CONDITIONAL NEGATIVE DEPENDENCE IN STOCHASTIC ORDERING AND INTERCHANGEABLE RANDOM VARIABLES

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A simple proof is provided for the result that when a subset of order statistics is given, the conditional joint distribution of the components of a random sample exhibit a certain form of negative dependence. It is also shown that the assertion remains valid even for the discrete distributions.

1. Introduction. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a random vector. $\mathbf{T}$ is said to be negatively dependent through stochastic ordering (NDS) if for every coordinatewise nondecreasing function h,

$$E[h(\mathbf{T}(i))|T_i = t_i] \downarrow t_i,\ i = 1 \ldots, n. \tag{1}$$

where $\mathbf{T}(i)$ denotes the $(n-1)$ vector obtained from $\mathbf{T}$ by removing the i^{th} component.

Block, Bueno, Savits, and Shaked (1987) established this negative dependence property for a random sample conditioned by a subset of its order statistics. It was shown that this conditional NDS has interesting applications in the study of the systems formed by *second hand* components. The proof of this result in their paper (see Theorem 3.1 in above) is quite involved and is derived under the assumption that the random variables are independent with a common *continuous distribution* function. Since this conditional NDS may manifest itself in other applications, it may be desirable to have a simpler proof of this property and to seek less restrictive conditions. The present article does both. A simpler proof is provided for this property where random variables are assumed to be independent but their common distribution may be discrete. It is shown that the interchangeability plays an important role and the relevant result is isolated and later used in the main proof.

2. Notation. Let $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ be a random vector and X be a random variable such that the n components of $\mathbf{Z}$ and X are identically distributed. It will

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be assumed that these $(n+1)$ random variables are interchangeable for deriving a basic result. A stronger condition of independence will be imposed for later results.

The order statistic vector for $(\mathbf{Z}, X)$ will be denoted by $\mathbf{V}$. We write I for the index set $[1, 2, \ldots, (n+1)]$. Since the common distribution of the random variables above could be discrete there is a possibility of ties. In such a case, we interpret $V_i = v_i$ as an event where exactly $(i-1)$ observations are less than v_i and *one or more* at v_i. In case of ties the possible ranks are assigned at random. $R(X)$ will denote the rank of X among the $(n+1)$ components of $(\mathbf{Z}, X)$.

Let $\mathbf{W}$ denote the order statistic corresponding to $\mathbf{Z}$ and $\mathbf{V}(i)$ be the n-vector obtained by removing the i^{th} component of $\mathbf{V}$.

3. Results. Let g be a coordinatewise nondecreasing function defined on R^n. We want to prove the conditional NDS property (1) for $(\mathbf{Z}, X)$, given $\mathbf{V}_B = \mathbf{v}_B$ (see relation (3) below). Here B is a nonempty subset of I and $\mathbf{v}_B$ denotes the set of values v_i, $i \in B$. Note that due to the permutation invariance of $(\mathbf{Z}, X)$, it suffices to consider conditioning on only one component, say X.

Our approach to prove the required monotonicity property is as follows. As a first step it is shown that the conditional expectation decreases as the rank $R(X)$ increases. This is shown under the minimal assumption of interchangeability. For the case where the underlying distribution is continuous, and the random variables are assumed to be independent, it is shown that the conditional expectation depends on the value of X only through its rank $R(X)$. Unfortunately, such a reduction is not possible for the case of discrete distributions and more detailed argument is needed for this case.

LEMMA. *Let the joint distribution of $(n+1)$ components of $(\mathbf{Z}, X)$ be interchangeable. Then*

$$E[g(\mathbf{Z}) | R(X) = i, \ \mathbf{V}_B = \mathbf{v}_B] \downarrow i.$$

PROOF. Due to the assumption of $(\mathbf{Z}, X)$ being interchangeable, it follows that the joint distribution of $(\mathbf{Z}, \mathbf{V}, X)$ is same as that of $(\mathbf{PZ}, \mathbf{V}(\mathbf{P}), X)$, where $\mathbf{P}$ is a permutation of the n components and $\mathbf{V}(\mathbf{P})$ is the order statistic corresponding to $(\mathbf{PZ}, X)$. However, $\mathbf{V}(\mathbf{P}) = \mathbf{V}$ and hence the expectation in the proposition does not change under permutations of $\mathbf{Z}$. Thus, without loss of generality, g may be assumed to be permutation invariant. Whenever convenient, we may replace the argument $\mathbf{Z}$ of g, by $\mathbf{W}$.

