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# Stochastic compactness of Lévy processes

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***This paper is dedicated to the memory of Sándor Csörgő***

**Abstract:** We characterize stochastic compactness and convergence in distribution of a Lévy process at "large times", i.e., as $t \to \infty$, by properties of its associated Lévy measure, using a mechanism for transferring between discrete (random walk) and continuous time results. We thereby obtain also domain of attraction characterisations for the process at large times. As an illustration of the stochastic compactness ideas, semi-stable laws are considered.

## 1. Introduction

Consider a Lévy process $(X_t)_{t\geq 0}$, having nondegenerate infinitely divisible (inf. div.) characteristic function (cf)

$$Ee^{\mathrm{i}\theta X_t} = e^{t\Psi(\theta)}, \quad \theta \in \mathbb{R}, \tag{1.1}$$

where

$$\Psi(\theta) = \mathrm{i}\gamma\theta - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} \left(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|\leq 1\}}\right) \Pi(\mathrm{d}x), \tag{1.2}$$

$\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$, and $\Pi$ is a measure on $\mathbb{R}$ with $\int_{\mathbb{R}\setminus\{0\}}(x^2 \wedge 1)\Pi(\mathrm{d}x)$ finite. We say that $X_t$ has *canonical triplet* $(\gamma, \sigma^2, \Pi)$. See Bertoin [1] and Sato [35] for properties of Lévy processes.

We wish to study conditions for stochastic compactness, convergence in distribution and relative stability for $X_t$ at "large times", i.e., as $t \to \infty$, and to express them in terms of the Lévy tail functions

$$\overline{\Pi}^{+}(x) = \Pi\{(x,\infty)\}, \ \overline{\Pi}^{-}(x) = \Pi\{(-\infty,-x)\}, \text{ and } \overline{\Pi}(x) = \overline{\Pi}^{+}(x) + \overline{\Pi}^{-}(x)$$

(all on $x > 0$), and the truncated mean and variance functions defined for $x > 0$ by

$$\nu(x) = \gamma + \int_{1<|y|\leq x} y\Pi(\mathrm{d}y) \quad \text{and} \quad V(x) = \sigma^2 + \int_{0<|y|\leq x} y^2\Pi(\mathrm{d}y). \tag{1.3}$$

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(Note that $\nu(x) = \gamma - \int_{x<|y|\le 1} y\Pi(\mathrm{d}y)$ for $0 < x < 1$.)

We shall begin with a couple of motivating observations.

**Observation 1.** Let $\xi, \xi_1, \xi_2, \ldots,$ be i.i.d. nondegenerate random variables (rvs) with cumulative distribution function $F$ and for each integer $n \ge 1$ denote their partial sum by $S_n = \sum_{i=1}^{n} \xi_i$. Suppose that there exist a subsequence $\{n_k\} \subset \{n\}$ and norming and centering constants $B(n_k)$ and $A(n_k)$ such that

$$\frac{S_{n_k} - A(n_k)}{B(n_k)} \xrightarrow{\mathrm{D}} X_1, \tag{1.4}$$

where here and elsewhere in this paper $X_1$ is a nondegenerate inf. div. rv with cf $e^{\Psi(\theta)}$. Associated with the $X_1$ that appears as the distributional limit in (1.4) is a Lévy process $(X_t)_{t\ge 0}$. It arises as the distributional limit,

$$\frac{1}{B(n_k)} \left\{ \sum_{i=1}^{\lfloor t n_k \rfloor} \xi_i - \frac{\lfloor t n_k \rfloor A(n_k)}{n_k} \right\} \xrightarrow{\mathrm{D}} X_t, \; t > 0, \tag{1.5}$$

where $X_t$ has cf $Ee^{\mathrm{i}\theta X_t} = e^{t\Psi(\theta)}$. (This fact goes back to Lévy [24]. See also Théoréme II of Doeblin [10]. Actually one can prove convergence in $D[0,T]$ for any $0 < T < \infty$. For the appropriate theory and methodology refer to Chapter IX of Gikhman and Skorokhod [17].) Note that the norming constants do not change with $t$. We also record here the well-known fact that if $X$ is an inf. div. rv then there exists a Lévy process $X_t$ such that $X_1 \stackrel{\mathrm{D}}{=} X$. See Lévy [24] and Sato [35].

**Observation 2.** Let $X_t$, $t \ge 0$, be a Lévy process and $\xi, \xi_1, \xi_2, \ldots,$ be i.i.d. $X_1$. Suppose for a subsequence $\{n_k\}$ of the integers $\{n\}$ there exist sequences of norming $B(n_k)$ and centering constants $A(n_k)$ such that

$$\frac{S_{n_k} - A(n_k)}{B(n_k)} \xrightarrow{\mathrm{D}} Y, \tag{1.6}$$

where $Y$ is an almost surely (a.s.) finite rv. Then obviously since $X_{n_k} \stackrel{\mathrm{D}}{=} S_{n_k}$,

$$\frac{X_{n_k} - A(n_k)}{B(n_k)} \xrightarrow{\mathrm{D}} Y.$$

Conversely suppose for a sequence of positive constants $t_k \to \infty$ there exist sequences of norming $B(t_k)$ and centering constants $A(t_k)$ such that

$$\frac{X_{t_k} - A(t_k)}{B(t_k)} \xrightarrow{\mathrm{D}} Y,$$

where $Y$ is an a.s. finite random variable. Set $n_k = \lfloor t_k \rfloor \vee 1$. Clearly, for $t_k \ge n_k$,

$$\frac{X_{t_k} - A(t_k)}{B(t_k)} - \frac{X_{n_k} - A(t_k)}{B(t_k)} \stackrel{\mathrm{D}}{=} \frac{X_{t_k - n_k}}{B(t_k)}.$$

Further for $t_k \ge n_k$, we see that

$$E\exp\left(\theta i X_{t_k - n_k}/B(t_k)\right) = e^{(t_k - n_k)\Psi(\theta/B(t_k))}.$$

Noting that necessarily $B(t_k) \to \infty$ and $0 \le t_k - n_k \le 1$ for $t_k \ge n_k$, we can conclude that $X_{t_k - n_k}/B(t_k) \xrightarrow{\mathrm{P}} 0$. Thus since $X_{n_k} \stackrel{\mathrm{D}}{=} S_{n_k}$, we have

$$\frac{S_{n_k} - A(t_k)}{B(t_k)} \xrightarrow{\mathrm{D}} Y.$$

**Definition 1.** We shall say that a Lévy process $X_t$, $t \geq 0$, is in the *Feller class* at infinity (*stochastically compact* at infinity) if there exist nonstochastic functions $B(t) > 0$, $A(t)$ such that every sequence $t_k \to \infty$ contains a subsequence $t_{k'} \to \infty$ with

$$\frac{X_{t_{k'}} - A(t_{k'})}{B(t_{k'})} \xrightarrow{\mathrm{D}} Y', \tag{1.7}$$

where $Y'$ is a finite nondegenerate rv, a.s. (The prime on $Y'$ signifies that in general it depends on the choice of subsequence $t_{k'}$.) We shall write this as "$X_t \in FC$". If the centering function $A(t)$ can be chosen to be identically equal to zero, we shall say that $X_t$ is in the centered Feller class at infinity, written "$X_t \in FC_0$".

**Definition 1′.** We shall also have occasion to talk about another kind of Feller class. We shall say that a Lévy process $X_t$, $t \geq 0$, is in the *Feller class* at zero if there exist nonstochastic functions $B(t) > 0$, $A(t)$ such that every sequence $t_k \downarrow 0$ contains a subsequence $t_{k'} \downarrow 0$ for which (1.7) holds. We shall write this as "$X_t \in FC$ at zero". If the centering function $A(t)$ can be chosen to be identically equal to zero, we shall say that $X_t$ is in the centered Feller class at zero, written "$X_t \in FC_0$ at zero".

