ESTIMATING A DISTRIBUTION FUNCTION BASED ON MINIMA-NOMINATION SAMPLING

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The nonparametric maximum likelihood estimator of a distribution function based on a maxima-nomination sample has been derived recently by Boyles and Samaniego (1986). In this article we study minima-nominations for the case of censored data.

1. Introduction. Let $X_{i1}, \ldots, X_{iK_i}$, $i = 1, \ldots, n$ be independent identically distributed (i.i.d.) random variables (r.v.'s) having a common continuous distribution function F with support $(0,\infty)$. Denote the vector $(X_{i1}, \ldots, X_{iK_i})$ by $\mathbf{X}_i$, $i = 1, \ldots, n$. Define the map $\Pi_i : \mathbb{R}^{K_i} \to \mathbb{R}$ such that Π_i maps $\mathbf{X}_i$ into a particular element in $\mathbf{X}_i$, say X_i $(i = 1, \ldots, n)$. We shall call X_i the nominee of $\mathbf{X}_i$ and the collection $\{X_i : i = 1, \ldots, n\}$ is called the nomination sample. The case when $\Pi_i(\mathbf{X}_i) = \max_{1 \le j \le K_i} X_{ij}$ has been studied by Willemain (1980) and Boyles and Samaniego (1986). Another important case is where $\Pi_i(\mathbf{X}_i) = \min_{1 \le j \le K_i} X_{ij}$; that is, when the nominee of $\mathbf{X}_i$ is the minimum. As an example of such a data generating process suppose that a factory has n identical machines; the i^{th} machine having K_i components $(i = 1, \ldots, n)$. Suppose also that each machine is set up as a series system of i.i.d. components with common d.f. F. Let $\mathbf{X}_i$ be the life lengths of the components in the i^{th} machine. As soon as the first component fails the entire machine fails, these first failure times for the entire factory are $(X_1, \ldots, X_n)$, the nomination sample. A reliability engineer may be interested in inference about the components of the machines, that is about F, rather than the machines itself.

Another example of such a data generating process is the following. Suppose a consumer has a known number of options from which he/she has to make a single decision. The wise consumer will usually choose the option that costs the least and hence the nominee will be the option of minimal cost. Although the distribution of all option costs is unknown, one would like to be able to draw some inference about this distribution from the nomination sample.

In this note we consider the estimation of the distribution function with a nomination sample in the presence of random censoring. This estimator is derived

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in Section 2. The asymptotic theory of this estimator and functionals of this estimator is studied in Section 3.

2. Estimation. Let K be a positive integer valued r.v. with probability mass function $p(\cdot)$ and the probability generating function (p.g.f.) $\psi(\cdot)$. Assume $E|K| < \infty$. Let F and G be continuous d.f.'s on $(0, \infty)$. Given $K = K_i$, let X_i be the minimum of the sample $\mathbf{X}_i$ of size K_i, $i = 1, \ldots, n$. Then X_i has conditional d.f. $1 - (1-F)^{K_i}$, $i = 1, \ldots, n$. Let $Z_1, \ldots, Z_n$ be i.i.d. G. Define

$$Y_i = \min\{X_i, Z_i\} = X_i \wedge Z_i \text{ and } \delta_i = 1[X_i \le Z_i], \quad i = 1, \ldots, n, \tag{1}$$

where $1[\cdot]$ is the indicator function of the event $[\cdot]$. One can see that Y_i is the nominee of the i^{th} sample if there is no censoring, otherwise we observe the censoring variable Z_i. Hence, if there is no nominee from the i^{th} sample, $\delta_i = 0$. In the reliability example discussed above, this would correspond to no failure in the i^{th} series system, surely this information should be accounted for when estimating F.