Given the order statistic $\mathbf{V}$ it is clear that all $(n+1)!$ permutations of I are equally likely as sets of ranks of $(\mathbf{Z}, X)$. This is clear for the case when the distribution is continuous. For the discrete case this follows from our choice of assigning ranks at random to the tied observations. In either case the rank vector is independent of $\mathbf{V}$. In particular $R(X)$ is independent of $\mathbf{V}$. If the rank $R(X) = i$ then the order statistic $\mathbf{W} = \mathbf{V}(i)$. Due to independence of $\mathbf{V}$ and $R(X)$, the event $\mathbf{V}_B = \mathbf{v}_B$, and $R(X)$ are also independent. Hence the difference

$$E[g(\mathbf{W})|R(X) = j,\ \mathbf{V}_B = \mathbf{v}_B] - E[g(\mathbf{W})|R(X) = i,\ \mathbf{V}_B = \mathbf{v}_B]$$

is the same as

$$E[g(\mathbf{V}(j))|\mathbf{V}_B = \mathbf{v}_B] - E[g(\mathbf{V}(i))|\mathbf{V}_B = \mathbf{v}_B]$$

or

$$E[\{g(\mathbf{V}(j)) - g(\mathbf{V}(i))\}|\mathbf{V}_B = \mathbf{v}_B], \tag{2}$$

where $\mathbf{V}(i)$ is defined above. The components of $\mathbf{V}(j)$ and $\mathbf{V}(i)$ are all the same except for a pair. If $j > i$, the smaller one of that pair belongs to $\mathbf{V}(j)$. Since g is increasing, the difference (2) must be nonpositive. This establishes the assertion of the Lemma.

THEOREM. *Suppose $(n+1)$ random variables $(\mathbf{Z}, X)$ are independent and identically distributed. Let $\mathbf{V}$ be the corresponding order statistic. For every nondecreasing function g and for every nonempty subset B of the index set I,*

$$E[g(\mathbf{Z})|X = x,\ \mathbf{V}_B = \mathbf{v}_B] \downarrow x, \text{ (almost surely)}. \tag{3}$$

PROOF. (a) In this part we assume that the underlying distribution is continuous. The condition $\mathbf{V}_B = \mathbf{v}_B$ creates a partition of the real line with $(b+1)$ open intervals, and b boundary points, where b denotes the cardinality of B. We illustrate our approach by a simple example. Suppose $n = 10$ and $B = \{3, 6, 8\}$ then the condition creates 4 open intervals $(-\infty, v_3)$, (v_3, v_6), (v_6, v_8), and (v_8, ∞). These contain respectively 2,2,1 and 3 observations, while the remaining 3 are at the boundaries. An observation is said to be of the i^{th} *category* if it falls in the i^{th} interval. It is important to note that once the boundaries of these intervals have been given, the observations taking values within these open intervals are independent.

Note that when the distribution is continuous there can be only one observation on the boundary.

Suppose that the conditioned value x of X is in the open interval corresponding to the highest category. In the example, this would imply that it is larger than v_8, or equivalently, $R(X) > 8$. Recalling that $\mathbf{W}$ denotes the order statistic corresponding to $\mathbf{Z}$, it is clear that in this case, $W_i = V_i$ for $i \in B$. Thus one Z value is assigned v_8, while two take values in (v_8, ∞) independently. Due to independence of $\mathbf{Z}$ and X, the expectation in (3) does not depend on the particular value of x but on the indicator of the event $R(X) > 8$. Suppose now that x is decreased to v_8. Then $R(X) = 8$. This changes the conditional distribution of Z. If x now decreases further to the open interval corresponding to the next lower category, in the example this would be the interval (v_6, v_8), then there is a change in the conditional distribution, and again the expectation remains the same as long as x is in this interval. Note that

$$x \in (v_i, v_j) \Longleftrightarrow R(x) \in (i, j).$$

The important observation is that the change in the expectation occurs only when x moves from an open interval to the boundary or from the boundary to the open interval. In either case this change can be expressed entirely in terms of $R(X)$ which is an increasing function of X. Thus the conditional expectation, as a function of x depends only on $R(x)$. The assertion of the Theorem for the continuous case now follows from the Lemma.

(b) Suppose now the common distribution has a discrete part. Then the rank $R(x)$ does not determine whether x is in the open interval or at the boundary. We have to show that in both cases the monotonicity property holds. Proceeding as in (a), suppose that the conditioned value x is in the open interval corresponding to the highest category, or $x > v_8$ in our example. One Z observation has to be at v_8, while two can take values in $[v_8, \infty)$, which includes a three way tie at v_8. When x is decreased to v_8, the conditional distribution has now three Z observations in $[v_8, \infty)$, as opposed to two in the previous case. As in the case of continuous distribution, conditionally, the observations are still independent, however, there is a possibility of a four way tie at the boundary. In any case, one of the observations has become stochastically larger making the conditional distribution of $\mathbf{Z}$ stochastically larger. If x now decreases further to the open interval corresponding to the next lower category, in the example this would be the interval (v_6, v_8), then this would result in promotion of W_7 to v_8. Again this would make the conditional distribution stochastically larger still. Repeating this argument, the Theorem for the discrete case is established.

REFERENCE

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