We shall also need the notion of a partial sum being in the Feller class.

**Definition 2.** We shall say that a sequence of partial sums $\{S_n\}_{n\geq 1}$ of i.i.d. $\xi$ rv with cumulative distribution function $F$ is in the *Feller class* (*stochastically compact*) if there exist norming and centering constants $B(n) > 0$, $A(n)$ such that every subsequence $\{n_k\}$ of $\{n\}$ contains a further subsequence $n_{k'} \to \infty$ with

$$\frac{S_{n_{k'}} - A(n_{k'})}{B(n_{k'})} \xrightarrow{\mathrm{D}} Y', \tag{1.8}$$

where $Y'$ is a finite nondegenerate rv, a.s. We shall write this as "$S_n \in FC$". If the centering function $A(n)$ can be chosen to be identically equal to zero, we shall say that $S_n$ is in the centered Feller class at infinity, written "$S_n \in FC_0$".

The classic Feller [14] condition for $S_n \in FC$ (stochastic compactness of $F$) is

$$\limsup_{y\to\infty} \frac{y^2 P\left(|\xi| > y\right)}{E\left(\xi^2 \mathbf{1}_{\{|\xi|\leq y\}}\right)} < \infty. \tag{1.9}$$

Here are two additional useful characterizations of $S_n \in FC$.

Feller [15] and Maller [26] show that $S_n \in FC$ if and only if there exist $c \geq 1$ and $0 < \alpha \leq 2$ such that for all $\lambda \geq 1$,

$$\limsup_{y\to\infty} V_\xi\left(\lambda y\right)/V_\xi\left(y\right) \leq c\lambda^{2-\alpha}, \tag{1.10}$$

where $V_\xi(y) = E(\xi^2 \mathbf{1}_{\{|\xi|\leq y\}})$ for $y > 0$. The equivalence of (1.9) and (1.10) can also be inferred from Lemma 1 below.

In the course of developing their quantile-empirical process approach to the asymptotic distribution of partial sums of i.i.d. rvs, Csörgő, Haeusler and Mason [9] show that $S_n \in FC$ (namely the Feller condition (1.9) holds) if and only if for all $\lambda > 0$

$$\limsup_{s\searrow 0} \frac{\sqrt{s}\left\{\left|F^{-1}\left(\lambda s\right)\right| + \left|F^{-1}\left(1-\lambda s\right)\right|\right\}}{\sigma\left(s\right)} < \infty, \tag{1.11}$$

where $F^{-1}$ is the *inverse* or *quantile function* of $F$ defined to be, for each $0 < s < 1$, $F^{-1}(s) = \inf\{x : F(x) \geq s\}$, and for $0 < s < 1/2$,

$$\sigma^2(s) = \int_s^{1-s}\int_s^{1-s}(u \wedge v - uv)\, F^{-1}(\mathrm{d}u)\, F^{-1}(\mathrm{d}v). \tag{1.12}$$

Csörgő et al. [9] also prove that whenever $S_n \in FC$ one can always choose in (1.8) for $n \geq 2$

$$B(n) = \sqrt{n}\sigma(1/n) \text{ and } A(n) = n\int_{1/n}^{1-1/n} F^{-1}(u)\,\mathrm{d}u. \tag{1.13}$$

Some standard norming and centering constants in (1.8), in terms of $F$, are

$$B(n), \text{ where } Q(B(n)) = 1/n, \text{ and } A(n) = nEX\mathbf{1}_{\{|X| \leq B(n)\}}, \tag{1.14}$$

where $Q$ is defined in (2.3) below. See Feller [14] complemented by Jain and Orey [21].

Clearly $S_n \in FC_0$ if and only if $S_n \in FC$ and $\limsup_{n\to\infty} |A(n)/B(n)| < \infty$. Maller [25] (see also Giné and Mason [18] and Griffin and Maller [20]) proved that $S_n \in FC_0$ if and only if

$$\limsup_{y\to\infty} \frac{y^2 P(|\xi| > y) + y\left|E\left(\xi\mathbf{1}_{\{|\xi|\leq y\}}\right)\right|}{E\left(\xi^2\mathbf{1}_{\{|\xi|\leq y\}}\right)} < \infty. \tag{1.15}$$

In terms of the quantile function, $S_n \in FC_0$ is equivalent to, for all $\lambda > 0$,

$$\limsup_{s\searrow 0} \frac{\sqrt{s}\left\{\left|F^{-1}(\lambda s)\right| + \left|F^{-1}(1-\lambda s)\right|\right\} + s^{-1/2}\left|\int_s^{1-s} F^{-1}(u)\,\mathrm{d}u\right|}{\sigma(s)} < \infty. \tag{1.16}$$

The notion of $S_n \in FC_0$, respectively $X_t \in FC_0$ at zero, plays an important role in the study of the asymptotic distribution of self-normalized sums in Giné and Mason [18] and Mason [30], respectively, self-normalized Lévy processes at small times in Maller and Mason [28].

Specializing to the case when $\xi, \xi_1, \xi_2, \ldots,$ are i.i.d. $X_1$, we can obviously readily conclude from Observation 2 that $X_t \in FC$ if and only if $S_n \in FC$ if and only if (1.9) holds with $\xi = X_1$; and $X_t \in FC_0$ if and only if $S_n \in FC_0$ if and only if (1.15) holds with $\xi = X_1$. However, these conditions in terms of the distribution of $X_1$ are not easy to check.

The main goal of this paper is to establish conditions equivalent to (1.9) and (1.15) in terms of the $\gamma$, $\sigma^2$ and $\Pi$ occuring in (1.2). We state these in the following theorem. Recall also (1.3).

**Theorem 1.** *Let $X$ be a nondegenerate inf. div. rv having cf $e^{\Psi(\theta)}$, where $\Psi$ is defined in (1.2), and let $X_t$ be a Lévy process with $X_1 \stackrel{\mathrm{D}}{=} X$.*

*(i) We have $X_t \in FC$ at infinity if and only if*

$$\limsup_{y\to\infty} y^2\overline{\Pi}(y)/V(y) < \infty. \tag{1.17}$$

*(ii) We have $X_t \in FC_0$ at infinity if and only if*

$$\limsup_{y\to\infty}\left(y^2\overline{\Pi}(y) + y|\nu(y)|\right)/V(y) < \infty. \tag{1.18}$$

By applying a result of Pruitt [34], Theorem 1 leads to the following corollary. In its statement, $S_n$, $n \geq 1$, denotes a sequence of partial sums $S_n = \sum_{i=1}^{n} \xi_i$, with $\xi_1, \xi_2, \ldots$, being i.i.d. $\xi$, which is assumed to be nondegenerate.

**Corollary 1.** *Whenever $S_n \in FC$, respectively, $S_n \in FC_0$, then each of its associated subsequential Lévy processes $X_t$, as introduced in (1.5) of Observation 1, is in both $FC$ (at infinity) and $FC$ (at zero), respectively, in $FC_0$ (at infinity) and $FC_0$ (at zero).*

The complete proof of Theorem 1 will be given in Section 4. It follows from a much more general result concerning transferring asymptotic distributional results from continuous time to discrete time and vice versa. This result is stated and proved in Section 3. A number of its consequences are detailed in Section 4. However in the next section we will digress to give a direct proof of part (i) of Theorem 1, which may be of separate interest. We shall also prove Corollary 1 in this section. In Section 5 we point out how semistable processes provide a nice example of Lévy processes, which possess the kind of asymptotic behavior that we will be describing.

