RELATIVE ERRORS IN RELIABILITY MEASURES

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A common assumption, in reliability and lifetesting situations when the components are installed in series system, is that they are independent and are exponentially distributed. In this paper we study the relative error in reliability measures such as the reliability function, the failure rate and the mean residual life under the erroneous assumption of independence when in fact lifetimes follow a bivariate exponential model. The behavior of these errors is discussed to examine their structure as a function of time. Some of the existing results in the literature follow as special cases.

1. Introduction. Klein and Moeschberger (1986, 1987) have studied the relative error (defined in Section 3) in system reliability and system mean life when the components follow the bivariate exponential distributions of Marshall and Olkin (1967), Freund (1961), Gumbel (1960), Downton (1970), and Oakes (1982). Moeschberger and Klein (1984) have studied the relative error in the Gumbel II bivariate exponential model.

In this paper we consider a series system whose components follow bivariate exponential models. The joint distribution of the component lives may not be uniquely determined from the observable data on (T, I), where $T = \min(X_1, X_2)$ and $I = I_{\{X_1<X_2\}}$, a problem of nonidentifiability as described by Tsiatis (1975) and others. If the data on T shows that T has an exponential distribution, then the component lives can be assumed to follow any one of the models described by Marshall and Olkin (1967), Freund (1961), and Block and Basu (1974). If the data shows simultaneous failure of both the components, the shock model developed by Marshall and Olkin will be more appropriate. If T is not exponential but the marginals are exponentials with the same parameters as for the independent case, then one may assume either one of Gumbel I or Gumbel II (1960). If in fact, the joint lifetimes follow any one of the five models mentioned above, the assumption of independence will lead to inappropriate conclusions.

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Section 2 describes the various bivariate models and Section 3 contains the definitions of the three reliability measures and the corresponding relative errors. The relative errors in the three measures for various models are given in Section 4. It also contains the analyses of these errors. The relative error in the system mean life studied by Klein and Moeschberger follows as a special case of the mean residual life at the origin.

2. The Models. Since the data on T is available, one can test to see whether the distribution of T is exponential. If it is found that the distribution of T is an exponential, then the joint distribution of X_1 and X_2 may be one of the following:

Independent:

$$\bar{F}_1(x_1,x_2)=e^{-\lambda_1 x_1-\lambda_2 x_2},\ \ \lambda_i>0,\ \ x_i>0,\ \ i=1,2.$$

Marshall and Olkin model (1967):

$$\bar{F}_2(x_1,x_2)=e^{-\lambda_1 x_1-\lambda_2 x_2-\lambda_{12}\max(x_1,x_2)},\ \ \lambda_i>0,\ \ \lambda_{12}>0, x_i>0,\ \ i=1,2.$$

Freund (1961):

$$\bar{F}_3(x_1,x_2)=\begin{cases} & (\lambda_1/(\lambda_1+\lambda_2-\theta_2))\,e^{-(\lambda_1+\lambda_2-\theta_2)x_1-\theta_2 x_2} \\ + & ((\lambda_2-\theta_2)/(\lambda_1+\lambda_2-\theta_2))\,e^{-(\lambda_1+\lambda_2)x_2},\ x_1\le x_2, \\ & (\lambda_2/(\lambda_1+\lambda_2-\theta_1))\,e^{-(\lambda_1+\lambda_2-\theta_1)x_2-\theta_1 x_1} \\ + & ((\lambda_1-\theta_1)/(\lambda_1+\lambda_2-\theta_1))\,e^{-(\lambda_1+\lambda_2)x_1},\ x_1>x_2, \\ & \theta_i>0,\ \lambda_i>0,\ x_i>0,\ i=1,2. \end{cases}$$

Block and Basu (1974):

$$\begin{aligned}\bar{F}_4(x_1,x_2) &= ((\lambda_1+\lambda_2+\lambda_{12})/(\lambda_1+\lambda_2))\,e^{-\lambda_1 x_1-\lambda_2 x_2-\lambda_{12}\max(x_1,x_2)} \\ &- (\lambda_{12}/(\lambda_1+\lambda_2))\,e^{-(\lambda_1+\lambda_2+\lambda_{12})\max(x_1,x_2)} \\ &\quad \lambda_i>0,\ \lambda_{12}\ge 0,\ x_i>0,\ i=1,2.\end{aligned}$$

An error may occur by assuming the independent model when in fact the joint distribution of X_1 and X_2 is described by one of the other three models. In case, however, due to various difficulties and resources, the data on T is not available at the designing stage of a system, it is not unreasonable to assume that the marginal distributions of X_1 and X_2 are exponential with parameters λ_1 and λ_2, respectively. The following models satisfy this condition:

Independent:

$$\bar{F}_1(x_1,x_2)=e^{-\lambda_1 x_1-\lambda_2 x_2},\ \ \lambda_i>0,\ \ x_i>0,\ \ i=1,2.$$

Gumbel I (1960):

$$\begin{aligned}\bar{F}_5(x_1, x_2) &= e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12} x_1 x_2}, \ \lambda_i > 0, \\ &\qquad \lambda_{12} \geq 0, \ x_i > 0, \ i = 1, 2.\end{aligned}$$

Gumbel II (1960):

$$\begin{aligned}\bar{F}_6(x_1, x_2) &= \left[1 + \alpha(1 - e^{-\lambda_1 x_1})(1 - e^{-\lambda_2 x_2})\right] e^{-\lambda_1 x_1 - \lambda_2 x_2} \\ &\quad \lambda_i > 0, \ x_i > 0, \ |\alpha| < 1, \ i = 1, 2.\end{aligned}$$

Once again an error may occur due to the erroneous assumption of independence.

3. Definitions. Suppose T is a non-negative random variable denoting the life of a component having distribution function $F(t)$ and probability density function (pdf) $f(t)$. Then the survival function $\bar{F}(t)$, the mean residual life function (MRLF) $r(t)$ and the failure rate $\lambda(t)$ of T are defined as follows:

DEFINITION 3.1. The mean residual life function $r(t)$ of T is defined by

$$r(t) = E(T - t | T > t) = \int_t^\infty \bar{F}(x) dx / \bar{F}(t) = \left[\mu - \int_0^t \bar{F}(x) dx\right] / \bar{F}(t).$$

DEFINITION 3.2. The failure rate $\lambda(t)$ is defined by

$$\lambda(t) = f(t)/\bar{F}(t) = [1 + r'(t)]/r(t)$$

DEFINITION 3.3. The survival function $\bar{F}(t)$ is given by

$$\bar{F}(t) = P(T > t) = (r(0)/r(t)) \exp\left[-\int_0^t dx/r(x)\right].$$

For the series system with two components $T = \min(X_1, X_2)$, it is clear from the above definitions that $\bar{F}(t)$, $\lambda(t)$ and $r(t)$ are equivalent in the sense that given any one of them the other two can be determined. We also define the relative error, in various reliability measures, incurred by erroneously assuming the independent bivariate exponential model when in fact the models are dependent. The relative errors in reliability measures are defined as follows:

Relative error in reliability (survival) function is

$$\left(\bar{F}_D(t) - \bar{F}_I(t)\right) / \bar{F}_I(t).$$

Relative error in failure rate is

$$\left(\lambda_D(t) - \lambda_I(t)\right) / \lambda_I(t),$$

Table I

Reliability Measures for the Two Component Series System

Model	Reliability Function	Failure Rate	Mean-Residual Life
Independent	$\exp(-(\lambda_1+\lambda_2)t)$	$\lambda_1+\lambda_2$	$1/(\lambda_1+\lambda_2)$
Marshall & Olkin	$\exp(-\lambda t)$	λ	$1/\lambda$
Freund	$\exp(-(\lambda_1+\lambda_2)t)$	$\lambda_1+\lambda_2$	$1/(\lambda_1+\lambda_2)$
Block & Basu	$\exp(-\lambda t)$	λ	$1/\lambda$
Gumbel I	$\exp(-(\lambda_1+\lambda_2+\lambda_{12}t)t)$	$\lambda_1+\lambda_2+$ $2\lambda_{12}t$	$\sqrt{\pi/\lambda_{12}}(1-\phi(\sqrt{2t'}))$ $/\exp(-t')$
Gumbel II	$\exp(-(\lambda_1+\lambda_2)t)h(t)$	$\lambda_1+\lambda_2-$ $[h'(t)/h(t)]$	$[(1+\alpha)/(\lambda_1+\lambda_2)-$ $\alpha g(t)]/h(t)$

where

$$\begin{aligned}
\lambda &= \lambda_1+\lambda_2+\lambda_{12}, t' = \lambda_{12}(t+[(\lambda_1+\lambda_2)/2\lambda_{12}])^2, \phi(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-x^2/2}dx \\
h(t) &= 1+\alpha(1-\exp(-\lambda_1 t))(1-\exp(-\lambda_2 t)) \\
g(t) &= \exp(-\lambda_1 t)/(2\lambda_1+\lambda_2)+\exp(-\lambda_2 t)/(\lambda_1+2\lambda_2)-\exp(-(\lambda_1+\lambda_2)t)/ \\
&\quad 2(\lambda_1+\lambda_2)
\end{aligned}$$

and the relative error in the mean-residual life is

$$(r_D(t)-r_I(t))\,/r_I(t),$$

where D stands for dependent and I for independent model.

