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Gaussian integrals involving absolute value functions

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Abstract: We provide general formulas to compute the expectations of absolute value and sign of Gaussian quadratic forms, i.e. $\mathbb{E}\,|\langle \mathbf{X}, A\mathbf{X}\rangle + \langle \mathbf{b}, \mathbf{X}\rangle + c|$ and $\mathbb{E}\,\mathrm{sgn}(\langle \mathbf{X}, A\mathbf{X}\rangle + \langle \mathbf{b}, \mathbf{X}\rangle + c)$ for centered Gaussian random vector $\mathbf{X}$, fixed matrix A, vector $\mathbf{b}$ and constant c. Products of Gaussian quadratics are also discussed and followed with several interesting applications.

1. Introduction

Evaluating the Gaussian integrals (expectation, moments, etc.) involving the absolute value function has been playing important roles in various contents. For example, in [10] and [12], the expected number of zeros of random harmonic functions, which is also the average number of images of certain gravitational lensing system, was associated with the expectation of absolute value of certain Gaussian quadratic forms. In [2], the dislocation point density of Gaussian random wave was expressed as the expectation of absolute value of certain Gaussian quadratic form. In [LMOS83], the authors were interested in the average of absolute multiplicative structures, e.g. $\mathbb{E}\,|X_1X_2\cdots X_n|$, which arised in the analysis of learning curves of many adaptive systems. Selberg's integral and Mehta's integral are also equivalent to this structure for certain Gaussian random vectors, see [14]. Very recently, an elegant Gaussian inequality $\mathbb{E}\,|X_1X_2\cdots X_n| \le \sqrt{\mathrm{perm}(\mathbb{E}\,X_iX_j)}$, due to the first author, was established in [13], where $\mathrm{perm}(\mathbb{E}\,X_iX_j)$ is the permanent of the covariance matrix of the centered Gaussian vector $(X_1, X_2, \ldots, X_n)$. The explicit expression of the simplest absolute multiplicative structure $\mathbb{E}\,|X_1X_2|$ and related series expansions were re-derived in [17], and were used to study the correlation between two dependent Brownian area integrals. Here we concentrate on exact evaluations which also appeared in the theory of Gaussian random matrices in various settings. In [1], the authors used the spectral analysis on the Gaussian Orthogonal Ensemble random matrix to compute the first order approximation for stationary isotropic process defined on a polyhedron, which provided an upper bound for the density of maximum of certain smooth Gaussian fields. In particular, they dealt with the expectation $\mathbb{E}\,|\det(G_n - \nu I_n)|$ where G_n was a GOE matrix and I_n standed for the $n \times n$ identity matrix. A Gaussian representation for the intrinsic volumes of convex body was given in terms of $\mathbb{E}\,|\det M|$, where M was the random matrix with independent standard Gaussian entries, see [19]. In [4], a special Gaussian integral involving absolute value function was studied to provide the density of critical points of given holomorphic section which was related to counting vacua in string theory.

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In this article, we also provide a formula for the expected sign of Gaussian quadratic forms, which is also useful in applications. For example, the best known constant in Grothendieck inequality was obtained by using the expectation $\mathbb{E}\,\mathrm{sgn}\,(XY)$ where X and Y are Gaussians, see [9]. In [20], the author also used this expectation to study the proportion of the time that a Brownian sheet on $[0,1]^d$ is positive. The explicit expression of $\mathbb{E}\,\mathrm{sgn}\,(XY)$, see Corollary 3.1, is often known as Sheppard's formula; see e.g. [3].

In general, evaluating Gaussian integrals involving absolute value functions or sign functions are technically difficult, and there is no universal method available. In this article, we provide a systematic study of techniques and associated examples. In particular, we focus on Gaussian quadratic forms. This paper is organized as follows: Section 2 is about the representations of absolute function and sign function which are helpful in dealing with quadratic forms of Gaussian random variables. Several interesting corollaries and examples are included in Section 3 based on these representations. Most of these results are new and of independent interest. In Section 4, we discuss other approaches for Gaussian integrals involving absolute value functions.

2. Representations and the main theorems

Gaussian quadratic forms appear in the problem of finding the number of zeros of random functions. In [12], we represent the number of zeros of random harmonic polynomials as an integral of expectation of absolute value of certain Gaussian quadratic form, according to the Rice formula. Our techniques are extended to obtain the following theorems.

Theorem 2.1. *Assume $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$ is a real centered Gaussian random vector with covariance matrix $\Sigma_{n\times n} = M_{n\times k}M_{n\times k}^T$, where $k = \operatorname{rank}\Sigma$. For any $n\times n$ symmetric matrix A, n-dim column vector $\mathbf{b}$, and constant c,*

$$\mathbb{E}\,|\langle \mathbf{X}, A\mathbf{X}\rangle + \langle \mathbf{b}, \mathbf{X}\rangle + c| = \frac{2}{\pi}\int_0^\infty t^{-2}\left(1 - F(t) - \overline{F(t)}\right)dt,$$

where

$$F(t) = \frac{\exp\left(\mathtt{i}tc - 2^{-1}t^2\langle \mathbf{b}, M\left(I - 2\mathtt{i}tM^TAM\right)^{-1}M^T\mathbf{b}\rangle\right)}{2\det(I - 2\mathtt{i}t\Sigma A)^{1/2}}. \tag{2.1}$$

Several remarks are needed here. First, for a general matrix A, other than symmetric matrices, we can replace A by $(A + A^T)/2$ which is symmetric, and then apply the theorem for $(A + A^T)/2$. This is because

$$\langle \mathbf{X}, A\mathbf{X}\rangle = \left\langle \mathbf{X}, \frac{1}{2}(A + A^T)\,\mathbf{X}\right\rangle.$$

Second, for a nonsingular covariance matrix Σ, M^{-1} exists. Thus the conclusion in Theorem 2.1 is reduced to

$$\mathbb{E}\,|\langle \mathbf{X}, A\mathbf{X}\rangle + \langle \mathbf{b}, \mathbf{X}\rangle + c| = \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{t^2} - \frac{\exp\left(\mathtt{i}tc - 2^{-1}t^2\langle \mathbf{b}, \left(\Sigma^{-1} - 2\mathtt{i}tA\right)^{-1}\mathbf{b}\rangle\right)}{2t^2\det(I - 2\mathtt{i}t\Sigma A)^{1/2}} - \frac{\exp\left(-\mathtt{i}tc - 2^{-1}t^2\langle \mathbf{b}, \left(\Sigma^{-1} + 2\mathtt{i}tA\right)^{-1}\mathbf{b}\rangle\right)}{2t^2\det(I + 2\mathtt{i}t\Sigma A)^{1/2}}\right\}dt.$$

This implies that for a nonsingular covariance matrix, we don't need to find $M_{n\times n}$ (it may be much more complicated than Σ). We can also see that although it is typical to make the transformation $\mathbf{X} = M_{n\times n}\xi_{n\times 1}$ when we are dealing with a correlated vector $\mathbf{X}$, it is easier in this case to keep the structure of $\mathbf{X}$. Here and throughout this paper, $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T$ and ξ_j's are independent standard Gaussian random variables. Third, if the covariance matrix of $\mathbf{X}$ is Σ, then

$$\langle \mathbf{X}, A\mathbf{X}\rangle = \langle M\xi, AM\xi\rangle = \langle \xi, (M^T AM)\xi\rangle. \tag{2.2}$$

Note that in general, $\langle \xi, (M^T AM)\xi\rangle \neq^d \langle \xi, (\Sigma A)\xi\rangle$ even we know $M^T AM$ and ΣA have the same nonzero eigenvalues, see [7]. However $\mathbb{E}\,|\langle \mathbf{X}, A\mathbf{X}\rangle| = \mathbb{E}\,|\xi, \Sigma A\xi|$, and we only need to consider the independent Gaussian vectors in the pure quadratic case. Fourth, all remarks above also apply to the next two results.