## 2. A digression on stochastic compactness

We begin this section with a direct analytic proof of the first part of Theorem 1. We do this by showing that a nondegenerate inf. div. rv $X$ is stochastically compact, meaning that it satisfies the classic Feller condition (1.9) with $\xi = X$, if and only if (1.17) holds.

**Proposition 1.** *Let $X$ be a nondegenerate inf. div. rv having cf function $e^{\Psi(\theta)}$. We have*

$$\limsup_{y\to\infty} \frac{y^2 P\left(|X| > y\right)}{E\left(X^2 \mathbf{1}_{\{|\xi|\leq y\}}\right)} < \infty \tag{2.1}$$

*if and only if (1.17) holds.*

*Proof.* As in Pruitt [34] for $y > 0$ let

$$G\left(y\right) = P\left(|X| > y\right) \tag{2.2}$$

and

$$Q\left(y\right) = P\left(|X| > y\right) + y^{-2} \int_{0<|u|\leq y} u^2 F\left(\mathrm{d}u\right). \tag{2.3}$$

Also set

$$Q^{\Pi}\left(y\right) = \overline{\Pi}\left(y\right) + y^{-2} \left(\sigma^2 + \int_{0<|u|\leq y} u^2 \Pi\left(\mathrm{d}u\right)\right). \tag{2.4}$$

For future reference we note that by integrating by parts,
(2.5)

$$Q\left(y\right) = 2y^{-2} \int_{0<x\leq y} x G\left(x\right) \mathrm{d}x \text{ and } Q^{\Pi}\left(y\right) = y^{-2}\left(\sigma^2 + 2\int_{0<x\leq y} x\overline{\Pi}\left(x\right)\mathrm{d}x\right).$$

We shall be applying the lemma on page 968 of Pruitt [34], which says that as $y \to \infty$,

$$Q^{\Pi}\left(y\right) \approx Q\left(y\right), \tag{2.6}$$

in the sense that, for some constants $c_1$ and $c_2$, $0 < c_1 \le Q^{\Pi}(y)/Q(y) \le c_2 < \infty$ for all large enough $y$.

Let $\rho$ be a nonnegative decreasing right continuous function such that for all $y > 0$

$$0 < U_\rho(y) := \int_0^y u\rho(u)\,\mathrm{d}u < \infty.$$

The next lemma is essentially familiar from "dominated variation" ideas. (For instance, refer to Theorem 2 of Feller [14].) For completeness we provide a short proof here.

**Lemma 1.** *There exist $c \ge 1$ and $0 < \alpha \le 2$ such that for all $\lambda \ge 1$,*

$$\limsup_{y\to\infty} U_\rho(\lambda y)/U_\rho(y) \le c\lambda^{2-\alpha} \tag{2.7}$$

*if and only if*

$$\limsup_{y\to\infty} \frac{y^2\rho(y)}{U_\rho(y)} = \tau < 2. \tag{2.8}$$

*Proof.* First assume (2.8). We see that for some $\tau < \kappa < 2$ for all $y > 0$ large enough $y^2\rho(y)/U_\rho(y) \le \kappa$. Therefore for all large enough $y > 0$ for any $\lambda \ge 1$

$$\log\left(\frac{U_\rho(\lambda y)}{U_\rho(y)}\right) = \int_y^{\lambda y} \frac{U'_\rho(u)}{U_\rho(u)}\mathrm{d}u$$

$$= \int_y^{\lambda y} \frac{u^2\rho(u)}{U_\rho(u)}\frac{\mathrm{d}u}{u} \le \kappa\log\lambda,$$

which implies that

$$\limsup_{y\to\infty} U_\rho(\lambda y)/U_\rho(y) \le \lambda^{2-\alpha},$$

where $2 - \alpha = \kappa$. Hence (2.7) holds.

Now assume (2.8) does not hold. We shall prove that then (2.7) cannot be satisfied. Arguing exactly as on page 864 of Mason [30] one can show that for any $0 < \kappa < 2$ there exists a strictly increasing sequence of positive constants $\eta_n$ such that for all $\lambda \ge 1$

$$\liminf_{n\to\infty} \inf_{\eta_n \le u \le \lambda\eta_n} \frac{u^2\rho(u)}{U_\rho(u)} \ge \kappa.$$

Thus for all $\lambda \ge 1$

$$\liminf_{n\to\infty} \log\left(\frac{U_\rho(\lambda\eta_n)}{U_\rho(\eta_n)}\right) = \liminf_{n\to\infty} \int_{\eta_n}^{\lambda\eta_n} \frac{u^2\rho(u)}{U_\rho(u)}\frac{\mathrm{d}u}{u} \ge \kappa\log\lambda.$$

This implies that (2.7) cannot hold for any $c \ge 1$ and $0 < \alpha \le 2$. □

Set

$$U_G(y) = \int_0^y uG(u)\,\mathrm{d}u \text{ and } U_{\overline{\Pi}}(y) = \int_0^y u\overline{\Pi}(u)\,\mathrm{d}u.$$

In the remainder of the proof of Proposition 1, we will assume that $U_{\overline{\Pi}}(y) > 0$ for all $y > 0$. Otherwise for $X$ to be nondegenerate it must be $N(0,\sigma^2)$ with $\sigma^2 > 0$, in which case (2.1) and (1.17) are trivially equivalent. Clearly by (2.5) and (2.6), as $y \to \infty$,

$$U_G(y) \approx U_{\overline{\Pi}}(y).$$

Thus there exist $c \geq 1$ and $0 < \alpha \leq 2$ such that for all $\lambda \geq 1$,

$$\limsup_{y\to\infty} U_G(\lambda y)/U_G(y) \leq c\lambda^{2-\alpha} \tag{2.9}$$

if and only if there exist $c' \geq 1$ and $0 < \alpha' \leq 2$ such that for all $\lambda \geq 1$,

$$\limsup_{y\to\infty} U_{\overline{\Pi}}(\lambda y)/U_{\overline{\Pi}}(y) \leq c'\lambda^{2-\alpha'}. \tag{2.10}$$

Applying Lemma 1 we see that

$$\limsup_{y\to\infty} \frac{y^2 G(y)}{U_G(y)} = \tau < 2 \text{ if and only if } \limsup_{y\to\infty} \frac{y^2\overline{\Pi}(y)}{U_{\overline{\Pi}}(y)} = \tau' < 2.$$

After integrating by parts, this proves the equivalence of (2.1) and (1.17). □

**Remark.** Notice that Lemma 1 provides a proof of $S_n \in FC$ if and only if (1.10) holds.

### *2.1. Proof of Corollary 1*

Let $S_n \in FC$ with norming and centering constants $B(n)$ and $A(n)$ and let $X_1$ be one of the nondegenerate subsequential limiting random variables. We shall assume that $E\xi^2 = \infty$. Otherwise the associated Lévy process is Brownian motion, which is clearly both in $FC$ at $\infty$ and at zero, with norming and centering functions $B(t) = \sqrt{t}$ and $A(t) = 0$ (and hence is in $FC_0$ in both cases). Now $X_1$ is inf. div., and we assume that $X_1$ has Lévy measure $\Pi$. According to the theorem in Pruitt [34], for some constant $C > 0$, for all $y > 0$,

$$y^2\overline{\Pi}(y) \leq CV(y). \tag{2.11}$$

By Theorem 1 this implies $X_t \in FC$ at infinity, and, by Theorem 2.1 in Maller and Mason [28], that $X_t \in FC$ at zero. (In Theorem 2.1 of [28] it is shown that $X_t \in FC$ at zero if and only if $\limsup_{x\downarrow 0} x^2\overline{\Pi}(x)/V(x) < \infty$.)