Let $Y_{1:n} \le, \ldots, \le Y_{n:n}$ denote the ordered values of observed Y_i's. Denote the $\{(Y_{i:n}, K_{i:n}, \delta_{i:n});\ i = 1, \ldots, n\}$ by $\mathcal{D}_c$, where $K_{i:n}$ and $\delta_{i:n}$ are the values of K_i and δ_i that correspond to $Y_{i:n}$, $i = 1, \ldots, n$. Proceeding as in Boyles and Samaniego (1986) (hereafter denoted B–S (1986)) we can obtain the nonparametric maximum likelihood estimator (NPMLE) of F by finding the d.f. F that maximizes

$$\begin{aligned} L(F|\mathcal{D}_c) &= \Pi_{i=1}^n \{[1-(1-F(Y_{i:n}))^{K_{i:n}}] - [1-(1-F(Y_{i-1:n}))^{K_{i:n}}]\}^{\delta_{i:n}} \\ &\quad \times \{\{1-F(Y_{i:n})\}^{K_{i:n}}\}^{(1-\delta_{i:n})} \\ &= \Pi_{i=1}^n \{\bar{F}^{K_{i:n}}(Y_{i-1:n}) - \bar{F}^{K_{i:n}}(Y_{i:n})\}^{\delta_{i:n}} \{\bar{F}(Y_{i:n})\}^{K_{i:n}(1-\delta_{i:n})}, \end{aligned} \tag{2}$$

where $\bar{F} \equiv 1 - F$ is the survival function.

Now, letting $p_i = \bar{F}(Y_{i:n})/\bar{F}(Y_{i-1:n})$ we have from (2) that

$$\begin{aligned} L(F|\mathcal{D}_c) &= \Pi_{i=1}^n (1-p_i^{K_{i:n}})^{\delta_{i:n}} p_i^{K_{i:n}(1-\delta_{i:n})} \Pi_{i'<i}\, p_{i'}^{K_{i':n}} \\ &= \Pi_{i=1}^n (1-p_i^{K_{i:n}})^{\delta_{i:n}} p_i^{\gamma_{[i]} - K_{i:n}\delta_{i:n}} \\ &= L(\mathbf{p}), \text{ say}, \end{aligned} \tag{3}$$

where

$$\gamma_{[i]} = \Sigma_{j=1}^n K_{j:n}, \quad i = 1, \ldots, n.$$

We maximize $L(\mathbf{p})$ in (3) by separate maximization of each factor. One can verify that the function $x^a(1-x^b)^c$ is concave and is uniquely maximized by $\hat{x} = (a/(a+bc))^{1/b}$. It follows that $L(\mathbf{p})$ is maximized by

$$\begin{aligned} \hat{p}_i &= ([\gamma_{[i]} - K_{i:n}\delta_{i:n}]/\gamma_{[i]})^{1/K_{i:n}} \\ &= ([\Sigma_{j=1}^n K_{j:n} - K_{i:n}\delta_{i:n}]/\Sigma_{j=1}^n K_{j:n})^{1/K_{i:n}}, \quad i = 1, \ldots, n. \end{aligned} \tag{4}$$

Therefore the NPMLE of F is given by

$$(5) \qquad \hat{F}_n(x) = \begin{cases} 0 & , \text{if } x < Y_{1:n} \\ 1 - \Pi_{j=1}^{i} \hat{p}_j & , \text{if } Y_{i:n} \le x < Y_{i+1:n},\ i = 1, \ldots, n-1 \\ 1 & , \text{if } Y_{n:n} \le x. \end{cases}$$

Note that $\hat{F}_n(x)$ given by (5) is closely related to the estimate developed in B–S (1986) where the nomination function Π_i was the maximum and there was no censoring. Note also that if $K_i = 1$, for all i, $\hat{F}_n$ reduces to the Kaplan-Meier estimate. Hence if $K_i = 1$ and $\delta_i = 1$, for all i, $\hat{F}_n$ reduces to the empirical distribution function of $(X_1, \ldots, X_n)$. These analogies will become more apparent in the next section when we discuss the asymptotic theory of the process $\sqrt{n}\,(\hat{F}_n - F)$.

3. Asymptotic Theory. The weak convergence results of $\hat{F}_n$ presented here are based on the methods of martingale based inference. The approach is to propose an estimator which is asymptotically equivalent to $\hat{F}_n$ and to demonstrate its limiting distribution.

