$ is positive and continuous and such that*

$$\int_0^\infty \frac{\sigma(t)}{H(t)} < \infty, \text{ then } \frac{X(t)}{H(t)} \overset{a.s.}{\to} 0.$$

PROOF. It is easy to show that we may choose points $0 \leq t_0 < t_1 < \cdots$ such that $\frac{\sigma(t_j)}{H(t_j)} < \frac{1}{2^{2j}}$ and $\int_{t_j}^\infty \frac{\sigma(t)}{H(t)} < \frac{1}{2^{2j}}$ for $j = 1, 2, \ldots$. For such a sequence we have for $t_{j-1} \leq t < t_j, j = 1, 2, \ldots$

$$\frac{|X(t)|}{H(t)} < \frac{1}{H(t)}\{|X(t_0)| + \sum_{i=1}^{j-1}\{\sup_{t_{i-1}<s\le t_i}|X(s)-X(t_{i-1})|\} + |X(t)-X(t_{j-1})|\}. \tag{1}$$

Trivially, $\frac{|X(t_0)|}{H(t)} \xrightarrow{\text{a.s.}} 0$. As for the second term on the right-hand side, we note that if $t_{j-1} = t_{j,0} < t_{j,1} < \cdots < t_{j,N_j} = t_j$ is any partition of $[t_{j-1}, t_j]$, then by Lemma 1,

$$P\left[\max_{1\le k\le N_j}\frac{|X(t_{j,k})-X(t_{j,0})|}{H(t_j)} \ge \frac{1}{2^j}\right] \le \frac{\sigma(X(t_j)-X(t_{j-1}))}{H(t_j)2^{-j-1}}.$$

Since $\{-X(t), t\ge 0\}$ has the same properties as $\{X(t), t\ge 0\}$ it follows that

$$\begin{aligned} P\left[\max_{1\le k\le N_j}\frac{|X(t_{j,k})-X(t_{j,0})|}{H(t_j)} \ge \frac{1}{2^j}\right] &\le \frac{\sigma(X(t_j)-X(t_{j-1}))}{H(t_j)2^{-j-1}} \\ &\le \frac{\sigma(t_j)}{H(t_j)2^{-j-1}} \le \frac{1}{2^{j-1}}. \end{aligned}$$

Using Theorem 3 we see that

$$P\left[\sup_{t_{j-1}<s\le t_j}\frac{|X(s)-X(t_{j-1})|}{H(t_j)} > \frac{1}{2^j}\right] \le \frac{1}{2^{j-1}}.$$

By the Borel-Cantelli theorem,

$$\sum_{j\ge 1}\sup_{t_{j-1}<s\le t_j}\frac{|X(s)-X(t_{j-1})|}{H(t_j)} < \infty \text{ a.s.}$$

From this and (1) it follows that we only need to show

$$X_j \equiv \sup_{t_{j-1}<s\le t_j}\frac{|X(s)-X(t_{j-1})|}{H(t)} \xrightarrow{\text{a.s.}} 0.$$

We proceed by picking points $\{s_{j,k} : 1 \le j < \infty, 0 \le k \le N_j\}$ as follows:

$$\begin{aligned} &t_{j-1} = s_{j,0} < s_{j,1} < \cdots < s_{j,N_j} \quad \text{with} \quad s_{j,N_j} - 1 < t_j \le s_{j,N_j} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{and} \quad H(s_{j,i}) = 2H(s_{j,i-1}) \end{aligned} \tag{2}$$

It follows that

$$|X_j| \le \max_{0\le i\le N_j-1}\left\{\sup_{s_{j,i}<s\le s_{j,i+1}}\frac{|X(s)-X(s_{j,0})|}{H(s_{j,i})}\right\}.$$