Now choose $S_n \in FC_0$. Once more we shall assume that $E\xi^2 = \infty$. Otherwise one readily argues that $E\xi = 0$ (see the proof of Part (ii) of Theorem 1 in Section 4) and that the associated Lévy process is Brownian motion, which is both in $FC_0$ at $\infty$ and at zero, as just mentioned. Recall, as we stated in the Introduction, that $S_n \in FC_0$ if and only if (1.15) holds. Assume that $F$ is in the centered Feller class with sequence of norming constants $B(n)$ and suppose that for a subsequence $\{n_k\}$ of $\{n\}$, $S_{n_k}/B(n_k)$ converges in distribution to a nondegenerate inf. div. rv $X_1$ with canonical triplet $(\gamma, \sigma^2, \Pi)$. We can infer by Gnedenko and Kolmogorov [19], Theorem 1, page 116, that necessarily for every choice of $x > 0$ and $\varepsilon > 0$ such that $\Pi\{-x, x\} = \Pi\{-\varepsilon, \varepsilon\} = 0$,

$$n_k P\{|X| > xB(n_k)\} \to \overline{\Pi}(x), \tag{2.12}$$

and for some $b_\varepsilon$, as $n_k \to \infty$,

$$\frac{n_k}{B(n_k)} \int_{-\varepsilon B(n_k)}^{\varepsilon B(n_k)} yF(\mathrm{d}y) \to b_\varepsilon; \tag{2.13}$$

further, Pruitt [34] shows that when $E\xi^2 = \infty$,

$$\frac{n_k}{B^2(n_k)} \int_{-xB(n_k)}^{xB(n_k)} y^2 F(\mathrm{d}y) \to V(x). \tag{2.14}$$

Thus by (1.15), (2.12), (2.13) and (2.14), for some $C > 0$,

$$C \geq \limsup_{k\to\infty} \frac{x^2 B^2(n_k) P\{|X| > xB(n_k)\} + xB(n_k) \left|\int_{-xB(n_k)}^{xB(n_k)} yF(\mathrm{d}y)\right|}{\int_{-xB(n_k)}^{xB(n_k)} y^2 F(\mathrm{d}y)}$$

$$= \frac{x^2\overline{\Pi}(x) + x\left|b_\varepsilon + \int_{\varepsilon\leq|y|\leq x} y\Pi(\mathrm{d}y)\,\mathbf{1}_{\{\varepsilon\leq x\}} - \int_{x\leq|y|\leq\varepsilon} y\Pi(\mathrm{d}y)\,\mathbf{1}_{\{x\leq\varepsilon\}}\right|}{V(x)}. \tag{2.15}$$

By right continuity of $\Pi$ this in fact holds for all $x > 0$. Observe that

$$m_\Pi(x) := b_\varepsilon + \int_{\varepsilon\leq|y|\leq x} y\Pi(\mathrm{d}y)\,\mathbf{1}_{\{\varepsilon\leq x\}} - \int_{x\leq|y|\leq\varepsilon} y\Pi(\mathrm{d}y)\,\mathbf{1}_{\{x\leq\varepsilon\}} \tag{2.16}$$

is independent of the particular $\varepsilon > 0$ satisfying $\Pi\{-\varepsilon, \varepsilon\} = 0$. Note that once $b_\varepsilon$ is defined for one $\varepsilon > 0$ satisfying $\Pi\{-\varepsilon, \varepsilon\} = 0$, then for any other $\delta > 0$ such that $\Pi\{-\delta, \delta\} = 0$,

$$b_\delta = \lim_{k\to\infty} \frac{n_k}{B(n_k)} \int_{-\delta B(n_k)}^{\delta B(n_k)} yF(\mathrm{d}y)$$

$$= b_\varepsilon + \int_{\varepsilon\leq|y|\leq\delta} y\Pi(\mathrm{d}y)\,\mathbf{1}_{\{\varepsilon\leq\delta\}} - \int_{\delta\leq|y|\leq\varepsilon} y\Pi(\mathrm{d}y)\,\mathbf{1}_{\{\delta\leq\varepsilon\}}.$$

Notice, moreover, that necessarily in the triplet $(\gamma, \sigma^2, \Pi)$,

$$\gamma = b_\varepsilon + \int_{\varepsilon\leq|y|\leq 1} y\Pi(\mathrm{d}y), \text{ and thus } m_\Pi(x) = \nu(x). \tag{2.17}$$

We now get that from inequality (2.15) and (2.17), that for all $x > 0$,

$$x^2\overline{\Pi}(x) + x|\nu(x)| \leq CV(x). \tag{2.18}$$

This implies by Theorem 1 that $X_t \in FC_0$ at infinity and by Theorem 2.3 of Maller and Mason [28] that $X_t \in FC_0$ at zero. (In Theorem 2.3 of [28] it shown that $X_t \in FC_0$ at zero if and only if $\limsup_{x\downarrow 0}(x^2\overline{\Pi}(x) + x|\nu(x)|)/V(x) < \infty$. )

**Remark.** Inequality (2.18) can be used to define a class of infinitely divisible distributions $\mathcal{I}$ . Say that a distribution function $H$ is in $\mathcal{I}$ if and only if $H$ is nondegenerate and infinitely divisible and has a Lévy measure $\Pi$ such that, for all $x > 0$, inequality (2.18) holds. Clearly if $X_t$ is a Lévy process such that the distribution of $X_1$ is in $\mathcal{I}$, then both $X_t \in FC_0$ (at infinity) and $X_t \in FC_0$ (at zero).

## 3. Transferring between continuous and discrete time

In this section we develop a mechanism for systematically transferring weak convergence results concerning $X_t$, with conditions specified in terms of its *canonical* *triplet* $(\gamma, \sigma^2, \Pi)$ (rather than in terms of its marginal distributions, or the distribution of its increments), to a random walk, with conditions specified in terms of the distribution of its increments, and vice versa.

*We shall assume throughout that* $\overline{\Pi}(x) > 0$ *for all* $x > 0$. Whenever this does not hold, $X_t$ has finite variance. (See for instance Kruglov [23] and Csörgő [3].) Define a distribution $F^{\Pi}(x)$ on $\mathbb{R}$ as follows. Take $x_0 > 1$ such that $\overline{\Pi}(x_0)/\overline{\Pi}(1) < 1$, and let

$$\widetilde{\gamma} = \gamma + \int_{1<|y|\le x_0} y\Pi(\mathrm{d}y).$$

This is $o(x_0)$ as $x_0 \to \infty$, so we can choose $x_0$ larger if necessary so that $x_0 > |\widetilde{\gamma}| \vee 1$. Let

$$F^{\Pi}(\mathrm{d}x) = \frac{\Pi(\mathrm{d}x)\mathbf{1}_{\{|x|>x_0\}}}{\overline{\Pi}(1)}, \text{ for } x \in \mathbb{R},$$

with mass $p := 1 - \overline{\Pi}(x_0)/\overline{\Pi}(1) > 0$ at point $\widetilde{\gamma}/(p\overline{\Pi}(1))$. We can further assume that

$$\frac{|\widetilde{\gamma}|}{p\overline{\Pi}(1)} < x_0,$$

because $p \to 1$ as $x_0 \to \infty$. Let $X^{\Pi}$ have distribution $F^{\Pi}$. Then we have, for $x > x_0$,

$$P(X^{\Pi} > x) = \frac{\overline{\Pi}^+(x)}{\overline{\Pi}(1)} \quad \text{and} \quad P(X^{\Pi} \le -x) = \frac{\overline{\Pi}^-(x)}{\overline{\Pi}(1)}. \tag{3.1}$$