Even though the three reliability measures described above are equivalent, the three relative errors do not exhibit such a property.

4. Relative Error in Reliability Measures. Table I lists the reliability measures for the two component series system. The relative errors in reliability measures under the assumption of independence are given in Table II.

Analysis of the errors.

(1) The relative error in all the three reliability measures is the same for Marshall and Olkin and Block and Basu models and there is no error for Freund model. For Marshall and Olkin, the relative error in reliability is negative and decreases from 0 to -1 as a function of t (λ_{12} fixed) or λ_{12} (t fixed). Thus in this case the independence assumption leads to an overassessment of reliability. For this model,

Table II

Relative Error in Reliability Measures Under the Assumption of Independence

True Model	Reliability Function	Failure Rate	Mean-Residual Life
Independent	0	0	0
Marshall & Olkin	$\exp(-\lambda_{12}t)-1$	$\lambda_{12}/(\lambda_1+\lambda_2)$	$-\lambda_{12}/\lambda$
Freund	0	0	0
Block & Basu	$\exp(-\lambda_{12}t)-1$	$\lambda_{12}/(\lambda_1+\lambda_2)$	$-\lambda_{12}/\lambda$
Gumbel I	$\exp(-\lambda_{12}t^2)-1$	$2\lambda_{12}t/(\lambda_1+\lambda_2)$	$[\sqrt{\pi/\lambda_{12}}(1-\phi(\sqrt{2t'}))$ $(\lambda_1+\lambda_2)/\exp(-t')]-1$
Gumbel II	$h(t)-1$	$-h'(t)/(\lambda_1+\lambda_2)$ $h(t)$	$([1+\alpha-\alpha(\lambda_1+\lambda_2)$ $g(t)]/h(t))-1$

where λ, t', $h(t)$, $\phi(t)$ and $g(t)$ are as above.

unlike the relative error in reliability, the other two relative errors are independent of t and are functions of all the three parameters λ_1, λ_2, and λ_{12}. Thus the wrong assumption leads to an underassessment in the case of failure rate and an overassessment in the case of mean residual life.

(2) For Gumbel I, the relative error in reliability is negative and decreases from 0 to -1 as a function of λ_{12} (t fixed) or $t(\lambda_{12}$ fixed), resulting in an overassessment under the wrong assumption of independence. In the case of Gumbel II, the relative error has the same sign as that of α and increases (decreases) from 0 to α if α is positive (negative).

(3) The relative error in failure rate for Gumbel I is positive and is a linear function of t (λ_{12} fixed). In this case the wrong assumption will lead to underassessment.

(4) For the case $\lambda_1 = \lambda_2$, the relative error in failure rate for Gumbel II is positive (negative) for negative (positive) values of α. If α is negative (positive) it increases (decreases) from 0 to $(1-\sqrt{1+\alpha})/2$. Thus the absolute maximum error in this case is $|1-\sqrt{\alpha+1}|/2$. The critical point is $t = -(1/\lambda_1)\ln[\sqrt{\alpha+1}(\sqrt{\alpha+1}-1)/\alpha]$.

(5) For the case $\lambda_1 = \lambda_2$, the relative error in the mean residual life for Gumbel II is positive (negative) for positive (negative) values of α. If α is positive (negative) it increases (decreases) from 0 to $(3\sqrt{\alpha+1}-\sqrt{\alpha+9})^2/12[\sqrt{(\alpha+9)(\alpha+1)}-(\alpha+3)]$. Thus the absolute maximum error is $(3\sqrt{\alpha+1}-\sqrt{\alpha+9})^2/12[\sqrt{(\alpha+9)(\alpha+1)}-(\alpha+3)]$.

(6) The relative error in the mean residual life for Gumbel I can be written as

$(\lambda_1 + \lambda_2)$ (1/(failure rate of normal random variable with mean = $-(\lambda_1 + \lambda_2)/2\lambda_{12}$ and variance = $1/2\lambda_{12}$)) $- 1$. Since the failure rate of a normal random variable is increasing, the relative error under discussion is decreasing. It is always negative and decreases from $(\sqrt{\pi/\lambda_{12}})(1 - \phi((\lambda_1 + \lambda_2)/\sqrt{2\lambda_{12}}))(\lambda + \lambda_2)/e^{-(\lambda_1+\lambda_2)^2/4\lambda_{12}}) - 1$ to -1. Thus in this case the independence assumption leads to overassessment of the mean residual life.

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