From (1), (2), and Lemma 1,

$$P\left\{|X_j| > \frac{1}{2^j}\right\}$$

$$\begin{aligned}
&\leq \sum_{i=0}^{N_j-1} P\left\{\sup_{s_{j,i}<s\leq s_{j,i+1}} \frac{|X(s)-X(s_{j,0})|}{H(s_{j,i})} > \frac{1}{2^j}\right\} \\
&\leq 2^{j+1}\sum_{i=0}^{N_j-1}\left[\frac{\sigma(X(s_{j,i+1})-X(s_{j,0}))}{H(s_{j,i})}\right] \\
&\leq 2^{j+1}\sum_{i=0}^{N_j-1}\left\{\frac{\sigma(s_{j,i+1})-\sigma(s_{j,0})}{H(s_{j,i})}\right\} + 2^{j+1}\sum_{i=0}^{N_j-1}\frac{\sigma(s_{j,0})}{H(s_{j,i})} \\
&\leq 2^{j+1}\sum_{i=0}^{N_j-1}\{\sigma(s_{j,i+1})-\sigma(s_{j,i})\}\left[\frac{1}{H(s_{j,i})}+\frac{1}{H(s_{j,i+1})}+\cdots\right] + \frac{2^{j+2}\sigma(s_{j,0})}{H(s_{j,0})} \\
&\leq 2^{j+3}\sum_{i=0}^{N_j-1}\frac{\{\sigma(s_{j,i+1})-\sigma(s_{j,i})\}}{H(s_{j,i+1})} + \frac{2^{j+2}\sigma(s_{j,0})}{H(s_{j,0})} \\
&\leq 2^{j+3}\int_{t_{j-1}}^{\infty}\frac{d\sigma(t)}{H(t)} + \frac{2^{j+2}\sigma(s_{j,0})}{H(s_{j,0})} < 2^{-j+6}.
\end{aligned}$$

From the Borel-Cantelli Theorem we get $X_j \overset{\text{a.s.}}{\to} 0$, completing the proof of the theorem.

REMARKS. The converse of Theorem 4 is not true. For a counterexample, let $\{X(t), t \geq 0\}$ be standard Brownian motion. In this case

$$\int_0^\infty d\sigma(t) = \frac{1}{2}\int_0^\infty t^{-1/2}dt = \infty,$$

and it is easy to construct $H(t) \uparrow 0$, such that $H(t) = b(t)t^{1/2}$ for $t \geq 1$ with $b(t) \uparrow \infty$ and $\int_0^\infty \frac{d\sigma(t)}{H(t)} = \infty$. It follows that $\frac{X(t)}{H(t)} \to 0$.

If $\{X(t), t \geq 0\}$ is as above except that $EX(t) \equiv 0$ does not necessarily hold, we would like to assert that $\frac{X(t)-EX(t)}{H(t)} \overset{\text{a.s.}}{\to} 0$.

Since the process $\{X(t) - EX(t), t \geq 0\}$ has associated increments if $\{X(t), t \geq 0\}$ does, what is needed are conditions that imply that $\{X(t) - EX(t), t \geq 0\}$ is a separable process.

It is easy to show (Wood, 1983, page 4), that if $\{X(t), t \geq 0\}$ is a separable process and f is a function that has finite right and left-hand limits at every point $t > 0$, then $\{X(t) + f(t), t \geq 0\}$ is also a separable process. This observation leads to the following:

COROLLARY 5. *Let* $\{X(t), t \geq 0\}$ *be a separable and centered stochastic process with associated increments. If* $\sup_{t\epsilon[0,T]} EX^2(t) < \infty$ *for all* $T > 0$ *and* $H(t) \uparrow \infty$

is positive and continuous and such that $\int_0^\infty \frac{d\sigma(t)}{H(t)} < \infty$, *then* $\frac{X(t)-EX(t)}{H(t)} \to 0$.

The proof follows from noting that $\{X(t), 0 \leq t \leq T\}$ is uniformly integrable over $[0,T]$ for each $T > 0$, and that $\lim_{s\uparrow t} X(s) = X(t-)$ and $\lim_{s\downarrow t} X(s) = X(t+)$ exist a.s. for each $t > 0$, implying that $\lim_{s\uparrow t} EX(t)$ and $\lim_{s\uparrow t} EX(t)$ exist for each $t > 0$.

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