$F^{\Pi}(x)$ is constant on $[-x_0, x_0]$ except for the jump at $\widetilde{\gamma}/(p\overline{\Pi}(1))$. Thus

$$E\left(X^{\Pi}\mathbf{1}_{\{|X^{\Pi}|\le x_0\}}\right) = \left(\frac{\widetilde{\gamma}}{p\overline{\Pi}(1)}\right)p = \frac{\widetilde{\gamma}}{\overline{\Pi}(1)}.$$

When $x > x_0$, we add to this a contribution from $[-x, -x_0) \cup (x_0, x]$ to get

$$\begin{aligned} \nu^{\Pi}(x) := E\left(X^{\Pi}\mathbf{1}_{\{|X^{\Pi}|\le x\}}\right) &= \frac{1}{\overline{\Pi}(1)}\left(\widetilde{\gamma} + \int_{x_0<|y|\le x} y\Pi(\mathrm{d}y)\right) \\ &= \frac{1}{\overline{\Pi}(1)}\left(\gamma + \int_{1<|y|\le x} y\Pi(\mathrm{d}y)\right) = \frac{\nu(x)}{\overline{\Pi}(1)} \end{aligned} \tag{3.2}$$

(recall (1.3)). Similarly, when $x > x_0$, letting

$$c_0 = \frac{\widetilde{\gamma}^2}{p\overline{\Pi}(1)} - V(x_0),$$

we have

$$\begin{aligned} V^{\Pi}(x) := E\left((X^{\Pi})^2\mathbf{1}_{\{|X^{\Pi}|\le x\}}\right) &= \left(\frac{\widetilde{\gamma}}{p\overline{\Pi}(1)}\right)^2 p + \frac{1}{\overline{\Pi}(1)}\int_{x_0<|y|\le x} y^2\Pi(\mathrm{d}y) \\ &= \frac{c_0 + V(x)}{\overline{\Pi}(1)}. \end{aligned} \tag{3.3}$$

(Note that the last expression is asymptotic to $V(x)/\overline{\Pi}(1)$, as $x \uparrow \infty$, when $EX_1^2 = \infty$.)

Now let $X_i^{\Pi}$ be i.i.d. with distribution $F^{\Pi}$, and set

$$S_n^{\Pi} = X_1^{\Pi} + \cdots + X_n^{\Pi}, \ n = 1, 2, \ldots.$$

**Proposition 2.** *Assume that* $\overline{\Pi}(x) > 0$ *for all* $x > 0$.

*(i) Assume* $EX_1^2 = \infty$. *Suppose there are nonstochastic functions* $A(t)$ *and* $B(t) > 0$ *and a sequence* $n_k \uparrow \infty$ *of integers for which*

$$\frac{X_{n_k} - A(n_k)}{B(n_k)} \xrightarrow{\mathrm{D}} Y, \tag{3.4}$$

*a finite random variable. Then*

$$\frac{S^{\Pi}_{n_k} - A^{\Pi}_{n_k}}{B^{\Pi}_{n_k}} \xrightarrow{\mathrm{D}} Y_{\Pi}, \tag{3.5}$$

*where* $A^{\Pi}_n = A(n)/\overline{\Pi}(1)$, $B^{\Pi}_n = B(n)$, *and* $Y_{\Pi}$ *is a finite random variable, related to* $Y$ *by*

$$E(e^{\mathrm{i}\theta Y_{\Pi}}) = \left(E(e^{\mathrm{i}\theta Y})\right)^{1/\overline{\Pi}(1)}, \ \text{for } \theta \in \mathbb{R}. \tag{3.6}$$

*Conversely, take any sequence* $t_k \to \infty$ *of real numbers and suppose (3.5) holds for nonstochastic sequences* $A^{\Pi}_n$, $B^{\Pi}_n$, *with* $n_k = \lfloor t_k \rfloor$ *and* $Y_{\Pi}$ *a finite rv. Then (3.4) holds with* $A(t) = \overline{\Pi}(1) A_{\lfloor t \rfloor}$ *and* $B(t) = B^{\Pi}_{\lfloor t \rfloor}$, *and again* $Y_{\Pi}$ *and* $Y$ *are related by (3.6).*

*(ii) Assume* $EX_1^2 < \infty$. *Then (3.4) holds with*

$$A(t) = tEX_1 = t\nu(\infty), \ B(t) = \widetilde{\sigma}\sqrt{t},$$

*where*

$$\widetilde{\sigma}^2 = EX_1^2 - (EX_1)^2 = V(\infty) - \nu^2(\infty),$$

*and* $Y \sim N(0,1)$, *while (3.5) holds with*

$$A^{\Pi}_n = nEX^{\Pi}_1 = n\nu^{\Pi}(\infty) = n\nu(\infty)/\overline{\Pi}(1), \ B^{\Pi}_n = \sigma^{\Pi}\sqrt{n},$$

*where*

$$(\sigma^{\Pi})^2 = E(X^{\Pi}_1)^2 - (EX^{\Pi}_1)^2 = V^{\Pi}(\infty) - (\nu^{\Pi}(\infty))^2,$$

*and* $Y_{\Pi} \sim N(0, 1/\overline{\Pi}(1))$.

*Proof of Proposition 2.* First we prove Part (ii). When $EX_1^2 < \infty$, the claimed distributional convergences hold by the correspondence between central limit theorems for $X_t$ and $S^{\Pi}_n$ that follows readily using the approach in Observation 2. (Alternatively, we could apply Theorem 2 below to obtain the central limit theorem for $X_t$.) The expressions for $EX_1$ and $\mathrm{Var}X_1$ given in the statement of Part (ii) follow from the classic representation of a Lévy process given in Chapter VI, Section 3, of Gikhman and Skorokhod [17].

Next we turn to the proof of Part (i). For the remainder of the proof we assume $EX_1^2 = \infty$. As a preliminary step, take any sequence $t_k \to \infty$ of real numbers and assume there are functions $A(t)$, $B(t) > 0$, such that

$$\frac{X_{t_k} - A(t_k)}{B(t_k)} \xrightarrow{\mathrm{D}} Y, \tag{3.7}$$

with $Y$ an a.s. finite rv. $Y$ must be inf. div., with triplet $(\beta,\tau^2,\Lambda)$, say, where $\Lambda$ is a Lévy measure on $\mathbb{R}$, and $B(t_k)\to\infty$. Thus, by (1.1),

$$\begin{aligned}
&E\left(e^{\mathrm{i}\theta\left(X_{t_k}-A(t_k)\right)/B(t_k)}\right)\\
&=\exp\left(\frac{\mathrm{i}\theta\,(t_k\gamma-A(t_k))}{B(t_k)}-\frac{t_k\theta^2\sigma^2}{2B^2(t_k)}\right)\times\\
&\qquad\times\exp\left(t_k\int_{\mathbb{R}\setminus\{0\}}\left(e^{\mathrm{i}\theta u/B(t_k)}-1-\frac{\mathrm{i}\theta u\mathbf{1}_{\{|u|\le 1\}}}{B(t_k)}\right)\Pi(\mathrm{d}u)\right),
\end{aligned}$$

which, by the change of variables $x=u/B(t_k)$, equals

$$\begin{aligned}
&\exp\left(\frac{\mathrm{i}\theta\,(t_k\gamma-A(t_k))}{B(t_k)}-\frac{t_k\theta^2\sigma^2}{2B^2(t_k)}+\frac{\mathrm{i}\theta t_k}{B(t_k)}\int_{1<|u|\le B(t_k)}u\Pi(\mathrm{d}u)\right)\times\\
&\times\exp\left(t_k\int_{\mathbb{R}\setminus\{0\}}(e^{\mathrm{i}\theta x}-1-\mathrm{i}\theta x\mathbf{1}_{\{|x|\le 1\}})\Pi(B(t_k)\mathrm{d}x)\right),
\end{aligned}\tag{3.8}$$

and by hypothesis this tends as $k\to\infty$ to

$$\exp\left(\mathrm{i}\theta\beta-\frac{\theta^2}{2}\tau^2+\int_{\mathbb{R}\setminus\{0\}}(e^{\mathrm{i}\theta x}-1-\mathrm{i}\theta x\mathbf{1}_{\{|x|\le 1\}})\Lambda(\mathrm{d}x)\right).\tag{3.9}$$