Note that the stochastic intensity (failure rate) of X_i given $K = K_i$ is given by $\lambda_i(t) = \lambda_0(t)K_i\delta_i$, where $\lambda_0(t)$ is the intensity of the distribution F. Let $\mathcal{H}_i$ be a history which satisfies "the usual conditions" (see Dellacherie (1972)). Embed K_i into an $\mathcal{H}_i$-predictable and locally bounded process $K_i(t)$. Also, embed δ_i into an $\mathcal{H}_i$-predictable process $\delta_i(t)$ taking values in {0,1}, indicating (by the value one) when the i^{th} sample is under observation; thus $\delta_i(\cdot)$ is the censoring process. Now, define the multiplicative intensity model $N_i(t)$ as the point process having stochastic intensity $\lambda_i(t)$. Also, define $N(t) = \Sigma_{i=1}^{n} N_i(t)$ with stochastic intensity $\lambda_0(t)\Sigma_{i=1}^{n} K_i(t)\delta_i(t)$ and history $\mathcal{H} = \vee_{i=1}^{n} \mathcal{H}_i$. Hence, this is related to the Cox regression model as studied by Anderson and Gill (1982) (hereafter denoted by AG (1982)). This is also the approach of B–S (1986), however, we will incorporate the censoring process in our development.

The theory of martingale based inference will give us an estimate of $\Lambda_0(t) = \int_0^t \lambda_0(s)ds$, the integrated hazard, and hence an estimate of $\bar{F}(t) = \exp(-\Lambda_0(t))$. From the results of Section 5.2 of Karr (1986) it can be seen that the estimator of $\Lambda_0(t)$ is given by

$$(6) \qquad \hat{\Lambda}_{0n}(t) = \int_0^t \frac{dN(s)}{\sum_{i=1}^{n} K_i(t)\delta_i(t)} = \sum_{i: X_i \le t} \delta_i \left[\sum_{j: X_j > X_i} K_j \right]^{-1}.$$

Define $H_n = -\log(1 - \hat{F}_n)$ as the NPMLE of the integrated hazard rate of F. A simple modification of B–S (1986) to allow for censoring yields

LEMMA 3.1. $\quad \sup_{t \in \mathbb{R}} \sqrt{n}\, |H_n(t) - \hat{\Lambda}_{0n}(t)| \xrightarrow{P} 0$ as $n \to \infty$.

This lemma implies that $\sqrt{n}(\hat{\Lambda}_{0n} - H)$ and $\sqrt{n}(H_n - H)$ will have the same asymptotic distribution where $H = -\log \bar{F}$.

The weak convergence of $\sqrt{n}(\hat{\Lambda}_{0n} - H)$ will follow from Rebolledo's (1980) martingale central limit theorem as found in the appendix of AG (1982). We need to verify their conditions (I.3) and (I.4) employed in its proof. Recall that $\psi(\cdot)$ is the p.g.f. of the r.v. K and that $\bar{G} \equiv 1 - G$ is the survival function of the censoring distribution. Define

$$\xi_n(t) = \sum_{i:X_i > t} K_i \delta_i. \tag{7}$$

Hence we have that $\hat{\Lambda}_{0n}(t) = \int_0^t \frac{dN(s)}{\xi_n(s)}$.

The following result will be used in verifying the conditions of Rebolledo's (1980) martingale central limit theorem.

LEMMA 3.2.
(a)

$$\begin{aligned}
\int_0^t \frac{\lambda_0(s)ds}{\xi_n(s)/n} &\xrightarrow{P} \int_0^t \frac{\lambda_0(s)ds}{\bar{F}(s)\bar{G}(s)\psi'(\bar{F}(s))} \\
&= \int_0^t \frac{dF(s)}{\bar{F}^2(s)\bar{G}(s)\psi'(\bar{F}(s))},
\end{aligned}$$

where ψ' is the first derivative of ψ.