Now we consider the expected sign function of Gaussian quadratic forms. We can apply a similar argument and obtain the following theorem:

Theorem 2.2. *Assume $\mathbf{X} = (X_1, X_2, \dots, X_n)^T$ is a real centered Gaussian random vector with covariance matrix $\Sigma_{n\times n} = M_{n\times k}M_{n\times k}^T$ where $k = \operatorname{rank}\Sigma$. For any $n\times n$ symmetric matrix A, n-dim column vector $\mathbf{b}$, and constant c,*

$$\mathbb{E}\operatorname{sgn}\left(\langle \mathbf{X}, A\mathbf{X}\rangle + \langle \mathbf{b}, \mathbf{X}\rangle + c\right) = \frac{2\mathtt{i}}{\pi}\int_0^\infty t^{-1}\left(\overline{F(t)} - F(t)\right) dt,$$

where $F(t)$ is given in (2.1).

Similar techniques can be used to find the expected absolute values of products of Gaussian quadratic forms. Actually, the study of the variance of zeros of random harmonic polynomial is associated with this expectation, see [12] for the connection.

Theorem 2.3. *Assume $\mathbf{X} = (X_1, X_2, \cdots, X_n)^T$ is a real centered Gaussian random vector with nonsingular covariance matrix $\Sigma_{n\times n}$. For any $n\times n$ symmetric matrix A_1 and A_2, we have*

$$\begin{aligned}
&\mathbb{E}\,|\langle \mathbf{X}, A_1\mathbf{X}\rangle \cdot \langle \mathbf{X}, A_2\mathbf{X}\rangle| \\
&\quad = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \frac{1}{t^2 s^2}\Bigg\{1 - \frac{1}{2\det(I - 2\mathtt{i}t\Sigma A_1)^{1/2}} - \frac{1}{2\det(I + 2\mathtt{i}t\Sigma A_1)^{1/2}} \\
&\qquad - \frac{1}{2\det(I - 2\mathtt{i}s\Sigma A_2)^{1/2}} - \frac{1}{2\det(I + 2\mathtt{i}s\Sigma A_2)^{1/2}} \\
&\qquad + \frac{1}{4\det(I - 2\mathtt{i}\Sigma(tA_1 + sA_2))^{1/2}} \\
&\qquad + \frac{1}{4\det(I - 2\mathtt{i}\Sigma(tA_1 - sA_2))^{1/2}} \\
&\qquad + \frac{1}{4\det(I - 2\mathtt{i}\Sigma(-tA_1 + sA_2))^{1/2}} \\
&\qquad + \frac{1}{4\det(I - 2\mathtt{i}\Sigma(-tA_1 - sA_2))^{1/2}}\Bigg\}\,dt\,ds.
\end{aligned}$$

Proof of the Theorems. To prove Theorem 2.1, we start with the case when $\Sigma = I$, the identity matrix, and then move to the general cases. We start with the following representation:

$$|x| = \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}\left(1 - \cos(xt)\right) dt = \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}\left(1 - \mathbb{E}_{\varepsilon} e^{\mathtt{i}\varepsilon xt}\right) dt, \tag{2.3}$$

where $\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = 1/2$. Then we can rewrite the expectation as

$$\mathbb{E}\,|\langle \mathbf{X}, A\mathbf{X}\rangle + \langle \mathbf{b}, \mathbf{X}\rangle + c| = \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}\left(1 - \mathbb{E}_\varepsilon \mathbb{E}_{\mathbf{X}} e^{\mathtt{i}\varepsilon t(\langle \mathbf{X}, A\mathbf{X}\rangle + \langle \mathbf{b}, \mathbf{X}\rangle + c)}\right) dt.$$

Since the covariance matrix $\Sigma = I$, the density function of $\mathbf{X}$ is

$$f_{\mathbf{X}}(\mathbf{x}) = (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\langle \mathbf{x}, \mathbf{x}\rangle\right).$$

Therefore we have

$$\begin{aligned}
&\mathbb{E}_{\mathbf{X}} \exp\left(\mathtt{i}\varepsilon t\left(\langle \mathbf{X}, A\mathbf{X}\rangle + \langle \mathbf{b}, \mathbf{X}\rangle + c\right)\right) \\
&\quad = \int_{\mathbb{R}^n} (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\langle \mathbf{x}, \mathbf{x}\rangle\right) \exp\left(\mathtt{i}\varepsilon t\left(\langle \mathbf{x}, A\mathbf{x}\rangle + \langle \mathbf{b}, \mathbf{x}\rangle + c\right)\right) d\mathbf{x} \\
&\quad = \int_{\mathbb{R}^n} (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\left(\langle \mathbf{x}, (I - 2\mathtt{i}\varepsilon tA)\mathbf{x}\rangle - \langle 2\mathtt{i}\varepsilon t\mathbf{b}, \mathbf{x}\rangle - 2\mathtt{i}\varepsilon tc\right)\right) d\mathbf{x} \\
&\quad = \det\left(I - 2\mathtt{i}t\varepsilon A\right)^{-1/2} \exp\left(\mathtt{i}\varepsilon tc - \frac{t^2}{2}\langle \mathbf{b}, \left(I - 2\mathtt{i}\varepsilon tA\right)^{-1}\mathbf{b}\rangle\right) \\
&\quad = \det\left(I - 2\mathtt{i}t\varepsilon A\right)^{-1/2} \exp\left(\mathtt{i}\varepsilon tc - \frac{t^2}{2}\langle \mathbf{b}, \left(I - 2\mathtt{i}\varepsilon tA\right)^{-1}\mathbf{b}\rangle\right).
\end{aligned}$$

Note that

$$\det(I - 2\mathtt{i}t\varepsilon A) \cdot \det(I + 2\mathtt{i}t\varepsilon A) = \det(I + 4t^2A^2) \neq 0,$$

since $A^2 = AA^T$ is positive definite. This implies that $\det(I \pm 2\mathtt{i}t\varepsilon A) \neq 0$ and $(I \pm 2\mathtt{i}t\varepsilon A)^{-1}$ exist. Hence

$$\mathbb{E}_\varepsilon \mathbb{E}_{\mathbf{X}} e^{\mathtt{i}\varepsilon t(\langle \mathbf{X}, A\mathbf{X}\rangle + \langle \mathbf{b}, \mathbf{X}\rangle + c)} = \frac{\exp\left(\mathtt{i}tc - 2^{-1}t^2\langle \mathbf{b}, \left(I - 2\mathtt{i}tA\right)^{-1}\mathbf{b}\rangle\right)}{2\det(I - 2\mathtt{i}tA)^{1/2}} + \frac{\exp\left(-\mathtt{i}tc - 2^{-1}t^2\langle \mathbf{b}, \left(I + 2\mathtt{i}tA\right)^{-1}\mathbf{b}\rangle\right)}{2\det(I + 2\mathtt{i}tA)^{1/2}}$$

which is real since the two terms are conjugate to each other. Thus for identity covariance matrix, the statement is true with $M = I$.