Recalling the definition of $\nu(\cdot)$ in (1.3), we see that (3.8) is an inf. div. cf with triple

$$\left(\frac{t_k\nu(B(t_k))-A(t_k)}{B(t_k)},\ \frac{t_k\sigma^2}{B^2(t_k)},\ t_k\Pi(B(t_k)\mathrm{d}x)\right).$$

It converges to an inf. div. cf with triple $(\beta,\tau^2,\Lambda(\mathrm{d}x))$, in (3.9). According to criteria for the convergence of inf. div. distributions (Kallenberg [22], Theorem 15.14, page 295), this means that, as $k\to\infty$,

$$t_k\Pi(B(t_k)\mathrm{d}x)\to\Lambda(\mathrm{d}x),$$

vaguely, on $\mathbb{R}\setminus\{0\}$, and, for each $x>0$ such that $\pm x$ are continuity points of $\Lambda(x)$,

$$\frac{t_k\nu(B(t_k))-A(t_k)}{B(t_k)}-t_k\int_{x<|y|\le 1}y\Pi(B(t_k)\mathrm{d}y)\to\beta-\int_{x<|y|\le 1}y\Lambda(\mathrm{d}y),$$

and

$$\frac{t_k\sigma^2}{B^2(t_k)}+t_k\int_{|y|\le x}y^2\Pi(B(t_k)\mathrm{d}y)\to\tau^2+\int_{|y|\le x}y^2\Lambda(\mathrm{d}y).$$

These say precisely that, as $k\to\infty$, at continuity points $x>0$ of the limits,

$$t_k\overline{\Pi}^{\pm}(xB(t_k))\to\overline{\Lambda}^{\pm}(x),\tag{3.10}$$

where $\overline{\Lambda}^{\pm}(x)$ are the positive and negative tails of $\Lambda$, and further,

$$\frac{t_k\nu(xB(t_k))-A(t_k)}{B(t_k)}\to\beta+\int_{1<|y|\le x}y\Lambda(\mathrm{d}y),\tag{3.11}$$

and

$$\frac{t_k V(xB(t_k))}{B^2(t_k)} \to \tau^2 + \int_{|y|\le x} y^2 \Lambda(\mathrm{d}y). \tag{3.12}$$

Now, assume (3.4), so that $t_k$ is restricted to be a sequence of integers, $t_k = n_k \uparrow \infty$. Then (3.10)–(3.12) hold with $t_k$ replaced by $n_k$. Let

$$\Lambda_\Pi(\mathrm{d}x) := \frac{\Lambda(\mathrm{d}x)}{\overline{\Pi}(1)},\ \beta_\Pi := \frac{\beta}{\overline{\Pi}(1)},\ \text{and } \tau_\Pi^2 := \frac{\tau^2}{\overline{\Pi}(1)},$$

and let $A_n = A(n)$, $A_n^\Pi := A(n)/\overline{\Pi}(1)$, and $B_n^\Pi := B(n)$. Because $EX_1^2 = \infty$, the convergence (3.12) implies $B_{n_k}^\Pi/\sqrt{n_k} \to \infty$, so, given any $x > 0$, we can choose $n_k$ large enough for $xB_{n_k}^\Pi > x_0$, where $x_0$ is as in (3.1)–(3.3). Then, as $k \to \infty$, at continuity points $x > 0$ of the limits,

$$n_k P(X^\Pi > xB_{n_k}^\Pi) = \frac{n_k \overline{\Pi}^+(xB(n_k))}{\overline{\Pi}(1)} \to \overline{\Lambda}_\Pi^+(x), \tag{3.13}$$

$$n_k P(X^\Pi \le -xB_{n_k}^\Pi) = \frac{n_k \overline{\Pi}^-(xB(n_k))}{\overline{\Pi}(1)} \to \overline{\Lambda}_\Pi^-(x), \tag{3.14}$$

(3.15)
$$\frac{n_k E\left(X^\Pi \mathbf{1}_{\{|X^\Pi| \le xB_{n_k}^\Pi\}}\right) - A_{n_k}^\Pi}{B_{n_k}^\Pi} = \frac{n_k \nu(xB(n_k)) - A_{n_k}}{B(n_k)\overline{\Pi}(1)} \to \beta_\Pi + \int_{1<|y|\le x} y \Lambda_\Pi(\mathrm{d}y),$$

and, because $n_k = o(B_{n_k}^2)$,
(3.16)
$$\frac{n_k E\left((X^\Pi)^2 \mathbf{1}_{\{|X^\Pi| \le xB_{n_k}^\Pi\}}\right)}{(B_{n_k}^\Pi)^2 \overline{\Pi}(1)} = \frac{n_k V\left(xB(n_k)\right)}{B^2(n_k)\overline{\Pi}(1)} + o(1) \to \tau_\Pi^2 + \int_{0<|y|\le x} y^2 \Lambda_\Pi(\mathrm{d}y).$$

Now when $EX_1^2 = \infty$, for the random walk $S_n^\Pi$ these mean, by Gnedenko and Kolmogorov [19], Theorem 1, page 116, that (3.5) holds, where $Y_\Pi$ is inf. div. with canonical triplet

$$\left(\beta_\Pi, \tau_\Pi^2, \Lambda_\Pi\right) = \left(\beta, \tau^2, \Lambda\right)/\overline{\Pi}(1).$$

So we also have (3.6), as asserted, in this case.

For the converse, take $t_k \uparrow \infty$, let $n_k := \lfloor t_k \rfloor$, and suppose (3.5) holds for some sequences $A_n^\Pi$ and $B_n^\Pi > 0$. Then $B_{n_k}^\Pi \to \infty$, and (3.13)–(3.16) hold for some triplet $(\beta_\Pi, \tau_\Pi, \Lambda_\Pi)$. Thus (3.10)–(3.12) hold, but with $t_k$ replaced by $n_k$, $A(t_k)$ and $B(t_k)$ replaced by

$$A_{n_k} := \overline{\Pi}(1) A_{n_k}^\Pi \text{ and } B_{n_k}^\Pi,$$

and with $(\beta, \tau, \Lambda) = (\beta_\Pi, \tau_\Pi, \Lambda_\Pi)\overline{\Pi}(1)$. Then by (Kallenberg [22], Theorem 15.14, page 295), the convergence in (3.9) holds with this replacement, so we deduce the existence of sequences $A_n$ and $B_n$ for which

$$\frac{X_{n_k} - A_{n_k}}{B_{n_k}} \xrightarrow{\mathrm{D}} Y, \text{ finite a.s.}$$

Finally, define $A(t) = A_{\lfloor t \rfloor}$ and $B(t) = B_{\lfloor t \rfloor}$, $t > 0$. Then

$$A(t_k) = A_{\lfloor t_k \rfloor} = A_{n_k},\ B(t_k) = B_{\lfloor t_k \rfloor} = B_{n_k},$$

and, by the argument in Observation 2,

$$\frac{X_{t_k} - A(t_k)}{B(t_k)} = \frac{X_{n_k} - A_{n_k}}{B_{n_k}} + \frac{X_{t_k} - X_{n_k}}{B_{n_k}} = \frac{X_{n_k} - A_{n_k}}{B_{n_k}} + o_P(1). \tag{3.17}$$

Thus we obtain (3.4), completing the proof of Proposition 2. □

**Remarks.** There are various ways of transferring results from the discrete time random walks to the continuous time Lévy processes. In Observation 2 we pointed out a simple connection that can sometimes be used, noting that $(X_t)_{t\geq 0}$ evaluated at integer times, $(X_n)_{n\geq 1}$, is a random walk with i.i.d. increments distributed as $X_1$. However as also pointed out in the Introduction, this method leads to conditions expressed in terms of the marginal distribution of $X_1$, which are inaccessible for most Lévy processes. The aim of our paper is to express conditions in terms of the canonical measure of $X_t$ since in practice, this is how most Lévy processes are specified.