(b) *For any $\epsilon > 0$*

$$\int_0^t \frac{n\lambda_0(s)}{\xi_n(s)} 1[\frac{\sqrt{n}}{\xi_n(s)} > \epsilon]\, ds \xrightarrow{P} 0.$$

PROOF. We will prove that

$$\Delta_n = \sup_{t \in \mathbb{R}} |\frac{\lambda_0(t)}{\xi_n(t)/n} - \frac{\lambda_0(t)}{\bar{F}(t)\bar{G}(t)\psi'(\bar{F}(t))}| \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Note that we have the identity

$$\Delta_n = \sup_{t \in \mathbb{R}} |\frac{\lambda_0(t)\bar{F}(t)\bar{G}(t)\psi'(\bar{F}(t)) - \lambda_0(t)\xi_n(t)/n}{\bar{F}(t)\bar{G}(t)\psi'(\bar{F}(t))\xi_n(t)/n}|. \tag{8}$$

The strong law of large numbers yields

$$\begin{aligned}
\frac{1}{n}\xi_n &\xrightarrow{\text{a.s.}} E\{K 1[X \geq t] 1[Z \geq X]\} \\
&= \bar{G}(t) \sum_{k=1}^{\infty} k \bar{F}^k(t) p(k) = \bar{G}(t)\bar{F}(t)\psi'(\bar{F}(t)).
\end{aligned} \tag{9}$$

Note that the expression in (9) is positive for all $t \epsilon \mathbb{R}$. Hence the numerator in (8) tends to zero and the denominator of (8) is $O_p(1)$. Therefore $\Delta_n \xrightarrow{P} 0$. The second equality follows since $\lambda_0(t) = \frac{f(t)}{\bar{F}(t)}$, where f is the density of F.
(b) The proof follows from a minor modification of Lemma 2.3b in B–S (1986). ||

Let θ be a positive real number such that $\theta < \tau^{-1}(1)$, where $\tau = (1-F)(1-G)$. By applying Rebolledo's (1980) theorem we will show that following result holds.

THEOREM 3.3. *The process* $\beta_n = \sqrt{n}\,(\hat{\Lambda}_{0n} - H)$ *converges weakly in* $D[0,\theta]$ *as* $n \to \infty$ *to a mean zero Gaussian martingale* B *with covariance function*

$$\text{(10)} \qquad \mathrm{Cov}(B(s), B(t)) = \int_0^s \frac{dF(u)}{\bar{F}^2(u)\bar{G}(u)\psi'(\bar{F}(u))} \quad 0 \le s \le t \le \theta.$$

PROOF. By the Doob-Meyer decomposition for submartingales (and hence for counting processes) we have

$$\text{(11)} \qquad dN_i(t) = \lambda_0(t)K_i(t)\delta_i(t) + dM_i(t)$$

$$\text{(12)} \qquad dN(t) = \lambda_0(t)\xi_n(t) + dM(t)$$

where $M(t) = \Sigma_{i=1}^n M_i(t)$ is an $\mathcal{H}$-martingale. Therefore the process in the theorem may be expressed as

$$\text{(13)} \qquad \sqrt{n}\,\{\hat{\Lambda}_{0n}(t) - H(t)\} = \bar{M}_n(t) + R_n(t),$$

where

$$\bar{M}_n(t) = \frac{1}{\sqrt{n}} \int_0^t \frac{dM(s)}{\xi_n(s)/n} \text{ and } R_n(t) = \int_0^t I(\xi_n(s) = 0)dH(s).$$

Note that (9) implies

$$\sup_{t\epsilon[0,\theta]} |R_n(t)| \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Also, note that $\bar{M}_n(t)$ is a square integrable martingale. Therefore to deduce that the process β_n converges to a Gaussian martingale, we will apply Rebolledo's (1980) martingale central limit theorem to the martingale $\bar{M}_n$. The version of Rebolledo's theorem we will use is found in AG (1982) with $p = 1$ and $H_{1\ell}(t) = n^{-1/2}(\xi_n(t)/n)^{-1}$. By Lemma 3.2a and (12) we have that

$$< \bar{M}_n, \bar{M}_n > (t) = \int_0^t \frac{\lambda_0(s)}{\xi_n(s)/n} ds \xrightarrow{P} \int_0^t \frac{\lambda_0(s)ds}{\bar{F}(s)\bar{G}(s)\psi'(\bar{F}(s))} =< B, B > (t).$$

