TESTS OF INDEPENDENCE AGAINST LIKELIHOOD RATIO DEPENDENCE IN ORDERED CONTINGENCY TABLES

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Tests of independence against positive likelihood ratio dependence in ordered contingency tables are reviewed. Some exact tests and an intuitive sign test are discussed.

1. Introduction. In the analysis of contingency tables, a classical test of independence is the chi-square test. However, if the chi-square test strongly rejects the null hypothesis of independence, the statistician receives little information as to what kind of dependency may exist in the data. When both categorical variables are ordinal, it might be interesting to check whether there is a monotonic relationship between the variables. Grove (1984) and Agresti (1984) explored a variety of definitions of positive dependence for contingency tables.

On testing the independence in contingency tables, various types of alternatives have been investigated. Armitage (1955) considered tests for linear trends in proportions. Goodman (1985) discussed association models, correlation models, and asymmetry models for contingency tables. Grove (1980, 1984) considered positive association and tests in a two-way contingency table. Nguyen and Sampson (1987) considered the alternative hypothesis of positive quadrant dependence. Agresti et al. (1979) and Patefield (1982) considered several exact tests of independence against positively likelihood ratio dependence. Lee (1988) investigated an intuitive sign test to test against likelihood ratio dependence.

In this review article, tests of independence against likelihood ratio dependence in ordered contingency tables are discussed.

2. Likelihood Ratio Dependence in Contingency Tables. Lehmann (1966) defined that random variables X and Y are said to be positively likelihood ratio dependent if their joint density f satisfies the following properties:

$$f(x,y)f(x',y') \geq f(x',y)f(x,y') \text{ for all } x \leq x',\ y \leq y'.$$

In this section, we shall discuss the concepts of positive likelihood ratio dependence in ordered contingency tables.

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Let X and Y be ordinal categorical random variables such that X has values $x_1 < x_2 < \ldots < x_r$ and Y has values $y_1 < y_2 < \ldots < y_c$. Consider a random sample (X_k, Y_k), $1 \leq k \leq N$ of size N. Collect the sample data into an $r \times c$ contingency table (n_{ij}) where n_{ij} denotes the number of observations (X_k, Y_k) such that $X_k = x_i$, $Y_k = y_j$, $k = 1, \ldots N$. Let $p_{ij} = P(X = x_i, Y = y_j)$ for $1 \leq i \leq r$, $1 \leq j \leq c$. For any two rows i and k and any two columns j and l, the corresponding odds ratio parameter is given by $(p_{i,j}p_{k,l})/(p_{i,l}p_{k,j})$. There are a total of $rc(r-1)(c-1)/4$ odds ratios formed this way. However, using all the odds ratios is redundant. It can be shown that local odds ratios, formed by using cells in adjacent rows and adjacent columns, determine all possible odds ratios that can be formed from any pairs of rows and any pairs of columns. Therefore, it suffices to work on the $(r-1)(c-1)$ local odds ratios defined on adjacent rows and columns, namely

$$\theta^{(i,j)} = (p_{i,j}p_{i+1,j+1})/(p_{i+1,j}p_{i,j+1}) \text{ for } 1 \leq i \leq r-1, \ 1 \leq j \leq c-1.$$

For cross-classifications of ordinal variables, positive likelihood ratio dependence corresponds to the property that all local odds ratios are greater or equal than 1. A lot of research has been done based upon the estimated odds-ratio

$$\hat{\theta}^{(i,j)} = (\hat{p}_{i,j}\hat{p}_{i+1,j+1})/(\hat{p}_{i+1,j}\hat{p}_{i,j+1}),$$

where $\hat{p}_{i,j} = n_{ij}/N$ is an unbiased estimator for $p_{i,j}$. See Bishop, Fienberg, and Holland (1975) and Agresti (1984) for a review of theories and methods based on odds ratios statistics.