Another, more sophisticated, method of transferring between discrete and continuous time is developed in Doney [11], and applied in Doney and Maller [13] to transfer moment and other conditions for exit times. In the present paper this method seems not to be useful for the kinds of results that we are interested in. But the straightforward method outlined in the present section suffices to transfer many weak convergence results.

## 4. Consequences of the discrete-continuous time transfer proposition

First we shall prove Theorem 1.

### *4.1. Proof of Theorem 1*

We shall begin by assuming that $\overline{\Pi}(x) > 0$ for all $x > 0$. Set, for $x > 0$,

$$H^{\Pi}(x) = P\left(\left|X^{\Pi}\right| > x\right)$$

(and recall definitions (3.1)–(3.3)).

Part (i): Notice by (1.9) that $S_n^{\Pi} \in FC$ if and only if

$$\limsup_{x\to\infty} x^2 H^{\Pi}(x)/V^{\Pi}(x) < \infty,$$

so the equivalence of $X_t \in FC$ at infinity and (1.17) follows from an application of Proposition 2. (Both cases, $EX_1^2 \leq \infty$, are covered.)

Part (ii): The equivalence of $X_t \in FC_0$ at infinity and (1.18) follows just as in the proof of Part (i), noting by (1.15) that $S_n^{\Pi} \in FC_0$ if and only if

$$\limsup_{x\to\infty} \left(x|\nu^{\Pi}(x)| + x^2 H^{\Pi}(x)\right)/V^{\Pi}(x) < \infty.$$

Now assume

$$\overline{\Pi}(x) = 0 \text{ for all large enough } x > 0. \tag{4.1}$$

As mentioned above, this implies that $0 < EX_1^2 < \infty$. In this case by the central limit theorem

$$\frac{X_n - nEX_1}{\sqrt{n\mathrm{Var}X_1}} \xrightarrow{\mathrm{D}} N(0,1). \tag{4.2}$$

Therefore by the argument in Observation 2, $X_t \in FC$ with $B(t) = \sqrt{t\mathrm{Var}X_1}$ and $A(t) = \lfloor t \rfloor EX_1$. Thus the equivalence of $X_t \in FC$ and (1.17) is trivial in this case.

Further note that whenever $X_t \in FC_0$ and $EX_1^2 < \infty$, then by an argument based on the convergence of types theorem, necessarily $EX_1 = 0$ in (4.2), which in combination with (4.1) forces $EX_1 = 0$ and $\nu(x) = 0$ for all large $x$, so that (1.18) holds. On the other hand, whenever (4.1) holds, then $EX_1^2 < \infty$ and (1.18) forces $EX_1 = 0 = \nu(x)$ for all large enough $x$, so by an elementary argument as in Observation 2, we see that (4.2) holds, which implies $X_t \in FC_0$ with $B(t) = \sqrt{t\mathrm{Var}X_1}$. Thus we see that in the case (4.1), we also have $X_t \in FC_0$ if and only if (1.18).

Using our discrete-continuous time transfer proposition we could also readily deduce from well-known convergence criteria for partial sums, the following asymptotic normality and stability theorem for a Lévy process $X_t$, as $t \to \infty$, after norming and possibly centering. It is a combination of Theorems 3.2 and 3.5 of Doney and Maller [12].

**Theorem 2.** *Assume* $\overline{\Pi}(x) > 0$ *for all* $x > 0$.

*(i) There are nonstochastic functions* $A(t)$ *and* $B(t) > 0$ *such that*

$$\frac{X_t - A(t)}{B(t)} \xrightarrow{\mathrm{D}} N(0,1), \tag{4.3}$$

*a standard normal rv, if and only if*

$$\lim_{x\uparrow\infty} \frac{V(x)}{x^2\overline{\Pi}(x)} = \infty. \tag{4.4}$$

*(ii) There is a nonstochastic function* $B(t) > 0$ *such that*

$$\frac{X_t}{B(t)} \xrightarrow{\mathrm{D}} N(0,1) \tag{4.5}$$

*if and only if*

$$\lim_{x\uparrow\infty} \frac{V(x)}{x|\nu(x)| + x^2\overline{\Pi}(x)} = \infty. \tag{4.6}$$

*(iii) There is a nonstochastic function* $B(t) > 0$ *such that*

$$\frac{X_t}{B(t)} \xrightarrow{\mathrm{P}} \pm 1 \tag{4.7}$$

*if and only if*

$$\lim_{x\uparrow\infty} \frac{|\nu(x)|}{x\overline{\Pi}(x)} = \infty. \tag{4.8}$$

**Remarks.** When (4.3) holds we say that $X_t$ is in the *domain of attraction of the normal distribution*, denoted $X_t \in D(N)$. When (4.5) holds we say that $X_t$ is in the *centered domain of attraction of the normal distribution*, denoted $X_t \in D_0(N)$. This is equivalent to $X_t \in D(N)$ (in which case $E|X_1| < \infty$), together with $EX_1 = 0$. The *relative stability* described by (4.7) is denoted by $X_t \in RS$. These are analogues of similar classes of random walks. Likewise, *Domains of Attraction* are characterized in the next theorem. The class of real functions regularly varying at infinity with index $\alpha$ will be denoted by "$RV(\alpha)$"; "$SV$" will be the slowly varying functions.

**Theorem 3.** *Assume $\overline{\Pi}(x) > 0$ for all $x > 0$. The following are equivalent:*

*(i) there are nonstochastic functions $A(t)$, $B(t) > 0$, such that*

$$\frac{X_t - A(t)}{B(t)} \xrightarrow{\text{D}} Y, \text{ as } t \to \infty, \tag{4.9}$$

*for an a.s. finite, nondegenerate random variable $Y$.*

*(ii) (a) $V(x) \in SV$ as $x \to \infty$, or (b) $\overline{\Pi}(x) \in RV(-\alpha)$ as $x \to \infty$, for some $\alpha \in (0, 2)$, and the limits $\lim_{x\to\infty} \overline{\Pi}^{\pm}(x)/\overline{\Pi}(x)$ exist.*

*Proof of Theorem 3.* When $\overline{\Pi}^{+}(x)$, $\overline{\Pi}^{-}(x)$, $\overline{\Pi}(x)$ and $V(x)$ are replaced by

$$1 - F^{\Pi}(x),\ F^{\Pi}(-x),\ 1 - F^{\Pi}(x) + F^{\Pi}(-x) \text{ and } V^{\Pi}(x),$$

in the notation of Section 3, we see from (3.1)–(3.3) and by Feller [16] (Theorem 1a, page 303) that Conditions (ii) of Theorem 3 are precisely the conditions for $S_n^{\Pi}$ to be in the domain of attraction of a stable law, i.e., for there to exist nonstochastic sequences $A_n$, $B_n > 0$, such that

$$\frac{S_n^{\Pi} - A_n/\overline{\Pi}(1)}{B_n} \xrightarrow{\text{D}} Y_{\Pi} \tag{4.10}$$

(convergence through the whole sequence $n$), where $Y_{\Pi}$ is a finite nondegenerate (stable) rv. If (4.9) holds then (3.4) holds with $\{n_k\} = \{k\}$, so (4.10) holds by (3.5), and we get Conditions (ii) of Theorem 3.