Next we consider a general covariance matrix $\Sigma = MM^T$. Here M can be uniquely determined by projecting $\mathbf{X} = (X_1, X_2, \ldots, X_n)^T$ onto $\xi = (\xi_1, \xi_2, \ldots, \xi_k)^T$ with $\xi_1 = X_1/\sqrt{\mathbb{E}\,X_1^2}$ and ξ_j's are independent standard Gaussian random variables for $j = 1, \ldots, k$. It is clear that $\mathbf{X} = M\xi$ from the definition. As a consequence, we have

$$\langle \mathbf{X}, A\mathbf{X}\rangle = \langle M\xi, AM\xi\rangle = \langle \xi, M^TAM\xi\rangle, \quad \langle \mathbf{b}, \mathbf{X}\rangle = \langle \mathbf{b}, M\xi\rangle = \langle M^T\mathbf{b}, \xi\rangle.$$

Define $\tilde{A} = M^TAM$ and $\tilde{\mathbf{b}} = M^T\mathbf{b}$ and applying the result from the first part of the proof, we have

$$
\begin{aligned}
&\mathbb{E}\,|\langle \mathbf{X}, A\mathbf{X}\rangle + \langle \mathbf{b}, \mathbf{X}\rangle + c| \\
&\quad = \mathbb{E}\,|\langle \xi, \tilde{A}\xi\rangle + \langle \tilde{\mathbf{b}}, \xi\rangle + c| \\
&\quad = \frac{2}{\pi}\int_0^\infty \left\{ \frac{1}{t^2} - \frac{\exp\left(\mathtt{i}tc - 2^{-1}t^2\langle \tilde{\mathbf{b}}, \left(I - 2\mathtt{i}t\tilde{A}\right)^{-1}\tilde{\mathbf{b}}\rangle\right)}{2t^2\det(I - 2\mathtt{i}t\tilde{A})^{1/2}} \right. \\
&\qquad\qquad \left. - \frac{\exp\left(-\mathtt{i}tc - 2^{-1}t^2\langle \tilde{\mathbf{b}}, \left(I + 2\mathtt{i}t\tilde{A}\right)^{-1}\tilde{\mathbf{b}}\rangle\right)}{2t^2\det(I + 2\mathtt{i}t\tilde{A})^{1/2}} \right\} dt \\
&\quad = \frac{2}{\pi}\int_0^\infty \left\{ \frac{1}{t^2} - \frac{\exp\left(\mathtt{i}tc - 2^{-1}t^2\langle \mathbf{b}, M\left(I - 2\mathtt{i}tM^TAM\right)^{-1}M^T\mathbf{b}\rangle\right)}{2t^2\det(I - 2\mathtt{i}tM^TAM)^{1/2}} \right. \\
&\qquad\qquad \left. - \frac{\exp\left(-\mathtt{i}tc - 2^{-1}t^2\langle \mathbf{b}, M\left(I + 2\mathtt{i}tM^TAM\right)^{-1}M^T\mathbf{b}\rangle\right)}{2t^2\det(I + 2\mathtt{i}tM^TAM)^{1/2}} \right\} dt.
\end{aligned}
$$

According to the fact that (see Theorem 1.3.20 in [7])

$$\det(I \pm 2\mathtt{i}tM^TAM) = \det(I \pm 2\mathtt{i}tMM^TA) = \det(I \pm 2\mathtt{i}t\Sigma A),$$

we complete the proof of Theorem 2.1.

To prove Theorem 2.2 we use the representation

$$\operatorname{sgn}(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{t}\sin(xt)dt = \frac{2}{\pi}\int_0^\infty \frac{1}{t}\,\mathbb{E}_\varepsilon \exp\left[\mathtt{i}\varepsilon(\pi/2 - xt)\right]\,dt, \tag{2.4}$$

which can be regarded as a differential form of (2.3). Following the similar argument of Theorem 2.1 we obtain Theorem 2.2. For Theorem 2.3, we apply (2.3) for both of the absolute values of Gaussian quadratic forms and the computation of expectation leads us to the result in Theorem 2.3. □

3. Consequences and examples

Based on different assumptions on A, Σ, $\mathbf{b}$ and c, we obtain several interesting corollaries. For example, when $\operatorname{rank} A = 1$ or 2, and when $\det(I \pm 2\mathtt{i}t\Sigma A)$ is the square of a polynomial of t, the explicit expressions of the expectations can be found.

3.1. Rank $A = 1$

If the rank of matrix A is one, then there exists $\mathbf{u} = (u_1, \ldots, u_n)^T$ and $\mathbf{v} = (v_1, \ldots, v_n)^T$ such that $A = \mathbf{u}\mathbf{v}^T$. Therefore we can change the quadratic form into the product of two new Gaussian random variables:

$$\langle \mathbf{X}, A\mathbf{X}\rangle = \langle \mathbf{X}, \mathbf{u}\mathbf{v}^T\mathbf{X}\rangle = \langle \mathbf{u}^T\mathbf{X}, \mathbf{v}^T\mathbf{X}\rangle = Y_1Y_2,$$

where $\mathbb{E}\,Y_1Y_2 = \sum_{j,k}\sigma_{j,k}u_jv_k$ and $\sigma_{j,k} = \mathbb{E}\,X_jX_k$. For the absolute value of the product of two Gaussians, we have the following well known proposition:

Corollary 3.1. *If* (Y_1, Y_2) *is a centered Gaussian vector with* $\mathbb{E}\,Y_1^2 = \sigma_1^2$, $\mathbb{E}\,Y_2^2 = \sigma_2^2$ *and* $\mathbb{E}\,Y_1Y_2 = \rho\sigma_1\sigma_2$, *then*

$$\mathbb{E}\,|Y_1Y_2| = \frac{2}{\pi}(\sqrt{1-\rho^2} + \rho\arcsin\rho)\sigma_1\sigma_2,\ \ \mathbb{E}\,\mathrm{sgn}\,(Y_1Y_2) = \frac{2}{\pi}\arcsin\rho.$$

Proof. Here we use Theorem 2.1 and 2.2 to prove the results. (One can also prove these results by polar coordinates substitution.) In this case we assume, without loss of generality, that $\sigma_1 = \sigma_2 = 1$, and therefore

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}.$$

Plugging $\det(I \pm 2\mathtt{i}t\Sigma A) = 1 \pm 2\mathtt{i}\rho t + (1-\rho^2)t^2$ into Theorem 2.1 and Theorem 2.2 we obtain the results after integrations. □

3.2. Rank $A = 2$

In this case, A can be decomposed into U^TV, where U and V are full-row-rank matrices, and $U = (u_{j,k})_{2\times n}$, $V = (v_{j,k})_{2\times n}$. Therefore we have

$$\langle \mathbf{X}, A\mathbf{X}\rangle = \langle \mathbf{X}, UV^T\mathbf{X}\rangle = \langle U^T\mathbf{X}, V^T\mathbf{X}\rangle = Y_1Y_2 + Y_3Y_4,$$

where

$$Y_1 = \sum_{k=1}^n u_{1,k}X_k,\ Y_2 = \sum_{k=1}^n v_{1,k}X_k,\ Y_3 = \sum_{k=1}^n u_{2,k}X_k,\ Y_4 = \sum_{k=1}^n v_{2,k}X_k.$$

It is clear that $\det(I \pm 2\mathtt{i}t\Sigma A) = \det(I \pm 2\mathtt{i}t\Sigma U^TV) = \det(I \pm 2\mathtt{i}tV\Sigma U)$, which leads to

$$\begin{aligned}
&\det(I \pm 2\mathtt{i}t\Sigma A)\\
&\quad= \det\left(I \pm 2\mathtt{i}t\begin{pmatrix} \sum_{j=1}^n v_{1,j}\sigma_{j,1} & \cdots & \sum_{j=1}^n v_{1,j}\sigma_{j,n} \\ \sum_{j=1}^n v_{2,j}\sigma_{j,1} & \cdots & \sum_{j=1}^n v_{2,j}\sigma_{j,n}\end{pmatrix} U^T\right)\\
&\quad= \det\left(I \pm 2\mathtt{i}t\begin{pmatrix} \sum_{j,k=1}^n v_{1,j}\sigma_{j,k}u_{1,k} & \sum_{j,k=1}^n v_{1,j}\sigma_{j,k}u_{2,k} \\ \sum_{j,k=1}^n v_{2,j}\sigma_{j,k}u_{1,k} & \sum_{j,k=1}^n v_{2,j}\sigma_{j,k}u_{2,k}\end{pmatrix}\right)\\
&\quad= \left(1 \pm 2\mathtt{i}t\sum_{j,k=1}^n v_{1,j}\sigma_{j,k}u_{1,k}\right)\left(1 \pm 2\mathtt{i}t\sum_{j,k=1}^n v_{2,j}\sigma_{j,k}u_{2,k}\right)\\
&\qquad + 4t^2\left(\sum_{j,k=1}^n v_{1,j}\sigma_{j,k}u_{2,k}\sum_{j,k=1}^n v_{2,j}\sigma_{j,k}u_{1,k}\right).
\end{aligned}$$