However, $\hat{\theta}^{(i,j)}$ may equal to 0 or ∞ if any of the $n_{ij} = 0$. One way to get around this is to collapse categories and therefore increase the cell values. But this kind of procedures would sometimes result in wasting information collected from the observed data. To avoid these situations, Lee (1988) investigated the cross-product difference parameters. Similarly as in the case of odds-ratios, it can be shown that it suffices to consider only the local cross-product difference parameters $u^{(i,j)}$, where

$$u^{(i,j)} = p_{i,j}p_{i+1,j+1} - p_{i,j+1}p_{i+1,j} \text{ for } 1 \leq i \leq r-1, \ 1 \leq j \leq c-1.$$

For cross-classification tables, positive likelihood ratio dependence corresponds to the property that all local cross-product differences are greater or equal to 0.

3. Some Exact Tests. Patefield (1982) develops exact tests of independence against trends of positive likelihood ratio dependence in $r \times c$ contingency tables. The exact test was constructed according to procedures outlined by Agresti, Wackerly, and Boyett (1979). This procedure is developed to construct four tests based on differing criteria for measuring departures from H_0 in favor of H_1, where the null hypothesis is $H_0 : \theta^{(i,j)} = 1$ for all $1 \leq i \leq r-1$, $1 \leq j \leq c-1$, and the alternative hypothesis is $H_1 : \theta^{(i,j)} \geq 1$ for all $1 \leq i \leq r-1$, $1 \leq j \leq c-1$, with strict inequality holding for at least one pair of (i,j). The tests Patefield considered are based on: (1) the likelihood ratio, (2) the maximized score correlation, (3) the natural score correlation, and (4) the measure of association proposed by Goodman and Kruskal (1954). For each of the four criteria, the test procedure requires calculation of the appropriate test statistic for each element of the conditional sample space, i.e. for each table having the same marginal totals as the observed table. Of the four tests Patefield considered, it was concluded that test 2, the maximized score correlation, should be preferred to tests 3 and 4 as it is more flexible and appropriate in practical applications. The statistic of Patefield's test 2 considers the correlation between row and column scores maximized over ordered values of those row and column scores, i.e.

$$\lambda_2 = \sup_R \left\{ n_{..}^{-1} \sum_{i=1}^{r} \sum_{j=1}^{c} n_{ij} w_i s_j \right\}$$

where

$$R = \left\{ w, s : \sum_i n_{i.} w_i = 0, \ \sum_j n_{.j} s_j = 0, \ \sum_i n_{i.} w_i^2 = n_{..}, \right.$$

$$\left. \sum_j n_{.j} s_j^2 = n_{..}, \ w_1 \le w_2 \le w_r, \ s_1 \le s_2 \le \ldots \le s_c \right\}$$

and

$$n_{i.} = \sum_j n_{ij}, \ n_{.j} = \sum_i n_{ij}, \ n_{..} = \sum_i \sum_j n_{ij}.$$

Through simulation and Monte Carlo power study, it was shown that test 2 and test 1, the likelihood ratio test, have similar power. Therefore, test 2 should be preferred to test 1, the likelihood ratio test, on the ground of computation feasibility. However, when the sample size is large or when tables have higher dimensions, full enumeration of the conditional sample space is impossible and the random sampling technique is used.

4. A Sign Test. In this section, an intuitive sign test is considered to test the hypothesis of $H_0 : u^{(i,j)} = 0$ for all i,j against the alternative hypothesis that $H_1 : u^{(i,j)} \ge 0$ for all i,j and $u^{(i,j)} > 0$ for at least one pair of (i,j).