Conversely, if Conditions (ii) of Theorem 3 are satisfied, then (4.10) holds, so (3.6) and hence (3.4) hold with $n_k = k$, $A(t) = A_{\lfloor t \rfloor}$ and $B(t) = B_{\lfloor t \rfloor}$. For $t > 1$ let $k = k(t) = \lfloor t \rfloor$, then

$$\frac{X_t - A(t)}{B(t)} = \frac{X_k - A_k}{B_k} + o_P(1),$$

where the $o_P(1)$ term is so just as in (3.17). Hence (4.9). □

**Remarks.** It is clear from the proof of Theorem 3 that the limit rv $Y$ in (4.9) is a stable rv, with index $\alpha \in (0, 2]$ (a normal rv, if $\alpha = 2$; in this case, (4.4) is equivalent to $V(x) \in SV$). The result of course is not unexpected but seems not to have been written out before.

## 5. Semistable laws and Lévy processes

An illustrative example of a class of random variables in the Feller Class are those in the domain of geometric partial attraction of a semistable law. We say that the random walk $S_n = \xi_1 + \cdots + \xi_n$ *is in the domain of geometric partial attraction of a semistable law* if there exists an increasing sequence of positive integers $n_k$ such that for some constant $c \geq 1$,

$$n_{k+1}/n_k \to c, \tag{5.1}$$

and centering and norming sequences $A_{n_k}$ and $B_{n_k}$ such that

$$\frac{\sum_{i=1}^{n_k} \xi_i - A_{n_k}}{B_{n_k}} \xrightarrow{\text{D}} Y, \tag{5.2}$$

for an a.s. finite, nondegenerate, rv $Y$. Notice that when $c = 1$, $S_n$ is in the domain of attraction of a stable law.

Whenever (5.1) and (5.2) hold, with $c > 1$, there exist centering and norming sequences $\widetilde{A}_n$ and $\widetilde{B}_n$ such that

$$\frac{\sum_{i=1}^{n} \xi_i - \widetilde{A}_n}{\widetilde{B}_n}$$

is stochastically compact and all of its subsequential limit random variables are contained in the class

$$\mathcal{Y}_c = \left\{\lambda^{-1} Y_\lambda : 1 \le \lambda \le c\right\}, \tag{5.3}$$

where for each $1 \le \lambda \le c$ the rv $Y_\lambda$ has cf

$$E \exp\left(\mathrm{i}\theta\lambda^{-1} Y_\lambda\right) = \exp\left(\lambda^{-1}\Psi\left(\theta\lambda\right)\right), \ \theta \in \mathbb{R},$$

where $E\exp(\mathrm{i}\theta Y_1) = E\exp(\mathrm{i}\theta Y) = \exp(\Psi(\theta))$ is the cf of the inf. div. rv $Y$ in (5.2). (This is a special case of Theorem 8.3.18 in Meerschaert and Scheffler [31], who give a thorough presentation of the theory of the domain of geometric partial attraction of a semistable law.)

This paper is dedicated to the memory of the late Sándor Csörgő. Much of his research from the 1990s to his passing was devoted to the study of the Saint Petersburg game. He uncovered an amazing variety of unexpected stochastic properties of the game. In fact, he was writing with Gordon Simons a monograph on this topic. The Saint Petersburg game is the classic example of a partial sum in the domain of geometric partial attraction of a semistable law. Here $\xi$ (denoting Paul's winnings) has distribution

$$P\left(\xi = 2^k\right) = 2^{-k} \text{ for } k = 1, 2, \ldots$$

Martin-Löf [29] has shown that (5.1) and (5.2) hold with $c = 2$, $n_k = 2^k$, $B_n = 2^n$ and $A_n = n$, $n = 1, 2, \ldots$, and $Y$ having the cf $\exp(\Psi(\theta))$, with

$$\Psi\left(\theta\right) = \int_0^\infty \left(e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x| \le 1\}}\right) \Lambda(\mathrm{d}x), \ \theta \in \mathbb{R}, \tag{5.4}$$

where $\Lambda$ is the Lévy measure on $\mathbb{R}^+$ with tail function

$$\overline{\Lambda}(x) = \Lambda\{(x, \infty)\} = \exp\left(-\left\lfloor \log_2(x) \right\rfloor\right), \ x > 0. \tag{5.5}$$

From Theorem 2.1 and Theorem 2.2 of Csörgő and Dodunekova [6] it can be inferred that

$$\frac{\sum_{i=1}^{n} \xi_i}{2^n} - \left\lfloor \log_2\left(n\right) \right\rfloor$$

is stochastically compact (hence, $S_n \in FC$), and every rv in $\mathcal{Y}_c$ is obtainable as a subsequential limit of an appropriately chosen subsequence of $\{n\}$. A subsequential limit rv $\lambda^{-1} Y_\lambda$ in $\mathcal{Y}_c$ has Lévy tail function

$$\overline{\Lambda}_\lambda(x) = \Lambda_\lambda\{(x, \infty)\} = \lambda^{-1} e^{-\lfloor \log_2(\lambda x) \rfloor}, \ x > 0. \tag{5.6}$$

On the other hand it is easy to check directly that $\xi$ satisfies Feller's necessary and sufficient condition for $S_n \in FC$, namely,

$$\limsup_{x \to \infty} \frac{x^2 P\left(|\xi| > x\right)}{E\left(\xi^2 \mathbf{1}_{\{|\xi| \le x\}}\right)} < \infty. \tag{5.7}$$

The Lévy process $X_t$ having cf $\exp(t\Psi(\theta))$ with $\Psi(\theta)$ as in (5.4) is a special case of a semistable Lévy process of index $\alpha = 1$. Since $\xi \in FC$, we know by Corollary 1 that both $X_t \in FC$ at infinity and $X_t \in FC$ at zero.

It turns out that any semistable Lévy process $X_t$ of index $\alpha \in (0, 2]$ satisfies both $X_t \in FC$ (at zero) and $X_t \in FC$ at infinity. For $\alpha = 2$ this is trivial since in this case $X_t$ is Brownian motion. For $0 < \alpha < 2$ it can be inferred from the readily established fact, using the representations of the Lévy measure of a semistable law given in Corollary 7.4.4 of Meerschaert and Scheffler [31], that, with $\Pi$ as usual denoting the Lévy measure of $X_t$,

$$\limsup_{x\downarrow 0} x^2\overline{\Pi}(x)/V(x) < \infty \text{ and } \limsup_{x\to\infty} x^2\overline{\Pi}(x)/V(x) < \infty,$$

from which we can infer by Theorem 1 that $X_t \in FC$ at infinity and by Theorem 2.3 of Maller and Mason [28] that $X_t \in FC_0$ at zero.

Csörgő [2] and Csörgö and Megyesi [8] show how for any choice of $0 < \alpha < 2$, a generalized version of the St. Petersburg game converges in distribution along subsequences to a semistable law of index $\alpha$. Csörgő [4] carried out a deep analysis of the analytic properties of semistable laws. Csörgő and coauthors have conducted a thorough study of merge theorems and central limit theorems for sums and trimmed sums in the domain of geometric partial attraction of a semistable law based on a mixture of quantile and Fourier methods. For details consult Csörgő [2] and [5], Csörgő and Dodunekova [6] and Csörgő and Megyesi [7] and [8]. Also refer to Megyesi [32] and [33] for further investigations along this line.

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