In order to obtain an explicit formula, we need to assume that

$$\sum_{j,k=1}^n v_{1,j}\sigma_{j,k}u_{1,k} = -\sum_{j,k=1}^n v_{2,j}\sigma_{j,k}u_{2,k}, \tag{3.5}$$

which means $\mathbb{E}\,Y_1Y_2 = -\mathbb{E}\,Y_3Y_4$, or equivalently $\mathbb{E}\,\langle \mathbf{X}, A\mathbf{X}\rangle = 0$. Then under the assumption (3.5), the determinant becomes

$$\begin{aligned}
&\det(I \pm 2\mathtt{i}t\Sigma A)\\
&= 1 + 4t^2\left(\left(\sum_{j,k=1}^n v_{1,j}\sigma_{j,k}u_{1,k}\right)^2 + \sum_{j,k=1}^n v_{1,j}\sigma_{j,k}u_{2,k}\sum_{j,k=1}^n v_{2,j}\sigma_{j,k}u_{1,k}\right).
\end{aligned}$$

Denote that

$$r = \left(\sum_{j,k=1}^{n} v_{1,j}\sigma_{j,k}u_{1,k}\right)^2 + \sum_{j,k=1}^{n} v_{1,j}\sigma_{j,k}u_{2,k} \cdot \sum_{j,k=1}^{n} v_{2,j}\sigma_{j,k}u_{1,k},$$

and the following integral can be computed:

$$\begin{aligned}\mathbb{E}\,|\langle \mathbf{X}, A\mathbf{X}\rangle| &= \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}\left(1 - \frac{1}{\sqrt{1+4rt^2}}\right) dt \\ &= \frac{2}{\pi}\int_0^\infty \frac{4r}{\sqrt{1+4rt^2}(\sqrt{1+4rt^2}+1)}\, dt \\ &= \frac{2}{\pi}\int_0^{\pi/2} \frac{2r^{1/2}}{1+\cos\theta}\, d\theta = \frac{4}{\pi} r^{1/2}.\end{aligned}$$

Thus we have

$$\mathbb{E}\,|\langle \mathbf{X}, A\mathbf{X}\rangle| = \frac{4}{\pi}\left(\left(\sum_{j,k=1}^{n} v_{1,j}\sigma_{j,k}u_{1,k}\right)^2 + \sum_{j,k=1}^{n} v_{1,j}\sigma_{j,k}u_{2,k} \sum_{j,k=1}^{n} v_{2,j}\sigma_{j,k}u_{1,k}\right)^{1/2}.$$

Corollary 3.2. *When* $\operatorname{rank} A = 2$*, under the assumption (3.5) we always have* $\mathbb{E}\operatorname{sgn}\langle \mathbf{X}, A\mathbf{X}\rangle = 0$.

Corollary 3.3. *Suppose* $(X_1, X_2, \ldots, X_{4n-1}, X_{4n})$ *is a Gaussian random vector with* $\mathbb{E}\,X_k^2 = 1$ *and* $\mathbb{E}\,X_jX_k = \rho$ *when* $j \neq k$*. Then*

$$\mathbb{E}\left|\sum_{(j,k)\in\Omega_1} X_jX_k - \sum_{(j,k)\in\Omega_2} X_jX_k\right| = \frac{4n}{\pi}(1-\rho),$$

where

$$\begin{aligned}\Omega_1 &:= \{(j,k) \mid j+k \equiv 3 \mod 4\},\\ \Omega_2 &:= \{(j,k) \mid j+k \equiv 1 \mod 4\}.\end{aligned}$$

Proof. In this case, we have the decomposition $A = A_0^T A_0$, where

$$A_0 = \begin{pmatrix} 0 & 1 & 0 & -1 & \cdots & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & \cdots & 1 & 0 & -1 & 0 \end{pmatrix}_{2\times 4n}.$$

Now $\det(I \pm 2\mathtt{i}t\Sigma A)$ is equal to

$$\det(I \pm \mathtt{i}t\left(A_0\Sigma A_0^T\right) = \det(I \pm \mathtt{i}t(1-\rho)\begin{pmatrix} 0 & 2n \\ 2n & 0\end{pmatrix} = 1 + 4n^2(1-\rho)^2t^2.$$

Applying Theorem 2.1, we have

$$\mathbb{E}\left|\sum_{(j,k)\in A} X_jX_k - \sum_{(j,k)\in B} X_jX_k\right| = \frac{4n}{\pi}(1-\rho).$$

□

Proposition 3.1. *Suppose* $(X_1, X_2, \ldots, X_{2n-1}, X_{2n})$ *is a Gaussian random vector with* $\mathbb{E}\,X_jX_k = \mathbb{E}\,X_{j+n}X_{k+n} = \alpha_{jk}$ *for* $j \le n$ *and* $k \le n$*, and* $\mathbb{E}\,X_jX_k = \beta_{jk}$ *for all* $|j-k| \ge n$*, we have*

$$\mathbb{E}\left|\sum_{j,k=1}^{n} X_jX_k - \sum_{j,k=n+1}^{2n} X_jX_k\right| = \frac{4}{\pi}\left[\left(\sum_{j,k=1}^{n}\alpha_{jk}\right)^2 - \left(\sum_{j,k=1}^{n}\beta_{jk}\right)^2\right]^{1/2}.$$

Proof. In this case we have

$$A = \begin{pmatrix} 1_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & -1_{n\times n} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2^T & \Sigma_1 \end{pmatrix} = \begin{pmatrix} (\alpha_{jk})_{n\times n} & (\beta_{jk})_{n\times n} \\ (\beta_{jk})_{n\times n}^T & (\alpha_{jk})_{n\times n} \end{pmatrix}.$$

An observation is that $A = A_1^T A_1$, where

$$A_1 = \begin{pmatrix} 1 \cdots 1 & 0 \cdots & 0 \\ 0 \cdots 0 & -1 \cdots & -1 \end{pmatrix}.$$

Now $\det(I \pm 2\mathtt{i}t\Sigma A)$ can be expressed as

$$\det(I \pm 2\mathtt{i}t\left(A_1 \Sigma A_1^T\right)) = \det\left(I \pm 2\mathtt{i}t \begin{pmatrix} \sum_{j,k=1}^n \alpha_{j,k} & \sum_{j,k=1}^n \beta_{j,k} \\ -\sum_{j,k=1}^n \beta_{j,k} & -\sum_{j,k=1}^n \alpha_{j,k} \end{pmatrix}\right).$$

Thus we have

$$\det(I \pm 2\mathtt{i}t\Sigma A) = 1 + 4\left[\left(\sum_{j,k=1}^n \alpha_{jk}\right)^2 - \left(\sum_{j,k=1}^n \beta_{jk}\right)^2\right] t^2,$$

which leads to the result by applying Theorem 2.1. □

3.3. $\det(I \pm 2\mathtt{i}t\Sigma A)$ *is the square of a polynomial of* t

In this subsection, we consider some of the most interesting consequences and examples based on our general approach.