Considering

$$\hat{u}_N^{(i,j)} = \frac{1}{N(N-1)} (n_{i,j} n_{i+1,j+1} - n_{i+1,j} n_{i,j+1})$$

as an unbiased estimator of the cross-product difference $u^{(i,j)}$, Lee (1988) introduced a test statistic as follows. Let

$$S = \sum_{\substack{1 \le i \le r-1 \\ 1 \le j \le c-1}} sgn\ \hat{u}_N^{(i,j)},$$

where $sgn\ x = 1\{x > 0\}$ denote the indicator function for positive values of x. That is, the test statistic S is simply the number of 2×2 subtables, formed by adjacent rows and adjacent columns, such that the corresponding cross product difference statistic $\hat{u}_N^{(i,j)}$ is positive. Since positive association in many 2×2 subtables (i.e. many $\hat{u}_N^{(i,j)} > 0$) will provide strong evidence for the alternative H_1, one rejects H_0, if the value of S is sufficiently large.

Instead of using the delta method which would lead to lengthy calculations, the method of U statistics can be used to derive the asymptotic distributions of the $\hat{u}_N^{(i,j)}s$. It can be shown that the joint distribution of

$$\sqrt{N}\{\hat{u}_N^{(i,j)} - u^{(i,j)},\ 1 \le i \le c-1,\ 1 \le j \le r-1\}$$

is asymptotically normal with mean vector zero and variance-covariance matrix 4Σ (see Serfling (1980)), where $\Sigma = (\sigma_{(i,j)(i',j')})$ is given by

$$\sigma_{(i,j)(i',j')} = cov(g^{(i,j)}(X_1, Y_1); g^{(i',j')}(X_1, Y_1)),\ \text{with}$$

$$\begin{aligned}
&var(g^{(i,j)}) = (p_{i,j}p_{i+1,j+1}^2 + p_{i+1,j+1}p_{i,j}^2 + p_{i,j+1}p_{i+1,j}^2 + p_{i+1,j}p_{i,j+1}^2)/4 - (u^{(i,j)})^2 \\
&cov(g^{(i,j)}, g^{(i-1,j-1)}) = (p_{i,j}p_{i-1,j-1}p_{i+1,j+1})/4 - u^{(i,j)}u^{(i-1,j-1)} \\
&cov(g^{(i,j)}, g^{(i-1,j+1)}) = (p_{i,j+1}p_{i+1,j}p_{i-1,j+2})/4 - u^{(i,j)}u^{(i-1,j+1)} \\
&cov(g^{(i,j)}, g^{(i-1,j)}) = -\left[p_{i,j+1}p_{i-1,j}p_{i+1,j} + p_{i,j}p_{i+1,j+1}p_{i-1,j+1}\right]/4 - u^{(i,j)}u^{(i-1,j)} \\
&cov(g^{(i,j)}, g^{(i,j-1)}) = -\left[p_{i,j}p_{i+1,j+1}p_{i+1,j-1} + p_{i+1,j}p_{i,j-1}p_{i,j+1}\right]/4 - u^{(i,j)}u^{(i,j-1)} \\
&cov(g^{(i,j)}, g^{(s,t)}) = 0 \text{ for all other } (s,t).
\end{aligned}$$

To derive the asymptotic null distribution of the test statistic S, it suffices to compute the probability $P(S = m)$ under H_0 for any $m = 1, 2, \ldots (r-1)(c-1)$.

$$\begin{aligned}
P(S &= m) \\
&= \sum_{C_m} P(\hat{u}_N^{(i_1,j_1)} > 0, \ldots, \hat{u}_N^{(i_m,j_m)} > 0,\ \hat{u}_N^{(i_{m+1},j_{m+1})} \le 0, \ldots, \\
&\qquad \hat{u}_N^{(i_{(r-1)(c-1)},j_{(r-1)(c-1)})} \le 0)
\end{aligned}$$

where $\sum_{C_m}$ denotes summation over all possible combinations of m distinct elements $\{(i_1, j_1), \ldots, (i_m, j_m)\}$ from $\{(1,1), \ldots, (r-1, c-1)\}$ and where $\{(i_{m+1}, j_{m+1}),$

$\ldots,(i_{(r-1)(c-1)}, j_{(r-1)(c-1)})\}\}$ is its complementary set. It can be shown that $P(S = m)$ converges to a sum of integrals of a multivariate normal over certain quadrants.