The Lindeberg condition of AG (1982) may be verified by applying Lemma 3.2b. Therefore the conditions of the theorem have been met and we have the desired result. ||

THEOREM 3.4. *The process $\chi_n(t) = \sqrt{n}\ \{\hat{F}_n(t) - F(t)\}$ converges weakly in $D[0,\theta]$ as $n \to \infty$ to $\chi(t) = \bar{F}(t)B(t)$, where $B(t)$ is the Gaussian martingale in Theorem 3.3. The covariance kernel of χ is given by*

$$K(s,t) = \bar{F}(s)\bar{F}(t) \int_0^s \frac{dF(u)}{\bar{F}^2(u)\bar{G}(u)\psi'(\bar{F}(u))} \quad 0 \leq s \leq t \leq \theta. \tag{14}$$

PROOF. By applying the Doléans-Dade exponential, $\mathcal{E}(\cdot)$ (see Liptser and Shiryayev, 1978, pp. 255–256) it is immediate that $\bar{F} = \mathcal{E}(-H)$ since $\bar{F}$ satisfies

$$\bar{F}(t) = 1 + \int_0^t \bar{F}(s)\, d(-H(s)).$$

Therefore,

$$\mathcal{E}(H(t) - \hat{\Lambda}_{0n}(t)) = (1 - \hat{F}_n(t))/\bar{F}(t)$$

so that

$$(1 - \hat{F}_n(t))/\bar{F}(t) = 1 + \int_0^t \frac{(1 - \hat{F}_n(s))}{\bar{F}(s)}\, d(H(s) - \hat{\Lambda}_{0n}(s)).$$

Using the decomposition in (13) it is clear that

$$\chi_n(t) = \bar{F}(t) \int_0^t \frac{(1 - \hat{F}_n(s))}{\bar{F}(s)}\, d\bar{M}_n(s) + R_n^*(t) \tag{15}$$

for some remainder term R_n^* which tends to zero in probability uniformly in $t \epsilon [0,\theta]$. Define $L_n(t)$ to be the integral in first term on the right hand side of (15). Note that $L_n(t)$ is a square integrable martingale with predictable variation process

$$< L_n, L_n > (t) = \int_0^t \left[\frac{1 - \hat{F}_n(s-)}{(1 - F(s))}\right]^2 \frac{\lambda_0(s)ds}{\xi_n(s)/n}. \tag{16}$$

By an application of Lenglart's (1977) inequality and the decomposition in (15) we have that the term on the right hand side of (16) in the square bracket tends to one. Therefore, it follows that

$$< L_n, L_n > (t) \xrightarrow{P} < B, B > (t). \tag{17}$$

As in the proof of Theorem 3.3, for $\bar{M}_n$, the Lindeberg condition for L_n may be verified. The result then follows by an application of Rebolledo's martingale central limit theorem. ||

As to be expected, if $\psi(u) \equiv u$, that is $K_i \equiv 1$ for all $i = 1, \ldots, n$ the covariance function in (11) reduces to the limiting covariance function of the Kaplan-Meier estimate. Similarly, if $\psi(u) \equiv u$ and $\bar{G}(u) \equiv 1$, that is, there is no censoring, the result reduces to the classical result for the empirical distribution function.

To apply Theorem 3.4 it is necessary to estimate the variance in (14), but this can also be done using martingale based methods. The process

$$W_n(t) = \int_0^t \frac{\lambda_0(s)ds}{\Sigma_{i=1}^n K_i(s)\delta_i(s)},$$

which converges to the variance in (10), has a martingale based estimator

$$\hat{W}_n(t) = \int_0^t \frac{1}{(\Sigma_{i=1}^n K_i(s)\delta_i(s))^2}\, dN(s) = \sum_{i:X_i \le t} \delta_i \left[\sum_{j:X_j > X_i} K_j\right]^{-2}.$$

By applying the results of Theorem 5.12 of Karr (1986) the estimate $\hat{W}_n(t)$ may be shown to be a consistent estimate of the variance in (10). Therefore,

$$\hat{U}_n(t) = (1 - \hat{F}_n(t))\hat{W}_n(t) \tag{18}$$

will consistently estimate the variance in (14). This is the type of estimator introduced by Tsiatis (1981) in the context of Cox regression models.