Proposition 3.2. *If* (X_1, X_2) *and* $(\tilde{X}_1, \tilde{X}_2)$ *are two centered i.i.d Gaussian random vectors with* $\mathbb{E}\, X_1^2 = \sigma_1^2$, $\mathbb{E}\, X_2^2 = \sigma_2^2$ *and* $\mathbb{E}\, X_1X_2 = \sigma_{12}$, *then*

$$\mathbb{E}\, |X_1^2 - X_2^2 + \tilde{X}_1^2 - \tilde{X}_2^2| = \frac{2\sigma_1^4 + 2\sigma_2^4 - 4\sigma_{12}^2}{\sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4\sigma_{12}^2}}.$$

Proof. In this case, we have

$$A = \mathrm{diag}\ (1, -1, 1, -1), \qquad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & 0 & 0 \\ \sigma_{12} & \sigma_2^2 & 0 & 0 \\ 0 & 0 & \sigma_1^2 & \sigma_{12} \\ 0 & 0 & \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

Therefore

$$\det(I \pm 2\mathtt{i}t\Sigma A) = \left(1 \pm (2\mathtt{i}\sigma_1^2 - 2\mathtt{i}\sigma_2^2)t + (4\sigma_1^2\sigma_2^2 - 4\sigma_{12}^2)t^2\right)^2.$$

Denote $p = 4\sigma_1^2\sigma_2^2 - 4\sigma_{12}^2$ and $q = \sigma_1^2 - \sigma_2^2$ and simple algebra gives us that

$$\begin{aligned} &\mathbb{E}\, |X_1^2 - X_2^2 + X_3^2 - X_4^2| \\ &= \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}\left(1 - \frac{1 + pt^2}{(1 + pt^2)^2 + 4q^2t^2}\right) dt \\ &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{p(1 + pt^2) + 4q^2}{(1 + pt^2)^2 + 4q^2t^2}\ dt \\ &= 2\mathtt{i}\sum\{\text{residues in upper half plane}\} + \mathtt{i}\sum\{\text{residues on } x\text{−axis}\}, \end{aligned} \tag{3.6}$$

and the four single poles of the integrand are at $(\pm p^{-1}\sqrt{p+q^2}\pm p^{-1}q)\mathtt{i}$ for nonsingular covariance matrix. Because $p>0$, $(p^{-1}\sqrt{p+q^2}\pm p^{-1}q)\mathtt{i}$ provides the upper half plane residues and no residues come from the x-axis. Therefore we have

$$\mathbb{E}\,|X_1^2-X_2^2+X_3^2-X_4^2|=\frac{p+2q^2}{\sqrt{p+q^2}}=\frac{2\sigma_1^4+2\sigma_2^4-4\sigma_{12}^2}{\sqrt{(\sigma_1^2+\sigma_2^2)^2-4\sigma_{12}^2}}. \qquad \square$$

Corollary 3.4. *If (X_1,X_2) and $(\tilde X_1,\tilde X_2)$ are two centered i.i.d Gaussian random vectors with $\mathbb{E}\,X_1^2=\sigma_1^2$, $\mathbb{E}\,X_2^2=\sigma_2^2$ and $\mathbb{E}\,X_1X_2=\sigma_{12}$, then*

$$\mathbb{E}\,\mathrm{sgn}\,(X_1^2-X_2^2+\tilde X_1^2-\tilde X_2^2)=\frac{\sigma_1^2-\sigma_2^2}{\sqrt{(\sigma_1^2+\sigma_2^2)^2-4\sigma_{12}^2}}.$$

Proof. Following the proof of Proposition 3.2, we have

$$\begin{aligned}\mathbb{E}\,\mathrm{sgn}\,(X_1^2-X_2^2+\tilde X_1^2-\tilde X_2^2)&=\frac{\mathtt{i}}{\pi}\int_0^\infty\frac{-4\mathtt{i}q}{(1+pt^2)^2+4q^2t^2}\,dt\\&=\frac{q}{\sqrt{p+q^2}}=\frac{\sigma_1^2-\sigma_2^2}{\sqrt{(\sigma_1^2+\sigma_2^2)^2-4\sigma_{12}^2}}. \qquad \square\end{aligned}$$

Proposition 3.3. *If (X_1,X_2) and $(\tilde X_1,\tilde X_2)$ are two centered i.i.d Gaussian random vectors with $\mathbb{E}\,X_1^2=\mathbb{E}\,X_2^2=1$, $\mathbb{E}\,X_1X_2=\rho$, then $\mathbb{E}\,|X_1\tilde X_1-X_2\tilde X_2|=\sqrt{1-\rho^2}$.*

Proof. In this case

$$A=\frac{1}{2}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix},\qquad \Sigma=\begin{pmatrix}1&\rho&0&0\\\rho&1&0&0\\0&0&1&\rho\\0&0&\rho&1\end{pmatrix}.$$

Therefore

$$\det(I\pm 2\mathtt{i}t\Sigma A)=(1+t^2-\rho^2t^2)^2.$$

Applying Theorem 2.1 we have

$$\mathbb{E}\,|X_1\tilde X_1-X_2\tilde X_2|=\frac{2}{\pi}\int_0^\infty\frac{1-\rho^2}{1+(1-\rho^2)t^2}\,dt=\sqrt{1-\rho^2}. \qquad \square$$

Remark 1. There is an alternative approach based on the special structure of $\mathbb{E}\,|X_1\tilde X_1-X_2\tilde X_2|$. We can write $X_2=\rho X_1+\sqrt{1-\rho^2}\xi$ and $\tilde X_1=\rho\tilde X_2+\sqrt{1-\rho^2}\eta$, where X_1, $\tilde X_2$, ξ, η are independent standard Gaussians. And using substitutions and identity in laws

$$(X_1,\eta)=^d\left(\frac{X_1+\eta}{2},\frac{X_1-\eta}{2}\right),\quad (\tilde X_2,\xi)=^d\left(\frac{\tilde X_2+\xi}{2},\frac{\tilde X_2-\xi}{2}\right),$$

the expectation is rewritten as

$$\begin{aligned}\mathbb{E}\,|X_1\tilde X_1-X_2\tilde X_2|&=\sqrt{1-\rho^2}\,\mathbb{E}\,|X_1\eta-\tilde X_2\xi|\\&=\sqrt{1-\rho^2}\,\mathbb{E}\left|\frac{X_1+\eta}{\sqrt2}\frac{X_1-\eta}{\sqrt2}-\frac{X_2+\xi}{\sqrt2}\frac{X_2-\xi}{\sqrt2}\right|\\&=\sqrt{1-\rho^2}\,\mathbb{E}\left|\frac{1}{2}(X_1^2+\xi^2)-\frac{1}{2}(X_2^2+\eta^2)\right|\\&=\sqrt{1-\rho^2}\,\mathbb{E}\,|e_1-e_2|\\&=\sqrt{1-\rho^2}\,\mathbb{E}\,(e_1+e_2-2\min(e_1,e_2))\\&=\sqrt{1-\rho^2}.\end{aligned}$$

Here e_1 and e_2 are i.i.d. exponential random variables with intensity 1.