Since this is essentially a test based on signs, it is not very powerful for 2×2 tables. The proposed sign test is not recommended for 2×2 tables. For the case of $(r-1)(c-1) = 2$, that is, for 3×2 (or 2×3) tables, explicit formulas for $P(S = m)$ can be derived.

EXAMPLE 4.1: 3×2 tables. In this case, the U-statistic considered has values in a two-dimensional space. It is known that the mass attributed to the positive quadrant by a standard bivariate normal random vector with correlation coefficient ρ is given by $1/4 + (\arcsin \rho)/2\pi$, (see Johnson and Kotz (1976)), and this probability is invariant under the scale transformation.

Therefore,

$$\begin{aligned} P(S=2) &= P(\hat{u}_N^{(1,1)} > 0,\ \hat{u}_N^{(2,1)} > 0) \\ &= \frac{1}{4} + \frac{1}{2\pi} \arcsin\left(-1(1 + \frac{p_{2.}}{p_{1.}p_{3.}})^{-\frac{1}{2}}\right), \\ P(S=1) &= 1 - 2P(S=2), \\ P(S=0) &= P(\hat{u}_N^{(1,1)} \le 0,\ \hat{u}_N^{(2,1)} \le 0) = P(S=2). \end{aligned}$$

Consider the following 3×2 table from the sample.

	y_1	y_2	total
x_1	23	13	36
x_2	1	1	2
x_3	20	31	51
total	44	45	89

The test statistic S is equal to 2. The proposed test of independence has a significance level of 0.048. In this case a chi-square test has a significance level of 0.077.

For higher dimensional tables, one may use Monte Carlo methods to evaluate multivariate normal orthant probabilities and then compute the asymptotic null distribution if the table dimensions satisfy the condition that $(r-1)(c-1) \le 20$. An efficient evaluation method is given by Evans and Schwartz (1986). Note that unlike the exact test procedures, the proposed sign test has the advantage that increasing sample sizes will not add difficulties to the computation of the null distribution. Hence the proposed sign test procedure is more appropriate for tables with large sample sizes.

EXAMPLE 4.2: Monte Carlo Simulations for Higher Dimensional Tables. Consider the following data set referred to by Kasser and Bruce (1969), BMDP (1979), and Nguyen and Sampson (1987).

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	Functional Class			
Active	None or Minimal	Moderate	Severe	Total
Very	4	2	0	6
Normal	8	14	2	24
Limited	1	2	4	7
Total	13	18	6	37

In this example the test statistic S is equal to 4. Using the Monte Carlo simulation method derived by Evans and Schwartz (1986), based on a sample of 10,000, the estimated level of significance is equal to 0.028492 with a standard error of 0.0000534. A chi-square test has a significance level of 0.011.

5. Discussion. For each exact test reviewed in Section 3, the test procedure requires calculation of the corresponding test statistic for each element of the conditional sample space, i.e. for each table having the same marginal totals as the observed tables. This kind of approach is feasible for small tables with small sample sizes, but when the sample size is large or when the table has much higher dimensions, full enumeration of the conditional sample space is impossible. See Gail and Mantel (1977) for an approximation of the number of $r \times c$ contingency tables with fixed marginals.

The sign test discussed in Section 4 does not have this restriction as it does not depend on the conditional sample space of tables having fixed marginals. The simulation method used for evaluation of the null distribution of the sign test for high dimensional tables is relatively efficient. See Evans and Schwartz (1986) for a discussion of the efficiency of the simulation method for evaluating the orthant probability of a multivariate normal distribution. Comparisons of the sign test with likelihood ratio tests using Goodman's association models are being considered. The comparison results will be discussed in another paper.

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