Using the results of Theorem 3.4 one may study the asymptotic behavior of the estimated quantiles. Define the quantiles of $\hat{F}_n$ as $\hat{F}_n^{-1}(t) = \inf\{x : \hat{F}_n(x) \ge t\}$. Applying the general results for the asymptotic behavior of quantiles by Tiwari and Wells (1988) we have

THEOREM 3.5. *Under the conditions of Theorem 3.4 the process* $\sqrt{n}\,[\hat{F}_n^{-1}(t) - F^{-1}(t)]f(F^{-1}(t))$ *converges weakly on* $D[0, F(\theta)]$ *to the mean zero Gaussian process* $\chi(F^{-1})$, *where* $\chi(\cdot)$ *has covariance function given by (14).*

The above theorems are stated under the assumptions that K is a random variable with finite expectation. In some applications the assumption that K is random may not be appropriate. B–S (1986) suggest a possible modification when K is not random. For further details see Section 3 of B–S (1986).

Many statistical procedures under the nomination sampling scheme can be viewed as a functional of the process $\sqrt{n}(\hat{F}_n - F)$ and the asymptotic properties of such procedures can be inferred from the process itself. In what follows, in the remainder of this section, we will consider the problem of estimation of a parameter of the unknown distribution F. Specifically, we will examine the properties of linear combinations of functions of estimated quantiles (lcfeq) under the nomination sampling scheme. In the case of simple random sampling the estimated quantiles are the order statistics and in that case parameter estimates are based on linear combinations of functions of order statistics (lcfos). In the case of nomination sampling we do not record the order statistics of the individual samples, thus we will use the estimated quantiles discussed in Theorem 3.5.

Let J_n be some known score generating function and let $h(\cdot)$ denote a known function of the form $h = h_1 - h_2$ with $h_i (i = 1, 2)$ increasing and left continuous. Consider the lcfeq

$$T_n = \int_0^t h(\hat{F}_n^{-1}(s))J_n(s)\,ds.$$

If $J_n \to J$, in some sense, T_n can be used as an estimate for the functional

$$\theta = \int_0^t h(F^{-1}(s))J(s)\,ds.$$

See Serfling (1980) for an extensive survey of functionals of this type.

Associated with the function $g = h(F^{-1})$ is a Lebesgue-Stieltjes signed measure; let $|g|$ denote the total variation measure of this measure. We shall need the following assumptions to demonstrate the asymptotic normality of $\sqrt{n}\,(T_n - \theta)$.

Assumption 1: (i) Suppose $|J| < \alpha(t)$ and, for all n, $|J_n| < \alpha(t)$ on (0,1) where $\alpha(t) = Mt^{-b_1}(1-t)^{-b_2}$ for $0 < t < 1$ with $M > 0$ and $(b_1 \wedge b_2) < 1$.

(ii) Suppose $h = h_1 - h_2$, with h_i increasing and left continuous on $\mathbb{R}$ with $|h_i(F^{-1})| < D(t)$ for $i = 1, 2$, where $D(t) = Mt^{-d_1}(1-t)^{-d_2}$, for $0 < t < 1$, with $M > 0$ and any fixed d_1, d_2.

Assumption 2: Except on a set of t's of $|g|$-measure zero we have both J is continuous at t and $J_n \to J$ uniformly in some small neighborhood of t as $n \to \infty$.

Under the above assumption we have a theorem which is an analog of the result of Shorack (1972) concerning lcfos in the simple random sampling set up. The proof of our result is quite similar to Shorack's and will be omitted.

THEOREM 3.6. *If $(b_1 + d_1) \vee (b_2 + d_2) < \frac{1}{2}$, then*

$$\sqrt{n}\,(T_n - \theta) \xrightarrow{d} N(0, \sigma^2) \text{ as } n \to \infty,$$

where

$$\sigma^2 = \int_0^1 \int_0^1 K(F^{-1}(s), F^{-1}(t))\; J(s)J(t)\; dg(s)dg(t)$$

with $K(\cdot,\cdot)$ being the covariance kernel given by (14).

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