Remark 2. Actually, in this setting, we have $\mathbb{E}\,|X_1\tilde{X}_2 - X_2\tilde{X}_1| = \mathbb{E}\,|X_1\tilde{X}_1 - X_2\tilde{X}_2| = \sqrt{1-\rho^2}$. Note that

$$\begin{aligned}\mathbb{E}\,|X_1\tilde{X}_2 - X_2\tilde{X}_1| &= \mathbb{E}\left|\det\begin{pmatrix} X_1 & \tilde{X}_1 \\ X_2 & \tilde{X}_2\end{pmatrix}\right| \\ &= \mathbb{E}\left|\det\begin{pmatrix} X_1 & \tilde{X}_1 \\ \rho X_1 + \sqrt{1-\rho^2}\xi_1 & \rho\tilde{X}_1 + \sqrt{1-\rho^2}\tilde{\xi}_1\end{pmatrix}\right| \\ &= \sqrt{1-\rho^2}\,\mathbb{E}\left|\det\begin{pmatrix} X_1 & \tilde{X}_1 \\ \xi_1 & \tilde{\xi}_1\end{pmatrix}\right| \\ &= \sqrt{1-\rho^2}.\end{aligned}$$

The last equality comes from Proposition 3.4, with zero correlations. To extend this idea, we have the following proposition:

Proposition 3.4. *If $(X_{1,j}, X_{2,j}, \cdots, X_{n,j})$, $j = 2, \ldots, n$, are independent copies of the Gaussian random vector $(X_{1,1}, X_{2,1}, \cdots, X_{n,1})$, and the covariance matrix of $(X_{1,1}, X_{2,1}, \cdots, X_{n,1})$ is Σ, then*

$$\mathbb{E}\,|\det(X_{j,k})_{n\times n}| = \sqrt{\det\Sigma}\;\mathbb{E}\,|\det G_n| = 2^{n/2}\sqrt{\det\Sigma}\prod_{j=1}^{n}\frac{\Gamma(j/2+1/2)}{\Gamma(j/2)},$$

where G_n is the $n\times n$ random matrix with i.i.d. standard Gaussian entries.

Proof. To simplify the problem we project $(X_{1,j}, X_{2,j}, \cdots, X_{n,j})$ onto a set of independent Gaussian random variables $(\xi_{1,j}, \xi_{2,j}, \cdots, \xi_{n,j})$ for $j = 1, 2, \ldots, n$ as

$$\begin{aligned} X_{1,j} &= m_{1,1}\xi_{1,j}, \\ X_{2,j} &= m_{2,1}\xi_{1,j} + m_{2,2}\xi_{2,j}, \\ &\cdots \\ X_{n,j} &= m_{n,1}\xi_{1,j} + m_{n,2}\xi_{2,j} + \cdots + m_{n,n}\xi_{n,j}. \end{aligned}$$

Therefore $\det(X_{j,k})_{n\times n} = \prod_{j=1}^n m_{j,j}\det(\xi_{j,k})_{n\times n}$. Set $m_{j,k} = 0$ if $j < k$ and $M = (m_{j,k})$ which is an lower triangle matrix, we have $\Sigma = MM^T$ and thus

$$\prod_{j=1}^{n} m_{j,j} = \det M = \sqrt{\det\Sigma}.$$

According to [6] and [14], (also see Proposition 4.2), it is well known that $|\det G_n| =^d \sqrt{\prod_{j=1}^n \chi_j^2}$ and thus

$$\mathbb{E}\,|\det G_n| = \mathbb{E}\sqrt{\prod_{j=1}^{n}\chi_j^2} = 2^{n/2}\prod_{j=1}^{n}\frac{\Gamma(j/2+1/2)}{\Gamma(j/2)},$$

where χ_j^2 is independent chi-square random variable with degree of freedom j. $\square$

Corollary 3.5. *If (X_1, X_2) and $(\tilde{X}_1, \tilde{X}_2)$ are centered i.i.d Gaussian random vectors with $\mathbb{E}\,X_1^2 = \mathbb{E}\,X_2^2 = 1$, $\mathbb{E}\,X_1X_2 = \rho$, then for any real c, we have*

$$\mathbb{E}\,|X_1^2 - X_2^2 + \tilde{X}_1^2 - \tilde{X}_2^2 + c| = 2\sqrt{1-\rho^2}\exp\left(-\frac{|c|}{2\sqrt{1-\rho^2}}\right) + |c|.$$

Proof. According to the assumption, $\det(I \pm 2\mathtt{i}t\Sigma A) = \big(1 + 4(1-\rho^2)t^2\big)^2$. Applying Theorem 2.1 we have

$$\begin{aligned}
&\mathbb{E}\,|X_1^2 - X_2^2 + \tilde{X}_1^2 - \tilde{X}_2^2 + c| \\
&= \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}\left(1 - \frac{\cos ct}{1+4(1-\rho^2)t^2}\right) dt \\
&= \frac{1}{\pi}\int_{-\infty}^\infty \frac{1+4(1-\rho^2)t^2 - \cos ct}{t^2(1+4(1-\rho^2)t^2)}\, dt \\
&= 2\mathtt{i}\sum\{\text{residues in upper half plane}\} + \mathtt{i}\sum\{\text{residues on } x\text{−axis}\}.
\end{aligned}$$

The residues come from the two poles at $(4-4\rho^2)^{-1/2}\mathtt{i}$ and 0. And therefore we obtain

$$\begin{aligned}
&\mathbb{E}\,|X_1^2 - X_2^2 + \tilde{X}_1^2 - \tilde{X}_2^2 + c| \\
&= 2\sqrt{1-\rho^2}\cosh\left(\frac{|c|}{2\sqrt{1-\rho^2}}\right) \\
&\quad - 2\sqrt{1-\rho^2}\sinh\left(\frac{|c|}{2\sqrt{1-\rho^2}}\right) + |c| \\
&= 2\sqrt{1-\rho^2}\exp\left(-\frac{|c|}{2\sqrt{1-\rho^2}}\right) + |c|. \qquad \square
\end{aligned}$$

Corollary 3.6. *If (X_1, X_2) and $(\tilde{X}_1, \tilde{X}_2)$ are centered i.i.d Gaussian random vectors with $\mathbb{E}\,X_1^2 = \mathbb{E}\,X_2^2 = 1$, $\mathbb{E}\,X_1X_2 = \rho$, then for any real c, we have*

$$\mathbb{E}\,\mathrm{sgn}\left(X_1^2 - X_2^2 + \tilde{X}_1^2 - \tilde{X}_2^2 + c\right) = \mathrm{sgn}(c)\left(1 - \exp\left(-\frac{c}{2\sqrt{1-\rho^2}}\right)\right).$$

Proof. Applying Theorem 2.2 we have

$$\begin{aligned}
\mathbb{E}\,\mathrm{sgn}\left(X_1^2 - X_2^2 + \tilde{X}_1^2 - \tilde{X}_2^2 + c\right) &= \frac{2}{\pi}\int_0^\infty \frac{1}{t}\frac{\sin ct}{1+4(1-\rho^2)t^2}\, dt \\
&= \mathrm{sgn}(c)\left(1 - \exp\left(-\frac{c}{2\sqrt{1-\rho^2}}\right)\right). \qquad \square
\end{aligned}$$

3.4. Infinite dimensional covariance matrix

Here we only consider the standard Brownian motion B_t for simplicity. In general, we can represent

$$B_t = \sum_{n=1}^\infty \xi_n \int_0^t \phi_n(u)\, du,$$

where $\{\phi_n(t)\}_{n\geq 1}$ is any complete orthogonal system in $L^2(0,1)$. Based on the above representation, the result below follows easily from Theorem 2.1 and we omit the proof.

Proposition 3.5. *Suppose B_t is a standard Brownian motion in $[0,1]$, then for $\kappa(t,s)$ satisfying*

$$\int_0^1\int_0^1 |\kappa(t,s)|\, dtds < \infty,$$

we have

$$\mathbb{E}\left|\int_0^1\int_0^1 \kappa(t,s)\,dB_t dB_s\right|$$
$$= \lim_{n\to\infty}\frac{2}{\pi}\int_0^\infty \frac{1}{u^2}\left(1-\frac{1}{2\det(I-2\mathtt{i}uA_n)^{1/2}}-\frac{1}{2\det(I+2\mathtt{i}uA_n)^{1/2}}\right)du,$$

where

$$A_n=(a_{jk})_{n\times n},\ a_{jk}=\int_0^1\int_0^1 \kappa(t,s)\phi_j(t)\phi_k(s)\,dtds, \qquad \text{for } j\geq 1,\ k\geq 1.$$

3.5. Products of Gaussian quadratic forms

According to Theorem 2.3, we can compute several special cases involving the product of quadratic forms. Here we present two interesting examples:

Corollary 3.7. *Let X_1 and X_2 be standard Gaussians with $\mathbb{E}\,X_1X_2=\rho$, then*

$$\mathbb{E}\,|X_1X_2(X_1+X_2)(X_1-X_2)|=\mathbb{E}\,|X_1^3X_2-X_2^3X_1|=\frac{4}{\pi}\sqrt{1-\rho^2}\,(1+\rho^2).$$

Proof. To find the expectation, we set $A_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $A_2=\begin{pmatrix}0 & 1/2\\ 1/2 & 0\end{pmatrix}$, and $\Sigma=\begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}$. Then it is easy to see that

$$\det(I\pm 2\mathtt{i}t\Sigma A_1)=1+4(1-\rho^2)t^2,$$
$$\det(I\pm 2\mathtt{i}s\Sigma A_2)=1\pm 2\mathtt{i}s\rho+(1-\rho^2)s^2,$$

which is denoted by $I^{\pm}(s)$ for convenience. Therefore

$$\det(I+2\mathtt{i}\Sigma(\pm tA_1+sA_2))=I^+(s)+4(1-\rho^2)t^2,$$
$$\det(I+2\mathtt{i}\Sigma(\pm tA_1-sA_2))=I^-(s)+4(1-\rho^2)t^2.$$

We first compute the inner integral (with respect to t) in Theorem 2.3, which is $I_1+I_2+I_3$ where

$$I_1=\int_0^\infty t^{-2}\left(1-\frac{1}{\sqrt{1+4(1-\rho^2)t^2}}\right)dt=2\sqrt{1-\rho^2},$$
$$I_2=\int_0^\infty t^{-2}\left(-\frac{1}{2\sqrt{I^-(s)}}+\frac{1}{2\sqrt{I^-(s)+4(1-\rho^2)t^2}}\right)dt=-\frac{\sqrt{1-\rho^2}}{I^-(s)},$$
$$I_3=\int_0^\infty t^{-2}\left(-\frac{1}{2\sqrt{I^+(s)}}+\frac{1}{2\sqrt{I^+(s)+4(1-\rho^2)t^2}}\right)dt=-\frac{\sqrt{1-\rho^2}}{I^+(s)}.$$

Therefore we have

$$\mathbb{E}\,|X_1^3X_2-X_2^3X_1|=\frac{4}{\pi^2}\sqrt{1-\rho^2}\int_0^\infty\frac{ds}{s^2}\left(2-\frac{1}{I^-(s)}-\frac{1}{I^+(s)}\right)$$
$$=\frac{4}{\pi^2}\sqrt{1-\rho^2}\int_0^\infty\frac{2+6\rho^2+2(1-\rho^2)^2s^2}{I^-(s)I^+(s)}\,ds.$$

Single poles of the integrand are at $\pm(1\pm\rho)^{-1}\mathtt{i}$, and among them $(1\pm\rho)^{-1}\mathtt{i}$ are in the upper half plane. By the Residue theorem (3.6), we finish the proof. ☐

Corollary 3.8. *Suppose X_1, X_2, X_3 and X_4 are standard Gaussian random variables with $\mathbb{E}\, X_1X_3 = \mathbb{E}\, X_2X_4 = \rho$ and other correlations between X_j and X_k $(j \neq k)$ are zero, then*

$$\mathbb{E}\,|(X_1^2 + X_2^2 - X_3^2 - X_4^2)(X_1X_4 - X_2X_3)| = \frac{8(1-\rho^2)}{\pi}.$$

Proof. In this case, following the notation in Theorem 2.3 we have

$$\det(I \pm 2\mathtt{i}t\Sigma A_1) = (1 - 4\rho^2t^2)^2,\ \det(I \pm 2\mathtt{i}s\Sigma A_2) = (1-\rho^2s^2)^2,$$
$$\det(I \pm 2\mathtt{i}t\Sigma A_1 \pm 2\mathtt{i}s\Sigma A_2) = (1-\rho^2s^2 - 4\rho^2t^2)^2.$$

Applying Theorem 2.3 and the Residue theorem we complete the proof. □

4. Other approaches

The techniques in the proof of the theorems in Section 2 work well for the absolute value of functions involving quadratic terms. For other types of functions, they might not be efficient. For example, it seems more difficult and not amenable to provide a derivation of Proposition 4.3 and 4.4, based on the method presented in Section 2 and 3. In this section, we briefly overview other techniques used in evaluating the Gaussian integrals involving absolute value functions.

4.1. Direct computations

Obviously we can compute the Gaussian integral directly in simple cases, and spherical coordinates transformation or series expansion can be used in the examples like Corollary 3.1. Generally, following the argument in [15], one can see that for real $\alpha, \beta > -1$,

$$\mathbb{E}\,|X|^\alpha|Y|^\beta = 2^{(\alpha+\beta)/2}\sigma_1^\alpha\sigma_2^\beta\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)F\left(-\frac{\alpha}{2}, -\frac{\beta}{2}; \frac{1}{2}; \rho^2\right),$$

where X and Y are correlated Gaussians with $\mathbb{E}\, X^2 = \sigma_1^2$, $\mathbb{E}\, Y^2 = \sigma_2^2$ and $\mathbb{E}\, XY = \rho\sigma_1\sigma_2$, and F is a hypergeometric function given by

$$F(a, b; \gamma; z) = 1 + \frac{ab}{1!\gamma}z + \frac{a(a+1)b(b+1)}{2!\gamma(\gamma+1)}z^2 + \cdots.$$

Other related computations and applications were given in [8] and [18]. Actually, in [18], the authors provided three different methods to find equivalent formulas for $\mathbb{E}\,|XY|$. The first series formula

$$\mathbb{E}\,|XY| = \frac{2}{\sqrt{\pi}}(1-\rho^2)^{3/2}\sum_{k=0}^{\infty}\rho^{2k}\frac{\Gamma(k+1)}{\Gamma(k+1/2)}\sigma_1\sigma_2$$

was obtained by conditioning and using the series representation for the non-central chi-square distribution. The second formula

$$\mathbb{E}\,|XY| = \frac{1}{2\pi}\left(4 + \sum_{k=1}^{\infty}\rho^{2k}(2k)!\left(\sum_{j=0}^{k}(-1)^j\frac{2^{k-j+1}(k-j)!}{j!2^j(2k-2j)!}\right)^2\right)\sigma_1\sigma_2$$

was obtained by using Mehler's formula for the Radon-Nikodym derivative of the bivariate normal distribution with correlation ρ with respect to the normal distribution with correlation 0 (independence). The third expression, which was the same as the first formula in Corollary 3.1, was obtained by representing (X, Y) in terms of independent $N(0,1)$ random variables and integrating via polar coordinates.

4.2. Product of Gaussians

In [16], the author used the representation similar to (2.2) to deal with the expectation of absolute value of the Gaussian products such as $\mathbb{E}\,|\prod_{j=1}^n X_j^{m_j}|$. In fact if $m_1,\ldots,m_p$ are odd, $m_{p+1},\ldots,m_n$ are even and $m = \sum_{j=1}^n m_j$, then

$$\mathbb{E}\left|\prod_{j=1}^n X_j^{m_j}\right| = \frac{1}{\mathrm{i}^{p+m}\pi^p}\int\cdots\int \frac{dt_1\cdots dt_p}{t_1\cdots t_p}\left(\frac{\partial^m \phi(t_1,t_2,\ldots,t_n)}{\partial t_1^{m_1}\cdots\partial t_n^{m_n}}\right)\Big|_{t_{p+1}=\cdots=t_n=0},$$

where ϕ is the characteristic function of $(X_1, X_2, \ldots, X_n)$. As an example, the author evaluated the following:

Proposition 4.1. *Let X_1, X_2 and X_3 be centered Gaussian random variables with $\mathbb{E}\,X_j^2 = 1$ and $\mathbb{E}\,X_jX_k = \rho_{jk}$ if $j \neq k$, then*

$$\begin{aligned}\mathbb{E}\,|X_1X_2X_3| = \left(\frac{2}{\pi}\right)^{3/2}\Bigg[&\sqrt{\det\Sigma} + (\rho_{12}+\rho_{13}\rho_{23})\sin^{-1}\left(\frac{\rho_{12}-\rho_{13}\rho_{23}}{\sqrt{(1-\rho_{13}^2)(1-\rho_{23}^2)}}\right)\\ &+ (\rho_{13}+\rho_{12}\rho_{23})\sin^{-1}\left(\frac{\rho_{13}-\rho_{12}\rho_{23}}{\sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)}}\right)\\ &+ (\rho_{23}+\rho_{13}\rho_{12})\sin^{-1}\left(\frac{\rho_{23}-\rho_{13}\rho_{12}}{\sqrt{(1-\rho_{13}^2)(1-\rho_{12}^2)}}\right)\Bigg],\end{aligned}$$

where Σ is the covariance matrix of (X_1, X_2, X_3).

However, this method does not seem to work for some of the examples in Section 3.

4.3. Diagonalization

In [4], the authors studied a Gaussian integral which is associated with the density of critical points of given holomorphic section which is related to counting vacua in string theory.

The Gaussian integral is of the form

$$I = \int_{Sym(m,\mathbb{C})\times\mathbb{C}} |\det(HH^* - |x|^2 I)|\exp\left(-\Lambda^{-1}\langle (H,x),(H,x)\rangle\right)\,dH dx, \tag{4.7}$$

where $\mathrm{Sym}(m,\mathbb{C})$ is the space of $m\times m$ complex symmetric matrices, dH and dx denote Lebesgue measures. As mentioned in [4], the integral contains an absolute value, which makes it difficult to evaluate the density explicitly when the dimension is greater than one, or even analyze its dependence. In particular, one cannot simplify it with Wick's formula. When the dimension is smaller than 5, they obtained results using Maple 10. For the general cases, they used a limiting procedure and the Itzykson-Zuber integral to rewrite (4.7) into a different Gaussian integral involving absolute value of products. The products come from an interesting diagonalization procedure, see [4] for details.

4.4. Properties of random matrices

Sometimes Gaussian integral related with random matrices can be obtained by applying certain properties of random matrices. Examples can be found in [6] and [14] such as the result we used in the proof of Proposition 3.5:

Proposition 4.2. *Assume M is a random matrix with i.i.d. standard complex Gaussian entries, then*

$$\det MM^* \sim \prod_{j=1}^{n} \chi_j^2 \quad \text{and} \quad |\det M| \sim \sqrt{\prod_{j=1}^{n} \chi_j^2}.$$

Remark. The expression can be proved by computing the characteristic function or the Gram-Schmidt process on the random matrix. This result is used in a proof of the Gaussian representation of intrinsic volume of convex body, see [19] for details.

For Gaussian Orthogonal Ensemble matrix, which has independent real Gaussian entries and is invariant under orthogonal transformation $M \to O^{-1}MO$, the following proposition about the absolute value function is proved:

Proposition 4.3. ([1]) *Let G_n be a Gaussian Orthogonal Ensemble matrix, then for $\nu \in R$ one has*

$$\mathbb{E}\left(|\det(G_n - \nu I_n)|\right) = 2^{3/2}\Gamma\left(\frac{n+3}{2}\right)\exp\left(\frac{\nu^2}{2}\right)\frac{q_{n+1}(\nu)}{n+1},$$

where $q_n(\nu)$ denotes the density of eigenvalues of $n \times n$ GOE matrices at the point ν, that is, $q_n(\nu)d\nu$ is the probability of G_n having an eigenvalue in the interval $(\nu, \nu + d\nu)$.

The basic idea of the proof uses the eigenvalues of G_n, which are denoted by $\nu_1, \ldots, \nu_n$. It is well known that the joint density f_n of the n-tuple of random variables $(\nu_1, \ldots, \nu_n)$ is given by the formula:

$$f_n(\nu_1, \ldots, \nu_n) = c_n \exp\left(-\frac{\sum_{j=1}^n \nu_j^2}{2}\right) \prod_{1 \le j < k \le n} |\nu_j - \nu_k|,$$

where $c_n := (2\pi)^{-n/2}(\Gamma(3/2))^n(\prod_{j=1}^n \Gamma(1 + j/2))^{-1}$. Then one has

$$\begin{aligned}
&\mathbb{E}\left(|\det(G_n - \nu I_n)|\right) \\
&\quad = \mathbb{E}\left(\prod_{j=1}^{n} |\nu_j - \nu|\right) \\
&\quad = \int_{\mathbb{R}^n} \prod_{j=1}^{n} |\nu_j - \nu| c_n \exp\left(-\frac{\sum_{j=1}^n \nu_j^2}{2}\right) \prod_{1 \le j < k \le n} |\nu_j - \nu_k| \, d\nu_1 \cdots d\nu_n \\
&\quad = e^{\nu^2/2} \frac{c_n}{c_{n+1}} \int_{\mathbb{R}^n} f_{n+1}(\nu_1, \ldots, \nu_n, \nu) \, d\nu_1 \cdots d\nu_n \\
&\quad = e^{\nu^2/2} \frac{c_n}{c_{n+1}} \frac{q_{n+1}(\nu)}{n+1}.
\end{aligned}$$

Note that for GOE matrices, $q_n(\nu)$ can be expressed via Hermite polynomials. In fact (see [14]),

$$\begin{aligned} e^{\nu^2/2} q_n(\nu) = e^{-\nu^2/2} \sum_{k=0}^{n-1} a_k^2 H_k^2(\nu) \\ + 1/2(n/2)^{1/2} a_{n-1} a_n H_{n-1}(\mu) \\ \times \left(\int_{-\infty}^{+\infty} e^{-y^2/2} H_n(y) dy - 2 \int_{\nu}^{+\infty} e^{-y^2/2} H_n(y) dy \right) \\ + \mathbf{1}_{\{n \text{ odd}\}} \frac{H_{n-1}(\nu)}{\int_{-\infty}^{+\infty} e^{-y^2/2} H_{n-1}(y) dy}, \end{aligned}$$

where $a_k := (2^k k! \sqrt{\pi})^{-1/2}$ and $H_k(x) := -\exp(x^2)(\exp(-x^2))^{(k)}$ is the Hermite polynomials.

Another example comes from the Selberg integral and its random matrix formulation in terms of Mehta's integral. From our point of view, they are Gaussian integrals for certain absolute value functions. The following proposition is discussed extensively in [14] with other related examples.

Proposition 4.4. *Assume that ξ_j's are independent standard Gaussian random variables, $j = 1, 2, \ldots, n$. Then*

$$\mathbb{E} \prod_{1 \le j < k \le n} |\xi_j - \xi_k|^{2\gamma} = \prod_{j=1}^{n} \frac{\Gamma(1 + j\gamma)}{\Gamma(1 + \gamma)}.$$

Note that when $\gamma = 1/2, 1$ and 2, this integral is related to the distributions of eigenvalues of Gaussian Orthogonal, Unitary and Symplectic Ensembles, respectively. The proof follows the standard argument of Selberg integral. For history and recent development, see the excellent survey [5]. There are also interesting connections with the so called linear polarization conjecture, see [11].

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